A Novel Multiscale Framework for Kinetic Plasmas: A Conditional Formulation of the Vlasov-Ampère System

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Model: Vlasov-Ampère System

Vlasov Equation

$$\partial_t f_\alpha + \vec{v} \cdot \nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha} \vec{E} \cdot \nabla_v f_\alpha = 0$$

$$f_\alpha(\vec{x}, \vec{v}, t)$$

Ampère Equation

$$\epsilon \partial_t \vec{E} + \vec{j} = \vec{0} \qquad \vec{E}(\vec{x}, t)$$

$$\epsilon \vec{j} = \sum_\alpha^N q_\alpha \int_{\mathbb{R}^3} d^3 v \, \vec{v} f_\alpha$$

$$\epsilon = \frac{\lambda_D}{L}$$

Design Goals of Numerical Method

1. Conserves mass, momentum, energy
2. Preserves Gauss' law
3. Quasi-neutral asymptotic preserving
4. Preserves positivity/monotonicity

Exploit the slow-manifold structure [Burby, CNSNS, 89 (2020)] in our system to ensure QN-AP property

Fast-slow system: Dynamical system with state variables $(x, y) \in X \times Y$

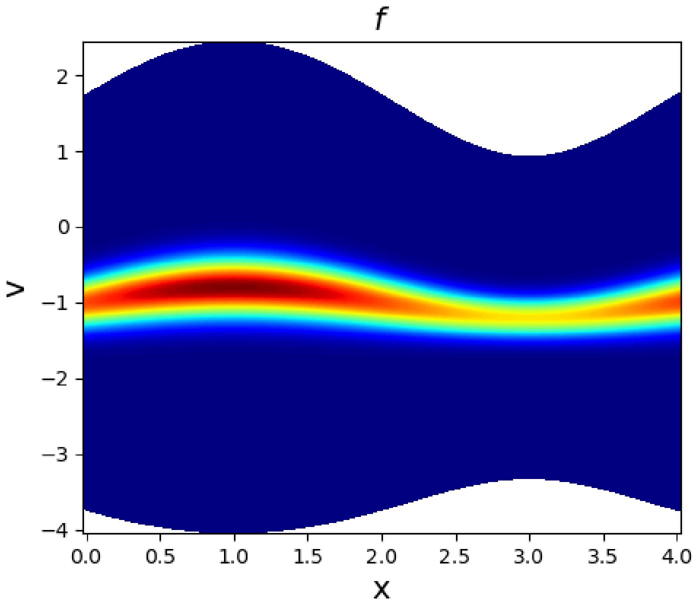
$$\epsilon \frac{dy}{dt} = f_\epsilon(x, y)$$

$$\frac{dx}{dt} = g_\epsilon(x, y)$$

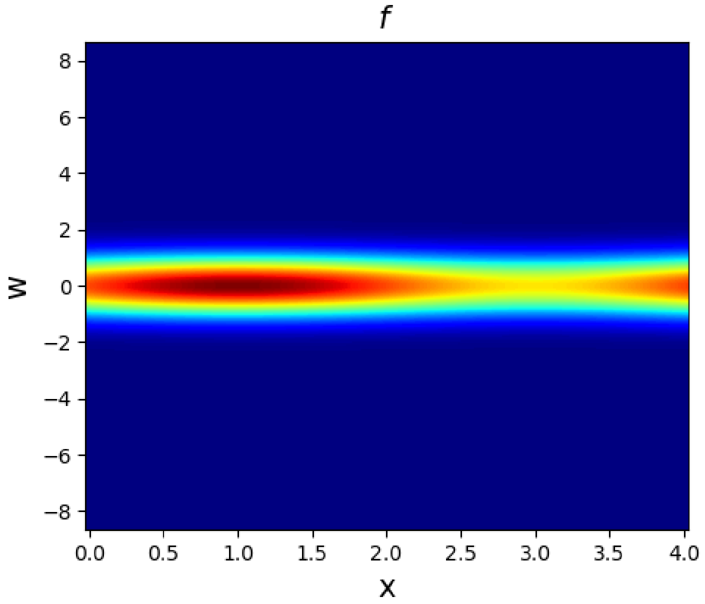
and

$D_y f_0(x, y): Y \to Y$ invertible whenever $f_0(x, y) = 0$

How do we do it?
Coordinate transformation and transformation of variable



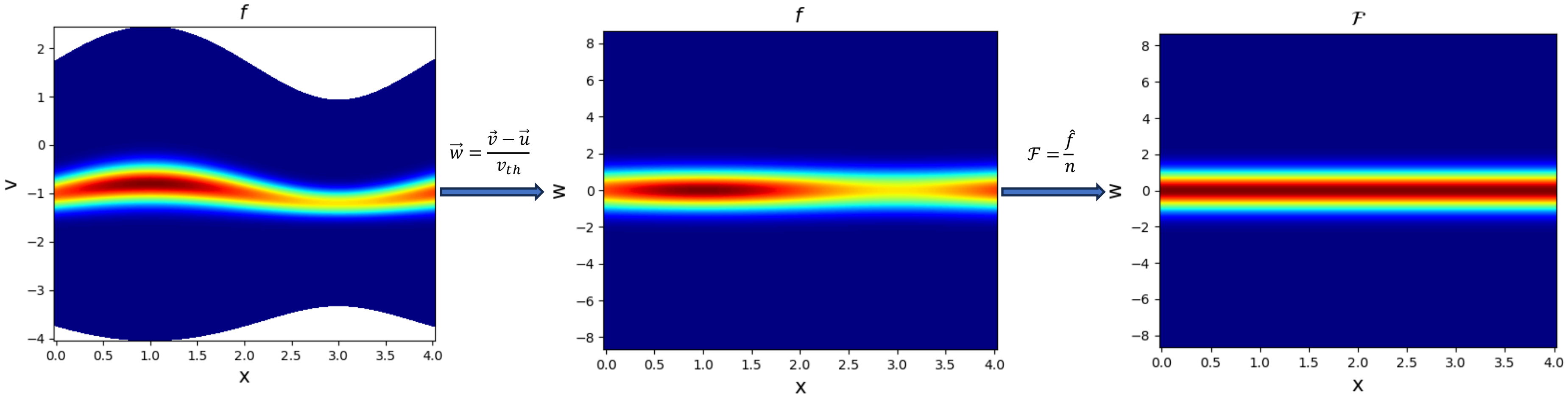
$$\vec{w} = \frac{\vec{v} - \vec{u}}{v_{th}}$$



How do we do it?
Coordinate transformation and transformation of variable

Scale the original PDF with density to define the conditional probability function (CDF):

$$\mathcal{F}(\vec{w}, t \mid \vec{x}\,) = \frac{\hat{f}}{n}$$



First explored (w/o v_{th} scaling) by J. Burby PoP, 25, 2018

Challenges have been shifted to moment-field sub-system

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot \left[\dot{\vec{x}} \mathcal{F}\right] + \frac{\partial}{\partial \vec{w}} \cdot \left[\dot{\vec{w}} \mathcal{F}\right] = \lambda \mathcal{F}$$

Positivity principle for $\mathcal{F}$ is preserved

$$\frac{\partial n_\sigma}{\partial t} + \nabla_x \cdot \vec{\gamma}_\sigma = 0$$

$$\frac{\partial \vec{\gamma}_\sigma}{\partial t} + \nabla_x \cdot \left[\frac{\vec{\gamma}_\sigma \vec{\gamma}_\sigma}{n_\sigma} + \frac{\mathbb{I} P_\sigma + \mathcal{T}_\sigma}{m_\sigma}\right] - \frac{q_\sigma}{m_\sigma} n_\sigma \vec{E} = 0$$

$$\frac{\partial \mathcal{E}_\sigma}{\partial t} + \nabla_x \cdot \left[\vec{u}_\sigma \left(\mathcal{E}_\sigma + \frac{P_\sigma}{m_\sigma}\right) + \frac{\vec{u}_\sigma \cdot \mathcal{T}_\sigma + \vec{Q}_\sigma}{m_\sigma}\right] - \frac{q_\sigma}{m_\sigma} \vec{\gamma}_\sigma \cdot \vec{E} = 0$$

$$\epsilon^2 \frac{\partial \vec{E}}{\partial t} + \sum_\sigma^{N_s} q_\sigma \vec{\gamma}_\sigma = 0$$

Fast time scales, conservation, and Gauss' law, have been isolated into this sub-system

$$\vec{u} = \frac{\vec{\gamma}}{n} \qquad P = \frac{2m}{3}\left(\mathcal{E} - \frac{\gamma^2}{2n}\right) \qquad \mathcal{T} = mnv_{th}^2 \left\langle \vec{w}\vec{w} - \frac{Tr(\vec{w}\vec{w})}{3}, \mathcal{F} \right\rangle_{\widehat{w}} \qquad \vec{Q} = \frac{mnv_{th}^3}{2} \langle \vec{w} w^2, \mathcal{F} \rangle_{\widehat{w}}$$

Slow manifold preconditioner for fast block and identity for slow blocks

slow

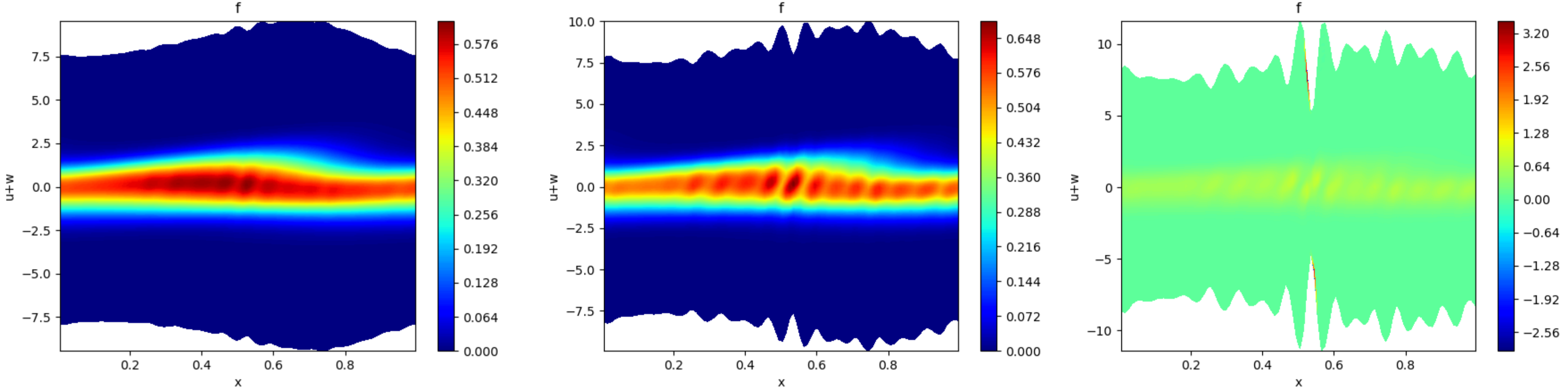
$$
\begin{array}{c}
\begin{array}{ccccccc} \delta n_i & \delta n_e & \delta \gamma_i & \delta \mathcal{E}_i & \delta \mathcal{E}_e & \delta j & \delta E \end{array} \\
\mathbb{P}^{(m)} = \begin{bmatrix}
\Delta t^{-1}\mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Delta t^{-1}\mathbb{I} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Delta t^{-1}\mathbb{I}_F & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta t^{-1}\mathbb{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta t^{-1}\mathbb{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \epsilon\Delta t^{-1}\mathbb{I}_F & -\widehat{\omega}^2 \\
0 & 0 & 0 & 0 & 0 & \mathbb{I}_F & \epsilon\Delta t^{-1}\mathbb{I}_F
\end{bmatrix}
\end{array}
$$

fast

Allows us to take $\Delta t \gg \epsilon$, but still subject to CFL condition

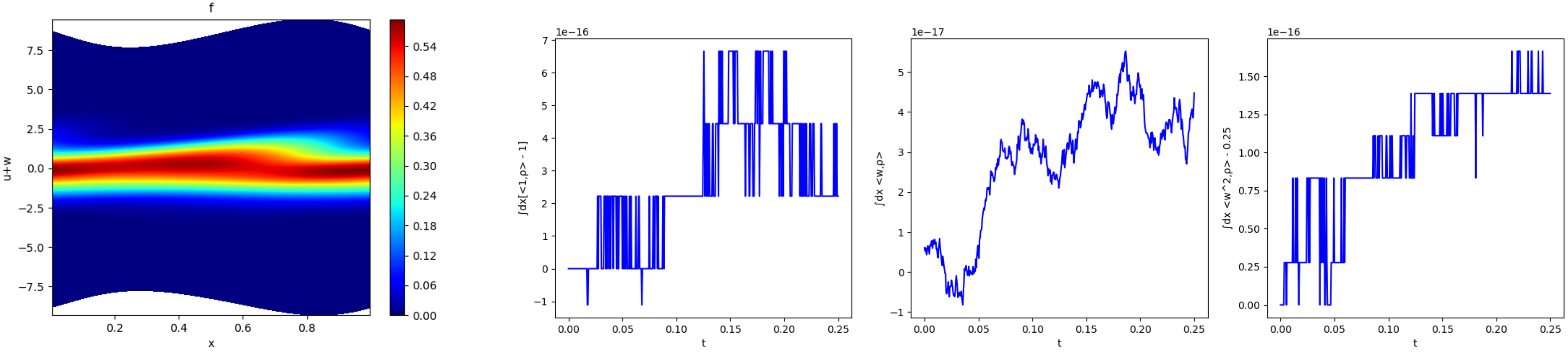
simulation unstable without discrete invariance preservation

$$\left\langle \vec{\phi}, \boldsymbol{\mathcal{F}}_l^{k+1} - \boldsymbol{\mathcal{F}}_l^{k} \right\rangle_w = \vec{\mathbb{E}}_{\mathcal{F}}$$



Discrete $\mathcal{F}$ invariance preservation allows for a stable integration

$$\left\langle \vec{\phi}, \boldsymbol{\mathcal{F}}_l^{k+1} - \boldsymbol{\mathcal{F}}_l^{k} \right\rangle_w = \vec{0}$$



Ion Acoustic Shockwave: A multiscale problem with non-trivial solution and strong gradients

$$\partial_t n_\sigma + \partial_x \gamma_\sigma = 0$$

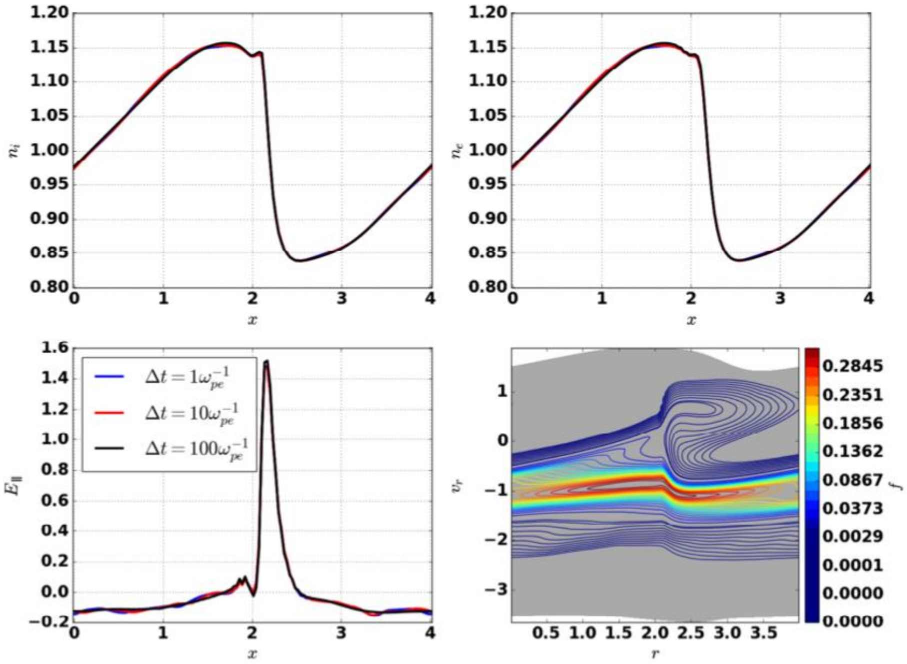
$$\partial_t \gamma_i + \partial_x \left[\frac{\gamma_i^2}{n_i} + \frac{P_i}{m_i}\right] - \frac{q_i}{m_i} n_i E = 0$$

$$\partial_t \mathcal{E}_{t,\sigma} + \partial_x \left[u_\sigma \mathcal{E}_{t,\sigma} + \frac{Q_\sigma}{m_\sigma}\right] = 0$$

$$\epsilon \partial_t j + \sum_\sigma^{N_s} q_\sigma \partial_x \left(\frac{\gamma_\sigma^2}{n_\sigma} + \frac{P_\sigma}{m_\sigma}\right) - \omega_p^2 E = 0$$

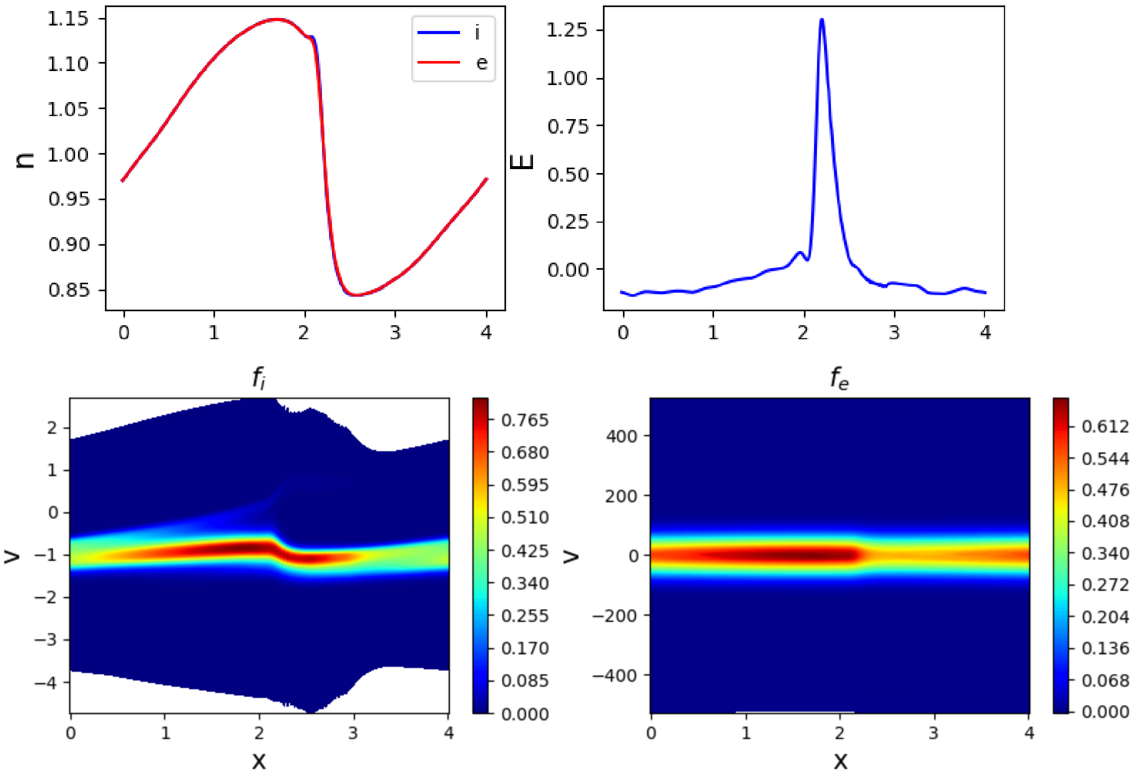
$$\epsilon \partial_t E + j = 0$$

$x \in [0,4\pi]$
$w \in [-8.5,8.5]$
$n_{0,i} = 1 + 0.2 \sin kx$
$n_{0,e} + 1 + 0.2(1 - k^2\epsilon^2) \sin kx$
$u_0 = -1 + 0.2 \sin kx$
$T_{0,e} = 1, T_{0,i} = 0.05$
$m_e = \frac{1}{1836}, m_i = 1$
$q_e = -1, q_i = 1$
$\epsilon = \frac{1}{36}$

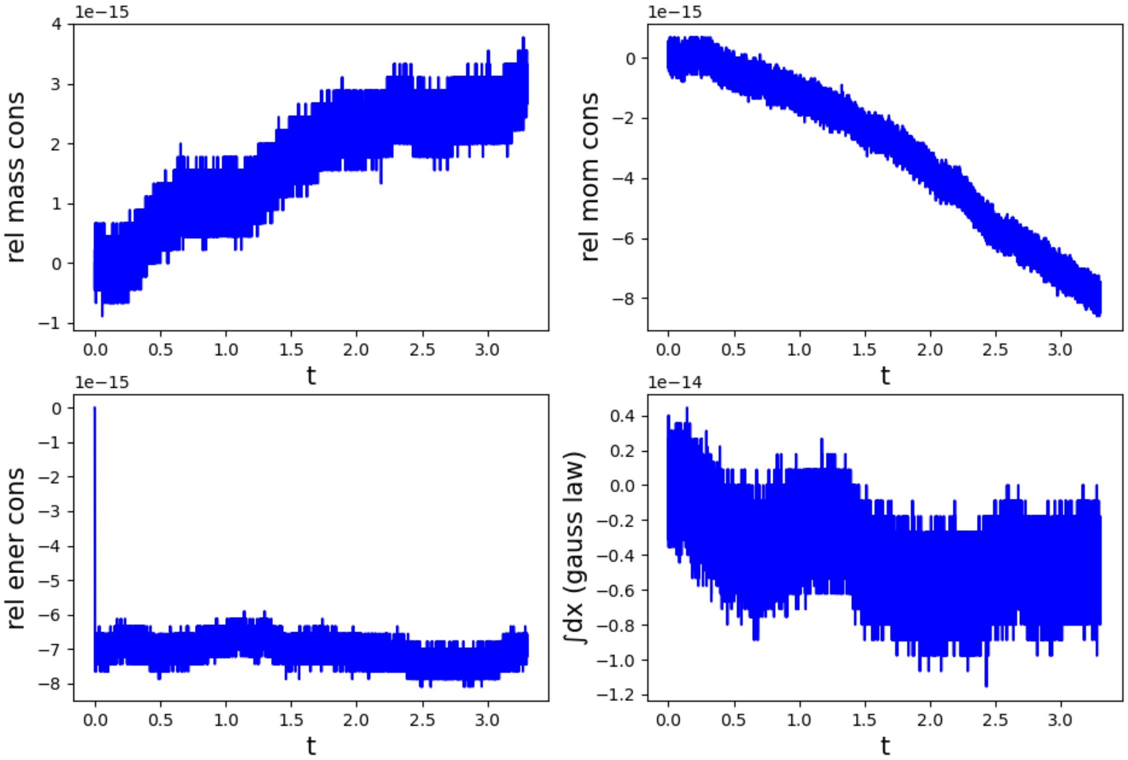


S.E. Anderson et al., JCP, **419**, 2020

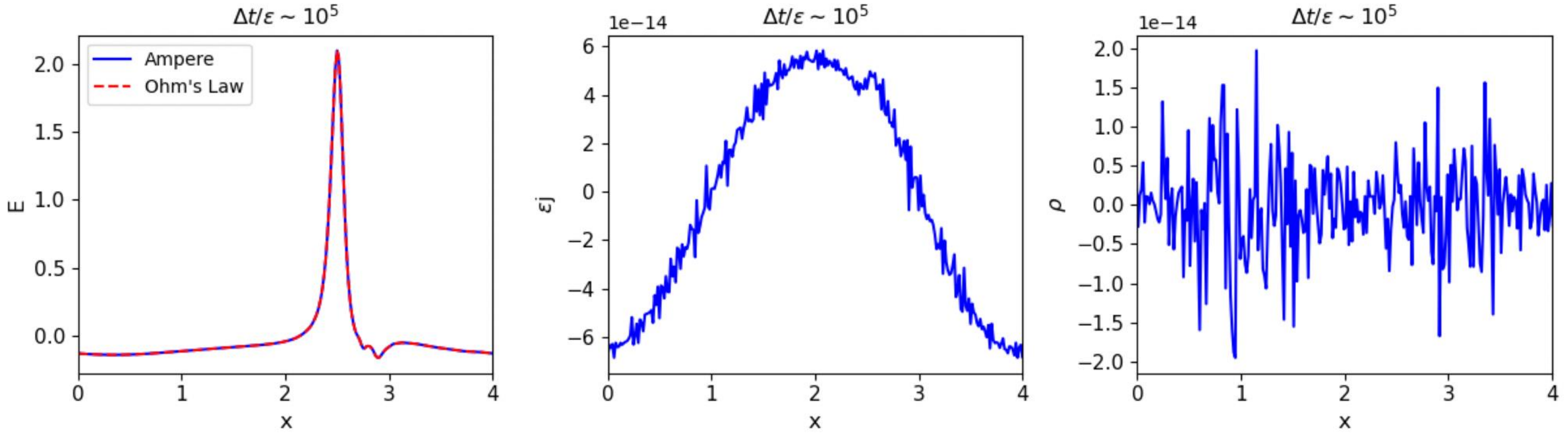
Ion Acoustic Shockwave:
New formulation reproduces the IASW solution



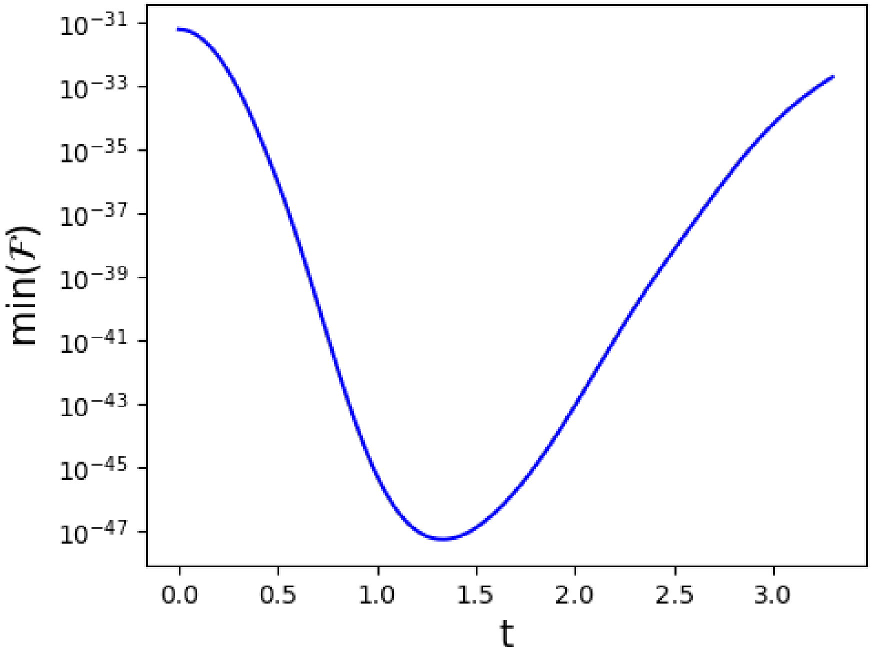
Ion Acoustic Shockwave:
1) Is conservative and 2) Gauss-law preserving



Ion Acoustic Shockwave:
3) Is quasi-neutrality asymptotic preserving



Ion Acoustic Shockwave:
4) Is positivity preserving



Ion Acoustic Shockwave: Slow manifold PC for moment-field system effectively deals with fast time-scale