# Pole Placement and Feedback Stabilization for Ensemble Systems

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# Overview

1. Background
2. Problem Formulation
3. Main Results

# Background of Ensemble Control Theory

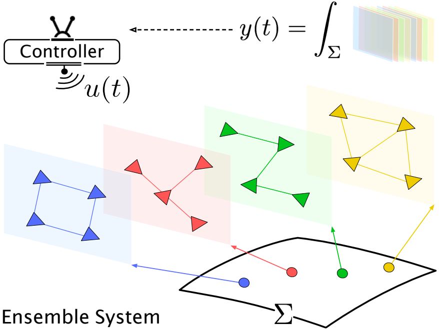
## System Model

- **Ensemble control** is about using a common control input to simultaneously steer a large population of dynamical systems

  It originated from quantum spin systems [Brockett, Khaneja, Li]

- **Mathematical model:**

$$\dot{x}_\sigma(t) = f(x_\sigma(t), u(t), \sigma), \quad \text{for all } \sigma \in \Sigma$$




  - common control input $u(t) \in \mathbb{R}^m$
  - continuum/discrete space $\Sigma$

  Integrated output (over $\Sigma$):

$$y(t) = \int_\Sigma \phi(x_\sigma(t)) \mathrm{d}\mu \in \mathbb{R}^\ell$$

  where $\mu$ describes population density

## Fundamental System Properties

- **Controllability:** ability of using $u(t)$ to steer the population from any initial condition to any target (within a given, but arbitrarily small error tolerance)
  - Linear ensemble systems: extended *Kalman rank condition* [Triggiani]
  - Control-affine ensemble systems: extended *Rachevsky-Chow* [Agrachev etc.]
- **Observability:** ability of using $u(t)$ and $y(t)$ to estimate $x_\sigma(t)$ for all $\sigma \in \Sigma$
  - Linear ensemble systems: *Duality theory* [Curtain]
  - Control-affine (nonlinear) ensemble systems: *Co-distribution algebra* [Chen]
- **Feedback stabilizability (focus today):** existence of a feedback control law $u(t) = k(x(t))$ to stabilize the population at an equilibrium point [Chen, 2024]

# Problem Formulation

## Setup

- Let $X$ be a Banach sequence space in $\mathbb{C}$ ($X = \ell^p$, for $1 \leq p < \infty$ or $X = c_0$)
- *Discrete* ensemble of unstable linear systems:

$$\dot{x}_n(t) = a_n x_n(t) + b_n u(t), \quad \text{for all } n \in \mathbb{N} \tag{1}$$

  - $x(t) := (x_n(t)) \in X$, $u(t) \in \mathbb{C}$
  - $a := (a_n) \in \ell^\infty$ with $a_n > 0$, $b := (b_n) \in X$
- Linear feedback control law: $u(t) = kx(t)$, where $k \in X^*$, which turns (1) into

$$\dot{x}(t) = (A + bk)x(t), \quad \text{where } A := \operatorname{diag}(a_1, a_2, a_3, \cdots) \tag{2}$$

- **Definitions:**
  - System (2) is **stable** if $\exists\, C > 0$ s.t. for any initial $x(0)$, $\|x(t)\| \leq C\|x(0)\|$
  - System (2) is **asymptotically stable** if it is stable and, moreover, $\lim_{t\to\infty} x(t) = 0$

## Feedback Stabilization and Pole Placement

- Feedback system:
$$\dot{x}(t) = (A + bk)x(t) \tag{3}$$
where $(A, b)$ is given and $k \in X^*$ is a free variable
- **Question 1:** *When is there a* $k \in X^*$ *such that* (3) *is (asymptotically) stable?*
- A necessary condition for (3) to be stable is that
$$\mathrm{spec}(A + bk) \subseteq H := \text{closed left half plane of } \mathbb{C} \tag{4}$$
- **Question 2:** *When is there a* $k \in X^*$ *such that* (4) *can be satisfied?*

# Results for Pole Placement

## A Necessary Condition

- **Theorem 1:** If there is a $k \in X^*$ s.t. $\operatorname{spec}(A + bk) \subseteq H$, then

  *1.* $(a_n) \in c_0$ and, moreover, $a_n \neq a_m$ for all $n \neq m$

  *2.* $b_n \neq 0$ for all $n \in \mathbb{N}$

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- **Sketch of proof:**
  - Since $a_n > 0$ and since $\operatorname{spec}(A + bk) \subseteq H$, $\operatorname{ess}(A) = \operatorname{ess}(A + bk) = \{0\}$
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- The two items are sufficient for **controllability:**
  If the above two items are satisfied, then the system $\dot{x} = Ax + bu$ is uniformly controllable (i.e., the linear span of $\{b, Ab, A^2b, \cdots\}$ is dense in $X$)

## Re-visit of the Finite-dimensional Case

- $N$-dimensional system: $\dot{x}(t) = A'x(t) + b'u(t)$, with $A' = \mathrm{diag}(a_1, \ldots, a_N)$
- If the $a_n$'s are distinct and the $b_n$'s are nonzero, then pole placement is feasible:
  - Given the target eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$, there is a *unique* (row) vector $k'$ s.t.

$$\mathrm{spec}(A' + b'k') = \{\lambda_1, \ldots, \lambda_N\}$$

  - The Ackermann's formula provides an explicit expression for $k' = (k_1', \ldots, k_N')$:

$$k_n' = -\frac{(a_n - \lambda_n)}{b_n} \prod_{m=1, m\neq n}^{N} \frac{1 - \lambda_m / a_n}{1 - a_m / a_n} \tag{5}$$

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- *Can we extend the result to the infinite-dimensional case?*

## Toward the Infinite-dimensional Case

- Let $c_H := \{\lambda = (\lambda_n) \in c_0 \mid \mathrm{re}(\lambda_n) \leq 0, \text{ for all } n \in \mathbb{N}\}$
- For each $\lambda = (\lambda_n) \in c_H$, we define $k(\lambda) = (k_n(\lambda))$ as

$$k_n(\lambda) := -\frac{(a_n - \lambda_n)}{b_n} \prod_{m=1, m \neq n}^{\infty} \frac{1 - \lambda_m / a_n}{1 - a_m / a_n}$$

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- *When does there exist such a $\lambda \in c_H$?*

## On Feasibility of Pole Placement: A Negative Result

- **Theorem 3:** If there is a $d < 2$ s.t. $(n^d a_n)_{n\in\mathbb{N}}$ is *eventually monotonically increasing*, then there does not exist any $\lambda \in c_H$ such that $k(\lambda) \in X^*$

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- **Sketch of proof:**
  - Focus on the special case $\lambda = 0$, which yields the minimum norm $\|k(\lambda)\|$
  - Evaluate $k_n(0)$ in the asymptotic regime ($n \to \infty$):

$$|1/k_n(0)| = |b_n| \cdot \frac{1}{a_n} \cdot \prod_{m=1, m\neq n}^{\infty} |1 - a_m/a_n| = O(n^d e^{-\alpha n})$$

    where $\alpha$ is some positive constant
  - $\lim_{n\to\infty} |k_n(0)| = \infty$, so $k(0) \notin X^*$

## On Feasibility of Pole Placement: A Positive Result

- **Theorem 4:** If $a$, $b$, and $\lambda$ satisfy the following:
  1. There is a $d > 2$ s.t. $(n^d a_n)_{n\in\mathbb{N}}$ is *eventually monotonically decreasing*
  2. $\limsup\limits_{n\to\infty} \frac{1}{n} \ln(a_n/|b_n|) \leq 0$ (i.e., $a_n/|b_n|$ does not grow exponentially fast)
  3. $\lim_{n\to\infty} \lambda_n/a_n = 0$

  then $|k_n(\lambda)|$ decays exponentially fast as $n \to \infty \implies k(\lambda) \in X^*$

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- **Examples:**
  - $a_n = 1/n^\alpha$ for $\alpha > 2$, $b_n = 1/n^\beta$, and $\lambda_n = -1/n^{\alpha+\epsilon}$ for $\epsilon > 0$
  - $a_n = e^{-\alpha n}$ for $\alpha > 0$, $b_n = e^{-\beta n}$ for $\beta \le \alpha$, and $\lambda_n = -e^{-(\alpha+\epsilon)n}$ for $\epsilon > 0$

# Result for Feedback Stabilization

## A Sufficient Condition:

- **Theorem 5:** If there are constants $0 < \nu_0 < \nu_1 < \nu_2 < 1$ such that $a_{n+1}/a_n < \nu_0$ and $\nu_1 < |b_{n+1}/b_n| < \nu_2$ for all $n \in \mathbb{N}$, then $k(-a) \in X^*$ and

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- **Sketch of proof:**
  - Define $Y := \{y \mid By \in X\}$ where $B := \operatorname{diag}(b_1, b_2, \cdots)$, and $\|y\|_Y := \|By\|_X$
  - Let $y(t) := B^{-1}x(t) \Longrightarrow \dot{y}(t) = B^{-1}(A + bk(-a))By(t)$ and $\|y(t)\|_Y = \|x(t)\|_X$
  - Define $P : Y \to Y$ as

$$P_{ij} := \left[ \frac{2a_j}{a_i + a_j} \prod_{k=1, k\neq j}^{\infty} \frac{1 + a_k/a_j}{1 - a_k/a_j} \right]$$

  - $P$ is bounded, $P^2 = I$, and $B^{-1}(A + bk(-a))B = -PAP$