Obstructions to global dynamic feedback stabilization

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Outline

1. Motivation
2. Fibre bundles and dynamic feedback
3. Main results and Sketch of proofs
4. Is it better to be twisted? (for a bundle)
5. Summary and Outlook

Motivation

- Systems evolving on closed manifolds (i.e., compact without boundary).
  E.g.: Attitude control, control of Spin systems
- Systems evolving on finite-dim. vector bundles on closed manifolds
  E.g.: Formation control, UAV control.

$\Rightarrow$ global state feedback stabilization is not possible:

Formal proof: S.P. Bhat and D.S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon", in *Systems & Control Letters*, 2000, vol 39.

Conventions: Maps are continuous. Stabilization is asymptotic. M, U are path-connected.

Notions of dynamic feedback stabilization

- State-space M: smooth manifold or vector bundle over a smooth manifold
  Control space U: smooth manifold.
- Dynamic feedback: arbitrary plant dynamics $\dot{x} = f(x, u)$ and controller dynamics $\dot{u} = g(x, u)$, defined in neighborhood $Op(x, u) \subset E$.

1. strongly stabilizes x^* if it stabilizes some combination system $+$ controller state.
2. weakly stabilizes x^* if it stabilizes system state and controller-state is arbitrary.
3. 1-point almost globally weakly/strongly stabilizes x^* if there exist one point z^* in closed-loop space E so that x^* is weakly/strongly globally stabilized over $E - \{z^*\}$.

Lemma

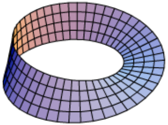
Assume that there exists a dynamic feedback controller (globally) weakly stabilizing x^*. Set $U^* := p^{-1}(x^*)$. Then, there exists a deformation retraction of E onto U^*.

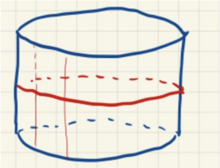
Motivation

This talk: Does there exist a dynamic controller that *globally* stabilizes a given state?

- Dynamic feedback evolves on a fibre bundle E with base space M and fibre U:

$$p : E \to M \text{ with fibre } U \text{ or } U \to E \xrightarrow{p} M.$$

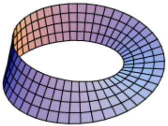


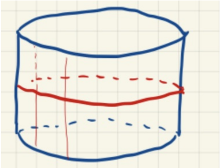


- From a topological viewpoint: state and dynamic feedback are locally the same, but globally different. A bundle E is called *trivial* if $E = M \times U$. Otherwise, we call it *twisted*.

Motivation

Key insights:





Twisting is a *design parameter* for global dynamic feedback law.

- Lemma: M is contractible $\Rightarrow$ *all* bundles over M are *trivial*.

We show that the topology of closed-loop system ($=$ system $+$ controller) matters!

Main results

Theorem (A)

Consider a control system with state-space M and control space U. Then, $x^* \in M$ is weakly/strongly globally stabilizable by dynamic feedback only if M is contractible.

Theorem (B)

Consider a control system with closed state-space M and control space U. Then, $x^* \in M$ is weakly/strongly 1-point globally stabilizable by dynamic feedback only if the closed-loop system has a total space E which is a nontrivial *bundle.*

Mathematical Background

- Exact sequence: $\cdots \longrightarrow A_{k+1} \xrightarrow{\ell_k} A_k \xrightarrow{\ell_{k-1}} A_{k-1} \longrightarrow \cdots$ with A_k abelian groups and $\operatorname{im} \ell_k = \ker \ell_{k-1}$
- Homology groups H_k and homotopy groups π_k are associated to a topological space.
- Theorem[Hurewicz] *Let M be a path connected space. Then for all $k \geq 1$, there exists a homomorphism $h_* : \pi_k(M) \to H_k(M;\mathbb{Z})$. Furthermore, for $k \geq 2$, if $\pi_i(M) = 0$ for all $1 \leq i \leq k-1$, then $h_* : \pi_k(M) \to H_k(M;\mathbb{Z})$ is an isomorphism.*
- A fibre bundle $U \hookrightarrow E \to M$ obeys the (Puppe) long exact sequence

$$\cdots \longrightarrow \pi_k(U) \longrightarrow \pi_k(E) \longrightarrow \pi_k(M) \longrightarrow \pi_{k-1}(U) \longrightarrow \cdots$$

- For $A, B \subseteq M$ so that $\operatorname{int} A \cup \operatorname{int} B = M$, we have the Mayer-Vietoris sequence

$$\cdots \to H_{k+1}(M) \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(M) \to \cdots$$

Mathematical background

- Künneth formula describes the homology of a product space: $M_1 \times M_2$

$$0 \xrightarrow{\ell_1} \bigoplus_{i+j=k} H_i(M_1) \otimes H_j(M_2) \xrightarrow{\ell_2} H_k(M_1 \times M_2) \xrightarrow{\ell_3} \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(M_1), H_j(M_2)) \xrightarrow{\ell_4} 0 \quad (1)$$

- Let $x^* \in M$; (M, x^*) is a *pointed space*. PM is the path space of (M, x^*)

$$PM := \{\gamma : [0,1] \to M \text{ with } \gamma(1) = x^*\}$$

  equipped with the compact-open topology.

- The loop space ΩM of (M, x^*) is the subspace of PM defined as

$$\Omega M := \{\gamma : [0,1] \to M \text{ with } \gamma(0) = \gamma(1) = x^*\}.$$

  It holds that $\pi_i(\Omega M) = \pi_{i+1}(M)$.

Proof sketch

The main theorems are consequences of the following propositions

Proposition (A)

Let M, U be smooth, finite dimensional manifolds. Let $p : E \to M$ be a fibre bundle with fibre U. Then E deformation retracts onto a point or a fibre only if M is contractible.

Let $E^* := E \backslash 1$ point.

Proposition (B)

Let M, U be smooth, finite dimensional manifolds with M closed. Let $p : E \to M$ be a fibre bundle with fibre U. Then, there exists a $x^* \in E$ such that $E^* := E - \{z^*\}$ deformation retracts onto a point $z^* \in E$ or a fibre $p^{-1}(x^*)$, for some z^*, x^*, only if $p : E \to M$ is a nontrivial bundle.

Proof sketch of Proposition A

Obstruction to weak stabilization: Prove that if E deformation retracts onto U_0, then M has homotopy type of a point. Then show that M deformation retracts to a point.

Claim: $\pi_k(M) = 0$ for all $k \geq 1$. Proof: Long exact sequence of a fibration.

$$\cdots \xrightarrow{\delta_*} \pi_k(U_0) \xrightarrow{\iota_*} \pi_k(E) \xrightarrow{p_*} \pi_k(M) \xrightarrow{\delta_*} \pi_{k-1}(U_0) \xrightarrow{\iota_*} \cdots$$

Claim: M is homotopy equivalent to a point.

Theorem:[Whitehead]*If X and Y are smooth manifolds (CW-complexes) and $f : X \to Y$ is so that $f_k : \pi_k(X) \to \pi_k(Y)$ are isomorphisms for all $k \geq 0$, then X is homotopic to Y.*

Proof sketch of Proposition A

Obstruction to strong stabilization:

Lemma 1: *Let $p : E \to M$ be a fibre bundle with contractible total space. Then M is simply-connected.*

Lemma 2: *Let $p : E \to M$ be a fibre bundle with contractible total space. Then its loop space ΩM is homotopy equivalent to U.*

Proof sketch of Proposition A

Proof of Prop. A (II):

- Assume a strongly stabilizing dynamic feedback exists on $p : E \to M$
  $\Rightarrow E$ deformation retracts onto a point $z^* \in p^{-1}(x^*)$.
- Lemma 2 $\Rightarrow U$ is homotopy equivalent to ΩM.
- Let $\dim U = r$. Then for any field F

$$H_i(\Omega M; F) \simeq H_i(U; F) = 0 \text{ for all } i \geq r + 1. \tag{2}$$

We now show that (2) can hold *only if* M is contractible.

Proposition [J-P Serre, Homologie Singulière des espaces fibrés, 1951, Annals of Math.]:
Let F be a field and M be simply connected. If for some $n \geq 2$, $H_n(M; F) \neq 0$ and $H_i(M; F) = 0$ for all $i > n$, then for all integers $k \geq 0$, there exists $0 < j < n$ so that $H_{k+j}(\Omega M; F) \neq 0$.

Proof sketch of Proposition A

Case 1: $H_n(M; F) \neq 0$ for some $n \geq 2$ and field F: In this case, the homology of ΩM does not vanish, which contradicts (2) and concludes the proof.

Case 2: $H_n(M; F) = 0$ for all $n \geq 2$ and all fields F:

- Lemma 1 $\Rightarrow H_1(M; \mathbb{Z}) = \pi_1(M) = 0$.
- From Universal Coefficients Theorem, if $H_n(M; F) = 0$ for all fields F and for all $n \geq 1$ then $H_n(M, \mathbb{Z}) = 0$.
- By *Hurewicz*: $h_* : \pi_2(M) \to H_2(M; \mathbb{Z})$ is an isomorphism $\Rightarrow \pi_2(M) = 0$. Next, $\pi_1(M) = \pi_2(M) = 0 \Rightarrow h_* : \pi_3(M) \to H_3(M; \mathbb{Z}) = 0$ is an isomorphism. Iterating, we get $\pi_k(M) = 0$ for all $k \geq 1$.
- Thus the inclusion map $j : \{x^*\} \to M$ is a weak homotopy equivalence and, by Whitehead Theorem, we conclude that M is contractible.

Proof Sketch Proposition B

Let $z^* \in E$. Denote $E^* := E - \{z^*\}$.

We show that E^* deformation retracts onto z^* only if E is twisted

By contradiction, assume that E is trivial and E^* is contractible $\Rightarrow H_k(E^*) = 0$ for $k \geq 1$.

- Lemma 3: Let $p : E \to M$ be a trivial bundle with fibre U and $\dim M = n$. If M is orientable, $H_n(E) \neq 0$ and $H_n(E) \neq H_n(U)$; otherwise $H_{n-1}(E) \neq 0$ and $H_{n-1}(E) \neq H_{n-1}(U)$.
- Lemma 4: Let $p : E \to M$ be a fibre-bundle with $\dim E = m$. Then, for $1 \leq k \leq m-2$, we have $H_k(E) = H_k(E^*)$.
- Consider three cases: $\dim U \geq 2$, $U = \mathbb{R}$ or $U = S^1$.

Proof Sketch Proposition B

Case $\dim U \geq 2$: Then, $m = \dim E \geq n+2$. Lemma 4 $\Rightarrow H_i(E) = H_i(E^*) = 0$ for $i = 1, \ldots, n$. But Lemma 3 says either $H_n(E) \neq 0$ or $H_{n-1}(E) \neq 0 \Rightarrow$ contradiction.

Case $U = S^1$: We have $E = M \times S^1$. From the Künneth of order 1, we obtain

$$0 \to H_0(M) \otimes H_1(S^1) \oplus H_1(M) \otimes H_0(S^1) \to H_1(M \times S^1) \to \mathrm{Tor}(H_0(M), H_0(S^1)) \to 0$$

Since $H_0(M) = H_0(S^1) = \mathbb{Z}$, the torsion term vanishes, and we get

$$0 \to \mathbb{Z} \oplus H_1(M) \to H_1(E) \to 0. \tag{3}$$

This shows that $H_1(E) \neq 0$. But $H_1(E^*)$ vanishes by assumption, so Lemma 4 yields a contradiction.

Proof Sketch of Proposition B

Case $U = \mathbb{R}$: Then, $E = M \times \mathbb{R} \simeq M \Rightarrow H_k(E) \simeq H_k(M)$ and in particular $H_m(E) = 0$.

- Let B be an open ball containing z^*, then $E = E^* \cup B$ and Mayer-Vietoris yields

$$\begin{aligned}\cdots \longrightarrow H_m(E) \longrightarrow H_{m-1}(E^* \cap B) \longrightarrow H_{m-1}(E^*) \oplus H_{m-1}(B) \longrightarrow \\ H_{m-1}(E) \longrightarrow H_{m-2}(E^* \cap B) \longrightarrow \cdots \end{aligned} \tag{4}$$

- We have $E^* \cap B \simeq S^{m-1}$, and $H_k(S^{m-1}) = \mathbb{Z}$ for $k = m-1$ and $k = 0$, and is zero otherwise; $H_k(B) = 0$ for all $k > 0$.
- From the Mayer-Vietoris sequence (4) starting at $H_m(E) = 0$, and using the above observations, we have

$$0 \xrightarrow{\ell_1} \mathbb{Z} \xrightarrow{\ell_2} H_{m-1}(E^*) \xrightarrow{\ell_3} H_{m-1}(E) \to 0$$

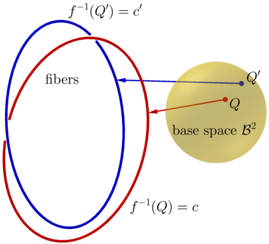
$\Rightarrow \ell_2$ is injective and thus $H_{m-1}(E^*) \neq 0$ and thus E^* is not contractible.

Does twisting help? Example: case $M = S^2$

- State feedback: $\dot{x} = f(x, u(x))$ so that x_0 is GAS?
- Dynamic feedback $\begin{cases} \dot{x} &= f(x,u) \\ \dot{u} &= g(x,u) \end{cases}$ so that x_0 is GAS?
- Dynamic feedback on $M \times U$ so that x_0 is almost globally (1-point) stable?
- Dynamic feedback on twisted bundle?

"Hopf system": Based on the Hopf fibration:

$$S^1 \longrightarrow S^3 \xrightarrow{p} S^2$$



Summary

- Natural habitat for dynamic feedback control: *fibre bundles*

$$p : E \to M \text{ with fibre } U.$$

  It can be trivial or twisted.

- Global dynamic feedback stabilization is possible only if the state-space M is contractible: same as state-feedback.

- Almost global dynamic feedback (1-point) stabilization is possible only if the state-space M is contractible or the closed-loop space E is a twisted bundle.

- Example of almost global 1-point stabilization on $M = S^2$ via "Hopf system".

Outlook

- How to design bundles over M that permit almost global (1-point) stabilization.
- Almost global stabilization = remove one point from E.
  $\Rightarrow$ hierarchy of sets to be removed from E to permit "global" stabilization.
- How to extend the results to stabilization to submanifolds of M?
  Synchronization for a pair of systems:

$$p : E \to M \times M$$

Goal: Stabilize the diagonal $M \subset M \times M$. Do similar obstructions exist?