Empirical Wavelets: Theory & Applications

Jérôme Gilles

Department of Mathematics and Statistics, SDSU
jgilles@sdsu.edu
http://jegilles.sdsu.edu
Gabor-Heisenberg incertitude principle limited TF:

- Short-time Fourier transform:
  \[ \mathcal{F}_f^w(m, n) = \int f(s)w(s - nt_0)e^{-im\omega_0s}ds. \]

- Wavelet transform:
  \[ \mathcal{WT}_f(m, n) = a_0^{-m/2} \int f(t)\psi(a_0^{-m}t - nb_0)dt. \]
Time-Frequency (TF) analysis (1/2)

Gabor-Heisenberg incertitude principle limited TF:

- Short-time Fourier transform:
  \[ \mathcal{F}_f^w(m, n) = \int f(s)w(s - nt_0)e^{-i\omega_0 s}ds. \]

- Wavelet transform:
  \[ \mathcal{WT}_f(m, n) = a_0^{-m/2} \int f(t)\psi(a_0^{-m}t - nb_0)dt. \]

How to go beyond this limitation? ⇒ Hilbert-Huang transform\(^1\):

Step 1  Empirical Mode Decomposition (EMD): decompose \( f \) as

\[
\{f_k\}_{k=0}^N \quad \text{s.t} \quad f(t) = \sum_{k=0}^N f_k(t)
\]

where \( f_k(t) = F_k(t) \cos(\varphi_k(t)) \quad \text{s.t} \quad F_k(t), \varphi'_k(t) > 0 \quad \forall t. \)

Main assumption: \( F_k \) and \( \varphi'_k \) vary much slower than \( \varphi_k \).

Step 2 Hilbert Transform (HT):

\[ \mathcal{H}_f(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{f_k(\tau)}{t - \tau} d\tau \]

Property: if \( f_k(t) = F_k(t) \cos(\varphi_k(t)) \) then

\[ f_k^*(t) = f_k(t) + i\mathcal{H}_f(t) = F_k(t)e^{i\varphi_k(t)} \]

\[ \Rightarrow \text{easy to extract } F_k(t) \text{ and the instantaneous frequency } \frac{d\varphi_k}{dt}(t). \]
Step 2. Hilbert Transform (HT):

\[ f_k(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi,t) e^{-i\xi t}}{\xi} d\xi \]

**Property:** if \( f_k(t) \) then 
\[ \hat{f}_k(t) = f_k(t) \cos \phi_k(t) \]

It is easy to extract \( f_k(t) \) and the instantaneous frequency \( \frac{d\phi_k}{dt}(t) \).
Step 2 Hilbert Transform (HT):

\[ \mathcal{H}_f(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{f(\tau)}{t - \tau} d\tau \]

Property: if \( f_k(t) = F_k(t) \cos(\varphi_k(t)) \) then

\[ f_k^*(t) = f_k(t) + i\mathcal{H}_f(t) = F_k(t) e^{i\varphi_k(t)} \]

⇒ easy to extract \( F_k(t) \) and the instantaneous frequency \( \frac{d\varphi_k}{dt}(t) \).

**Drawbacks:**
- EMD is purely algorithmic
- Lacks of mathematical foundation
Step 2 Hilbert Transform (HT):

\[ \mathcal{H}_f(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{f_k(\tau)}{t - \tau} d\tau \]

Property: if \( f_k(t) = F_k(t) \cos(\varphi_k(t)) \) then

\[ f_k^*(t) = f_k(t) + i\mathcal{H}_f(t) = F_k(t)e^{i\varphi_k(t)} \]

\[ \Rightarrow \text{easy to extract } F_k(t) \text{ and the instantaneous frequency } \frac{d\varphi_k}{dt}(t). \]

**Drawbacks:**

- EMD is purely algorithmic
- Lacks of mathematical foundation

**Observation:** behaves like a data-driven filter bank
Empirical wavelet transform (EWT): Concept

Idea: Build an adaptive (i.e data-driven) wavelet filter bank to replace the EMD.
Empirical wavelet transform (EWT): Concept

**Idea:** Build an adaptive (i.e data-driven) wavelet filter bank to replace the EMD.

**How?** AM-FM components $\Leftrightarrow$ “compactly supported” spectral modes.
Empirical wavelet transform (EWT): Concept

**Idea:** Build an adaptive (i.e data-driven) wavelet filter bank to replace the EMD.

**How?** AM-FM components $\leftrightarrow$ “compactly supported” spectral modes.

$\Rightarrow$ segment Fourier spectrum to find the filter supports
Empirical wavelet transform (EWT): Concept

**Idea:** Build an adaptive (i.e data-driven) wavelet filter bank to replace the EMD.

**How?** AM-FM components $\leftrightarrow$ “compactly supported” spectral modes.

$\Rightarrow$ segment Fourier spectrum to find the filter supports

$\Rightarrow$ define set of boundaries $\{\omega_n\}$
**Empirical wavelet transform (EWT): Concept**

**Idea:** Build an adaptive (i.e data-driven) wavelet filter bank to replace the EMD.

**How?** AM-FM components ⇔ “compactly supported” spectral modes.
⇒ segment Fourier spectrum to find the filter supports
⇒ define set of boundaries \( \{ \omega_n \} \)
⇒ define transition areas and then wavelet filters

\[
\hat{\phi}_n(\omega) = \begin{cases} 
1 & \text{if } |\omega| \leq (1 - \gamma) \omega_n \\
\cos \left( \frac{\pi}{2} \beta \left( \frac{1}{2 \gamma \omega_n} (|\omega| - (1 - \gamma) \omega_n) \right) \right) & \text{if } (1 - \gamma) \omega_n \leq |\omega| \leq (1 + \gamma) \omega_n \\
0 & \text{otherwise}
\end{cases}
\]

\[
\hat{\psi}_n(\omega) = \begin{cases} 
1 & \text{if } (1 + \gamma) \omega_n \leq |\omega| \leq (1 - \gamma) \omega_{n+1} \\
\cos \left( \frac{\pi}{2} \beta \left( \frac{1}{2 \gamma \omega_{n+1}} (|\omega| - (1 - \gamma) \omega_{n+1}) \right) \right) & \text{if } (1 - \gamma) \omega_{n+1} \leq |\omega| \leq (1 + \gamma) \omega_{n+1} \\
\sin \left( \frac{\pi}{2} \beta \left( \frac{1}{2 \gamma \omega_n} (|\omega| - (1 - \gamma) \omega_n) \right) \right) & \text{if } (1 - \gamma) \omega_n \leq |\omega| \leq (1 + \gamma) \omega_n \\
0 & \text{otherwise}
\end{cases}
\]
Experiment: ECG

![ECG Waveform](image1)

-0.2 0.0 0.2 0.4 0.6

-0.2 0.2 0.4 0.6

50 100 150 200 250

0.02 0.04 0.06 0.08 0.1 0.12 0.14

0.02 0.04 0.06 0.08 0.1 0.12 0.14

0.03 0.05 -0.15 -0.05 0.05 0.15 -0.05 0.05 -0.05 0.05 -0.06 -0.02 0.02 0.06 -0.1 0.1 0.4
Experiment: ECG
2D Tensor product extension

1DFFT

Average

1DFFT

Average

MFB

MFB

MFB

...
2D Tensor EWT - Example

\[ N_C = N_R = 3 \]
Wavelets defined on concentric dyadic annuli in the Fourier plane $\rightarrow$ radial profile $\leftrightarrow$ 1D dyadic wavelet
Wavelets defined on concentric dyadic annuli in the Fourier plane $\rightarrow$ radial profile $\leftrightarrow$ 1D dyadic wavelet

Empirical extension: detect annuli positions
Wavelets defined on concentric dyadic annuli in the Fourier plane $\rightarrow$ radial profile $\leftrightarrow$ 1D dyadic wavelet

Empirical extension: detect annuli positions

Detect the boundaries over a radial line
Wavelets defined on concentric dyadic annuli in the Fourier plane $\rightarrow$ radial profile $\leftrightarrow$ 1D dyadic wavelet

Empirical extension: detect annuli positions

Detect the boundaries over a radial line $\rightarrow$ average spectrum for all $\theta$
Wavelets defined on concentric dyadic annuli in the Fourier plane $\rightarrow$ radial profile $\leftrightarrow$ 1D dyadic wavelet

Empirical extension: detect annuli positions

Detect the boundaries over a radial line $\rightarrow$ average spectrum for all $\theta$

Useful tool: Pseudo-Polar Fourier Transform (PPFT)
2D Empirical Littlewood-Paley Transform - Example

$N = 4$
Empirical “Curvelet” transforms

Idea: fix scales and angular positions empirically.

\[ \mathcal{F}_2(\psi_{nm})(\omega, \theta) = W_n(\omega) V_m(\theta) \]

Different options:

1. independent detections:
   \[ \Omega_\omega = \{\omega^n\}_{n=0,...,N_s}, \ \Omega_\theta = \{\theta^m\}_{m=1,...,N_\theta} \]

2. scales first and then angles per scale:
   \[ \Omega_\omega = \{\omega^n\}_{n=0,...,N_s}, \ \Omega_\theta^{\omega} = \{\theta^{n,m}\}_{m=1,...,N_\theta} \]

3. angles first and then scales per angular sector:
   \[ \Omega_\theta = \{\omega^{n,m}\}_{n=0,...,N_s}, \ \Omega_\omega = \{\theta^m\}_{m=1,...,N_\theta} \]
Empirical Curvelet Transform I - Examples

\[ N_s = 4 \quad N_\theta = 4 \]
Empirical Curvelet Transform II - Examples

\[ N_s = 4 \quad \text{and} \quad N_\theta = 4 \]
“Two dominant modes” within the $\beta$-range → confirmed!

“Synchronization issues between Thalamus and Global Pallidus” → coherence metrics.
Application - Texture analysis

unsupervised segmentation

77-88% of good classification (> 3% than any other wavelets)
Application - Texture analysis

Supervised segmentation

98.36% good classification (only 68.64% for the state of the art!)
Current and Future investigations

From the mathematical side . . .

- Current: Theoretical framework for continuous EW,
- Current: Create 3D Empirical Curvelets,
- Current: More arbitrary 2D supports → Voronoi EW,
- Future: revisit Littlewood-Paley theory and the definition of Besov like spaces,

From the applications side . . .

- Pursue Parkinson’s disease investigations,
- EEG analysis of epileptic patients (characterization, prediction),
- Image restoration (denoising, deblurring, . . .).
Current and Future investigations

From the mathematical side . . .

- Current: Theoretical framework for continuous EW,
- Current: Create 3D Empirical Curvelets,
- Current: More arbitrary 2D supports → Voronoi EW,
- Future: revisit Littlewood-Paley theory and the definition of Besov like spaces,

From the applications side . . .

- Pursue Parkinson’s disease investigations,
- EEG analysis of epileptic patients (characterization, prediction),
- Image restoration (denoising, deblurring, . . .).


J.Gilles, K.Heal, “A parameterless scale-space approach to find meaningful modes in histograms - Application to image and spectrum segmentation”, International Journal of Wavelets, Multiresolution and Information Processing, Vol.12, No.6, 1450044-1–1450044-17, December 2014

Y.Huang, V.De Bortoli, F.Zhou, J.Gilles, "Review of wavelet-based unsupervised texture segmentation, advantage of adaptive wavelets", IET Image Processing, Vol.12, No.9, 1626–1638, August 2018

Y.Huang, F.Zhou, J.Gilles, "Empirical curvelet based Fully Convolutional Network for supervised texture image segmentation", submitted


MATLAB Toolbox available in Fileexchange in Matlab Central website


J.Gilles, K.Heal, “A parameterless scale-space approach to find meaningful modes in histograms - Application to image and spectrum segmentation”, International Journal of Wavelets, Multiresolution and Information Processing, Vol.12, No.6, 1450044-1–1450044-17, December 2014

Y.Huang, V.De Bortoli, F.Zhou, J.Gilles, "Review of wavelet-based unsupervised texture segmentation, advantage of adaptive wavelets", IET Image Processing, Vol.12, No.9, 1626–1638, August 2018

Y.Huang, F.Zhou, J.Gilles, "Empirical curvelet based Fully Convolutional Network for supervised texture image segmentation", submitted


MATLAB Toolbox available in Fileexchange in Matlab Central website

THANK YOU!
### Original methods

- Midpoint between consecutive local maxima
- Lowest minimum between consecutive local maxima
- Fine To Coarse algorithm

Eventually combined with some spectrum preprocessing (global trend removal, smoothing)
About the Fourier support detection

Original methods

- Midpoint between consecutive local maxima
- Lowest minimum between consecutive local maxima
- Fine To Coarse algorithm

Eventually combined with some spectrum preprocessing (global trend removal, smoothing)

Drawbacks: many possible combinations and many parameters to choose, the number of modes must be known.
About the Fourier support detection

Original methods

- Midpoint between consecutive local maxima
- Lowest minimum between consecutive local maxima
- Fine To Coarse algorithm

Eventually combined with some spectrum preprocessing (global trend removal, smoothing)

Drawbacks: many possible combinations and many parameters to choose, the number of modes must be known.

Latest investigation: a scale-space approach
Principle

Let $f(x)$ be a signal, $t$ be the “scale-parameter”, then the scale-space representation of $f$ is given by

$$S_f(x; t) = \int_{\mathbb{R}} g(y; t)f(x - y) \, dy$$

$\leftrightarrow$ convolution of $f$ with the Gaussian kernel $g(x; t) = \frac{1}{\sqrt{2\pi}t} e^{-x^2/(2t)}$
Consistency of minima through scales

We detect the local minima at each scale $\Rightarrow$ binary scale-space map of minima

Idea: meaningful modes correspond to consistent minima through scales $\rightarrow$ curves longer than a certain threshold $T$
Clustering approach: Otsu’s method

Goal: split an histogram into two classes $H_1$ and $H_2$ such that we have

- minimum variance within each class
- maximum variance between each class

⇒ easy to implement and extremely fast.