# Mean Field Games on Sparse and Dense Networks: The Graphexon MFG Equations

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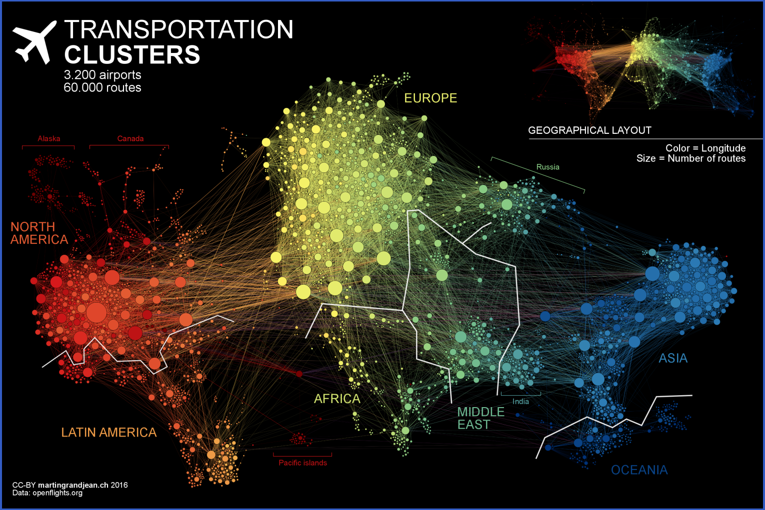
# Graphon Mean Field Games: Motivation

sp Non-uniform Network of Clusters



# Finite Network Finite Population Mean Field Games

Consider a finite population distributed over a finite graph $G_k$ with $M_k$ clusters of agents at the $M_k$ nodes.

This gives a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ agents.

For $\mathcal{A}_i$ in the node cluster $\mathcal{C}(i)$, there are two dynamical input terms (scalar states for simplicity):

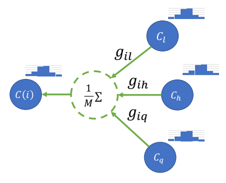
$$f_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} f_0(x_i, u_i, x_j) \tag{1}$$

$$f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \left[\frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} f(x_i, u_i, x_j)\right] \tag{2}$$

NB: $f_{G_k}$ uses the sectional (i.e.vertex neighbourhood) information $g^k_{\mathcal{C}(i)\bullet}$.

# Finite Network Finite Population MF Games: Agent Dynamics in Clusters at Nodes

**The state process of $\mathcal{A}_i$ in its cluster $\mathcal{C}(i)$ is given by the SDE**




$$dx_i(t) = \frac{1}{|C(i)|} \sum_{j \in \mathcal{C}(i)} f_0(x_i, u_i, x_j)dt$$

$$+ \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \left[\frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} f(x_i, u_i, x_j)\right]dt + \sigma dw_i \qquad (3)$$

$$= f_0(x_i, u_i, \mathcal{C}(i))dt + f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})dt + \sigma dw_i \qquad (4)$$

$$1 \leq i \leq N$$

# Nash Equilibria

**Fundamental Notion of Non-cooperative Game Equilibrium:**

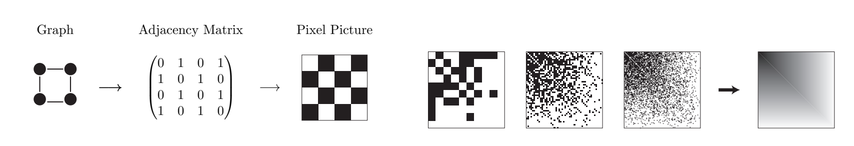
**The controls $\mathcal{U}^0 = \{u_i^0; u_i^0$ adapted to $\mathcal{U}_{loc,i}, 1 \leq i \leq N\}$, generate an $\varepsilon$*-Nash Equilibrium* w.r.t. $\{J_i; 1 \leq i \leq N\}$ if, for all $i$, a unilateral control law $u_i$ utilizing the global information pattern $\mathcal{U}$ satisfies**

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \leq \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^0) \leq J_i(u_i^0, u_{-i}^0)$$

**So, by definition, a unilateral move against a population of agents all of whom are utilizing a Nash strategy cannot yield a benefit of more than $\varepsilon > 0$ for the unilateral agent.**

# From Graphons to Graphexons

## Graph Sequence Convergence to Graphons




Definition: A graphon (Lovasz, AMS 2012) is a bounded symmetric Lebesgue measurable function $\mathbf{W} : [0,1]^2 \to [0,1]$.

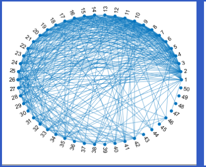
Graphons may be interpreted as weighted undirected edge graph limits on the vertex set $[0,1]$.

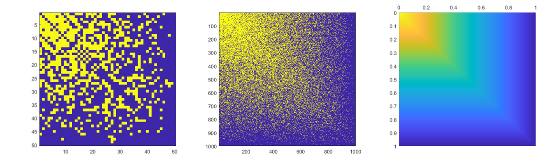
Theorem [Lovasz and Szegedy, 2006; LL AMS2012]
Under the cut metric the graphon space $(\mathcal{W}_{[0,1]}, \delta_\square)$ is compact.

# Graphons

Graphon [Lov2012]: A measurable function $[0,1]^2 \to$ [0,1]

Example: Unif. Attachment Graphon: $g(\alpha, \beta) = 1 - \max(\alpha, \beta)$






Convergence of “adjacency matrices” to graphons is in the cut metric. The space of graphons is compact.
We use stronger topology of $L_2[0,1]$ operator convergence.

# Infinite Network Infinite Population MF Games: Drifts of Each Agent in its Cluster

**Assume :**
**(i) The graph sequence $G_k; 1 \leq k < \infty$ has a unique graphon limit function $g(\alpha, \beta), \alpha, \beta \in [0, 1]$.**
**(ii) The subpopulation at each node tends to infinity, giving the local mean field $\mu_\alpha$ and the global set of mean fields $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$. Hence resulting in the drifts:**

$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}^n} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{5}$$

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] \; := \; \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) g(\alpha, \beta)\mu_\beta(dz)d\beta, \tag{6}$$

**yielding the local limit graphon drift dynamics (and similarly for costs $\widetilde{l}[x, u, \mu_G; g_\alpha]$):**

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] \quad := \quad f_0[x_\alpha, u_\alpha, \mu_\alpha] \; + \; f[x_\alpha, u_\alpha, \mu_G; g_\alpha]$$

# Graphon Mean Field Game (GMFG) Equations

HJB generates the value function $V^\alpha$ for the representative agent at node $\alpha$ at Nash equilibrium.

$$\begin{aligned}
\text{[HJB]}(\alpha) \quad -\frac{\partial V^\alpha(t,x)}{\partial t} &= \inf_{u \in U} \left\{ \widetilde{f}[x, u, \mu_G; g_\alpha] \frac{\partial V^\alpha(t,x)}{\partial x} \right. \\
&\quad \left. + \widetilde{l}[x, u, \mu_G; g_\alpha] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V^\alpha(t,x)}{\partial x^2}, \\
V^\alpha(T,x) &= 0, \quad (t,x) \in [0,T] \times \mathbb{R}, \quad \alpha \in [0,1], \qquad (7)
\end{aligned}$$

FPK generates the mean field density for the representative agent at node $\alpha$ at Nash equilibrium.

$$\begin{aligned}
\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} &= -\frac{\partial \{ \widetilde{f}[x, u^0, \mu_G; g_\alpha] p_\alpha(t,x) \}}{\partial x} \\
&\quad + \frac{\sigma^2}{2} \frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \quad p_\alpha(0) = p_0 \qquad (8) \\
\text{[BR]}(\alpha) \quad u^0 &= \varphi(t, x | \mu_G; g_\alpha).
\end{aligned}$$

# GMFG Eqns: Existence, Uniqueness, epsilon-Nash

Theorem: GMFG Existence and Uniqueness [PEC-Minyi Huang CDC2018,CDC 2019, SICON 2021]

For $U$ compact, subject to regularity conditions there exists a unique Nash equilibrium solution $(V^{\alpha}, \mu_{\alpha}(\cdot))_{\alpha \in [0,1]}$ to the GMFG equations (7) and (8).

Moreover subject to a graph convergence condition, the GMFG epsilon - Nash Property holds.

The feedback control best response *(BR)* strategy $\varphi(t, x_{\alpha} | \mu_G(\cdot); g_{\alpha})$ for each agent depends only upon the agent's state and the graphon mean fields: $(x_{\alpha}, \mu_G)$.

For networked LQG and control affine systems the GMFG equation solvability depends on the complexity of the graphon.

GMFG refs: CDC 2018-23, IEEE TAC 2020, 2021,2023, MTNS 2022, ESAIM 2022, IFAC 2023, Automatica + SCL 2024

# Inconvenient Truths Provoking ”Graphexon Response”

Graphon theory is widely employed in large Network Mean Field Control and Games studies.

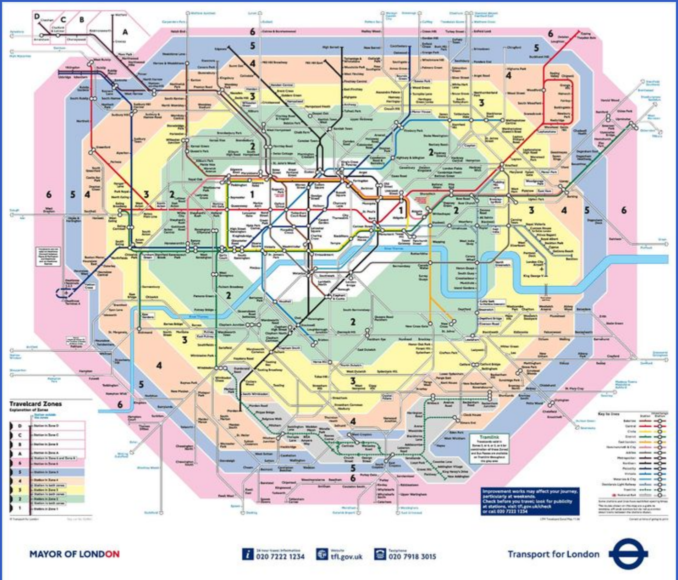
Fact: non-zero graphon limits exist only if ”dense”, i.e. the number of edges scales quadratically with node cardinality, equivalently, have strictly positive asymptotic density. (Shall call non-dense network sequences sparse.)

Hence bounded node degree network sequences have Zero Limit graphons. True even with refined definitions of sparse.

But (i) NO large real world networks are dense - all are sparse due to low bounded node degree: Metros, Power grids, etc.

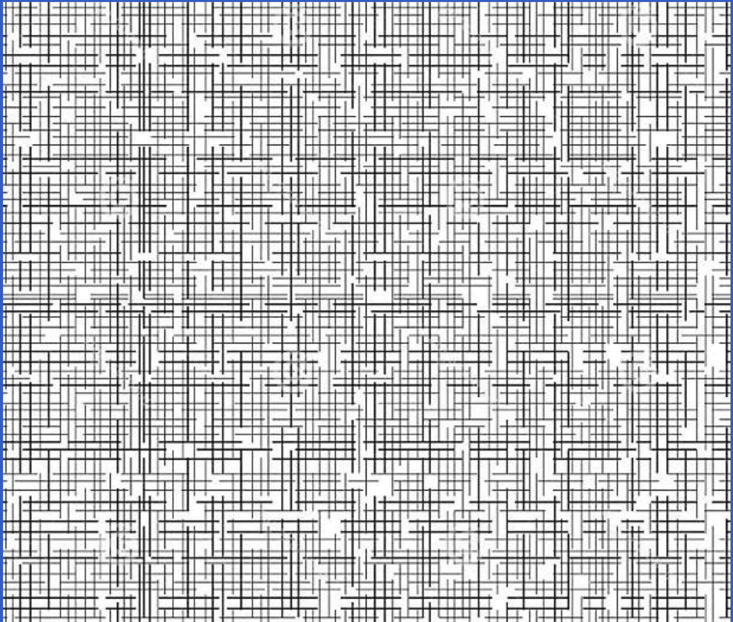
(ii) And there is NO metric based topology for graphs in $R^m, 2 \leq m$, since nodes of a limit graphon $G$ are combinatorically indexed by reals in $[0, 1]^m$.

# London Underground Bounded Degree Real Network



# Large Sparse Network : Node degree = 4, $n = 1, 2, ..$

## Rectangular Lattice Vertices and Edges in the Unit Square: A Sparse Graph



## Vertexon and Graphexon Sequences

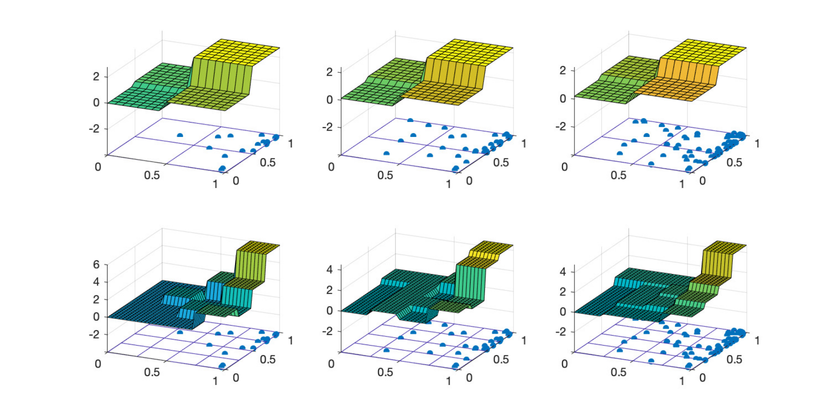
Let each of a sequence of simple graphs $\{G_n; n \in \mathbb{N}\}$ have its vertices embedded in $[0,1]^m$, and each of the corresponding edge vertex pairs be embedded in $R^{2m}$.

Take cubic partitions (voxels) of $[0,1]^m$ and $[0,1]^{2m}$ of edge length $1/k$.

Definition: Vertexon and Graphexon $\{n,k\}$-sequences:

$k-$ indexed step functions on $[0,1]^m$ (resp. $[0,1]^{2m}$) with steps of height proportional to the fraction of vertices (resp. vertex pairs connected by edges) of $G_n$ contained in the corresponding $k-$voxels.

# Networks in Space: Embedded Graphs: Vertexon and Graphexon $\{n, k\}$-Sequences




$k$ indexes the partition level corresponding to a row above, while $n$ indexes the total number of elts. per graph passing L to R.

# Vertexon and Graphexon Limits

Vertexon (limits) are weak (double) limit measures on $[0,1]^m$ of Vertexon $\{n,k\}$ sequences of embedded graph sequences.

Graphexon (limits) are weak (double) limit measures on $[0,1]^{2m}$ of Graphexon $\{n,k\}$ sequences of embedded graph sequences.

# Networks in Space: Embedded Graphs: Vertexons and Graphexons

Theorem: Embedded Vertexon-Graphon Stepping Function Limits (PEC CDC 2022)

Consider an $M = [0,1]^m$ embedded graph sequence $\{G_n = (V_n, E_n), n \in \mathbb{N}\}$.

Then there exists a joint vertexon-graphon sub-sequence converging weakly in measure to a limit vertexon - graphon measure pair:

$$G_{n_\nu} = (U_{n_\nu}, E_{n_\nu})(z,w) \to (V_\infty, W_\infty)(dz, dw) \ \ a.e. M \times M^2 \ \text{as} \ \ n_\nu \to \ \infty$$

# Networks in Space: Embedded Graphs: Vertexons and Graphexons

(a) Graphexon limits have topologies: the metrics on spaces of distributions.

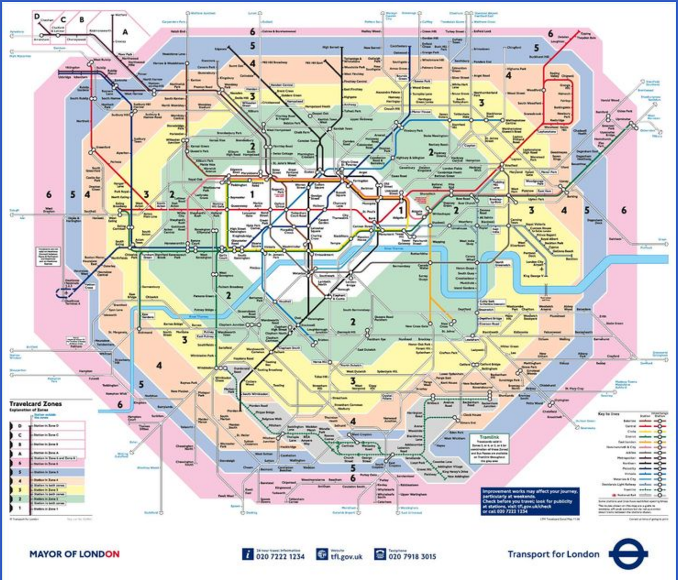
(b) All (non-empty) graph sequences have non-zero subsequential limits.

Overview:
Graphon (limits) map $[0, 1]$ node indexed graph sequences to bounded measurable function (limits) on the unit square.

Graphexon (limits) map ($m$ dim) embedded graph sequences to measure limits on the ($2m$ dim) unit cube.

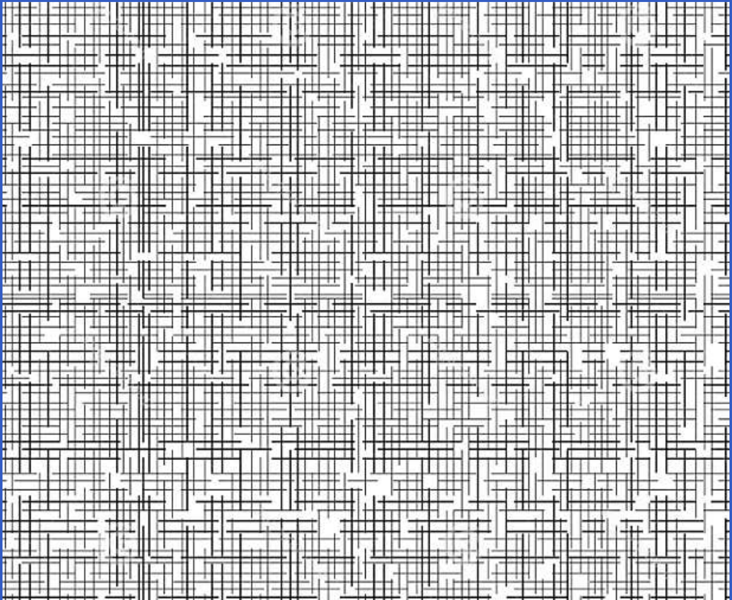
Normalizing any non-zero graphon $g(\alpha, \beta)$ on $[0, 1]^2$ yields an a.c. graphexon $\frac{g(\alpha, \beta)}{\iint (g(\alpha, \beta))}$ on $[0, 1]^2$.

# Recall London Underground Network Example



# Recall Large Sparse Network : Node degree = 4.

## Rectangular Lattice Vertices and Edges in the Unit Square: A Sparse Graph



# Infinite Network Infinite Population MF Games: Drifts of Each Agent in its Cluster

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$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}^n} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{9}$$

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] \ := \ \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) g(\alpha, \beta)\mu_\beta(dz) d\beta, \tag{10}$$

**yielding the local limit graphon drift dynamics (and similarly for costs $\widetilde{l}[x, u, \mu_G; g_\alpha]$):**

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] \ := \ f_0[x_\alpha, u_\alpha, \mu_\alpha] \ + \ f[x_\alpha, u_\alpha, \mu_G; g_\alpha]$$

# Graphexon Mean Field Game (GMFG) Equations

Example: (MH-PEC) Vertexon-Graphexon of the Infinite 2-dimensional Rectangular Lattice) - in 4-dimensions!!

Consider the limit of a uniformly distributed uniformly rectangular (i.e. square) grid in $[0,1]^2$.

Vertexon: A uniform unit density on $[0,1]^2$ which is evidently not singular.

Graphexon: The sum of two singular measures supported on two foliations of $[0,1]^4$, namely

$$\left\{\frac{1}{2}\delta(p-x)\delta(q-y) \; + \; \frac{1}{2}\delta(p-y)\delta(q-x); 0 \leq x, y, p, q \leq 1\right\}.$$

# Infinite Network Infinite Population MF Games: Drifts of Each Agent in its Cluster: Graphexon Case

**Assume :**
**(i) The graphexon sequence $G_k; 1 \leq k < \infty$ has a unique graphexon limit measure $N_\alpha(d\beta), \alpha, \beta \in [0,1]$.**
**(ii) The subpopulation at each node tends to infinity, giving the local mean field $\mu_\alpha$ and the global set of mean fields $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$. Hence resulting in the drifts:**

$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}^n} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{11}$$

$$f[x_\alpha, u_\alpha, \mu_G; N_\alpha] \; := \; \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) N_\alpha(d\beta)\mu_\beta(dz), \tag{12}$$

**yielding the local limit graphon drift dynamics (and similarly for costs $\widetilde{l}[x, u, \mu_G; g_\alpha]$):**

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] \quad := \quad f_0[x_\alpha, u_\alpha, \mu_\alpha] \; + \; f[x_\alpha, u_\alpha, \mu_G; N_\alpha]$$

# Graphexon MFG Equations (MH-PEC CDC24 sub.)

On a Graphexon with singular measure $N_\alpha$ the GXMFG HJB generates a value function $V^\alpha$ for the representative agent at node $\alpha$ at Nash equilibrium with prescribed dynamics on the support of measure.

$$\text{[HJB]}(\alpha) \quad -\frac{\partial V^\alpha(t,x)}{\partial t} = \inf_{u\in U}\left\{\widetilde{f}[x,u,\mu_G;N_\alpha]\frac{\partial V^\alpha(t,x)}{\partial x} + \widetilde{l}[x,u,\mu_G;N_\alpha]\right\} + \frac{\sigma^2}{2}\frac{\partial^2 V^\alpha(t,x)}{\partial x^2},$$

FPK generates the mean field for the representative agent at $\alpha$

$$\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial\{\widetilde{f}[x,u^0,\mu_G;N_\alpha]p_\alpha(t,x)\}}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \quad p_\alpha(0) = p_0$$

$$\text{[BR]}(\alpha) \quad u^0 = \varphi(t,x|\mu_G;N_\alpha).$$

# Example: Heat Equation Influence Between NNs on a Ring Graphexon (MH-PEC CDC24 sub.)

Scalar nonlinear model with affine control drift dynamics at node $\alpha$

$$
\begin{aligned}
f[x_\alpha, u_\alpha, \mu_G; N_\alpha] \; &:= \int_0^{2\pi} \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) N_\alpha(d\beta)\mu_\beta(dz) \\
&:= \int_0^{2\pi} \int_{\mathbb{R}^n} D\partial_\beta^2 m_\beta(t)\delta(\alpha - \beta)\mu_\beta(dz),
\end{aligned}
$$

Hence:

$$
\begin{aligned}
dx_\alpha(t) \; &= \; f_0(\alpha, x_\alpha(t), u_\alpha(t), \mu_\alpha(t))dt \\
&\quad + D\partial_\alpha^2 m_\alpha(t)dt + \sigma dw_\alpha(t),
\end{aligned}
$$

where $\alpha \in [0, 2\pi)$, $x_\alpha(t) \in \mathbb{R}$, $u_\alpha(t) \in \mathbb{R}$, $w_\alpha(t) \in \mathbb{R}$, and

$$
m_\alpha(t) = \int_{\mathbb{R}} x\mu_\alpha(t, dx).
$$

# Example: Heat Equation Influence Between NNs on a Ring Graphexon (MH-PEC CDC24 sub.)

How does the coupling term $\partial_\alpha^2 m_\alpha$ arise?

- In this case we consider a ring graphexon corresponding to a ring in $[0,1]^2$ with paramterixed singular graphexon $\delta(\alpha-\beta)$ on $[0,2\pi)^2$. Agent interactions are modelled as nearest neighbour influences.

- Each node's local mean field $m_{\alpha_i}$ receives an averaging effect with respect to two neighbouring nodes' $m_{\alpha_{i-1}}$ and $m_{\alpha_{i+1}}$.

- Suitable scaling leads to the second order term $\partial_\alpha^2 m_\alpha$, which also has a heat equation diffusion interpretation.

# Example: Heat Equation Influence Between NNs on a Ring Graphexon (MH-PEC CDC24 sub.)

$$
\begin{aligned}
HJB \qquad \partial_t V_\alpha(t,x) =& \partial_x V_\alpha(t,x) f_0(x,\mu) - \frac{1}{4}(\partial_x V_\alpha(t,x))^2 \\
&+ D\partial_\alpha^2 m_\alpha(t) + L_0(x,\mu) + \frac{\sigma^2}{2}\partial_x^2 V_\alpha(t,x).
\end{aligned}
$$

$$
\begin{aligned}
FPK \qquad \partial_t p_\alpha(t,x) =& -\partial_x\{[f(x,\mu_\alpha(t)) - (1/2)\partial_x V_\alpha(t,x) \\
&+ D\partial_\alpha^2 m_\alpha(t)] p_\alpha(t,x)\} \\
&+ \frac{\sigma^2}{2}\partial_x^2 p_\alpha(t,x),
\end{aligned}
$$

where $p_\alpha(t,\cdot)$ is the probability density function of $\mu_\alpha(t)$.

**In the linear dynamics quadratic costs case existence and uniquness of solutions have been established (MH+PEC, CDC24 submitted) but $\epsilon$-Nash remains to complete!**

## Summary and Conclusions

(1) Work with Alex Dunyak on Q-noise driven Graphon/Graphexon LQG control, games, filtering and its connections to Network Science and Low-rank graphon approximations (following Shuang Gao) has not been presented here.

(2) Recall that *all* spatially embedded sparse and dense graphs have graphexon (weak measure) limits. To Explore!

(3) In principle, the graphexon framework permits the application of diff. calculus in GXMFG/C systems. Explore!

(4) In the absolutely continuous case the graphexon framework enables the analysis of maxima, minima, and saddle points of mean field game equilibrium values as functions of location. Hence can define "Nash-optimal places". Explore!