Safe and Constrained Feedback Optimization of Dynamical Systems

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Team and Acknowledgements

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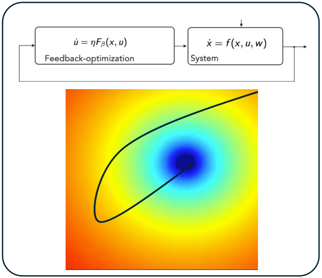
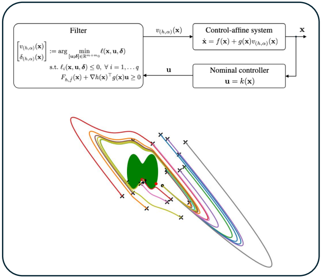
Safe online feedback optimization



Online feedback optimization

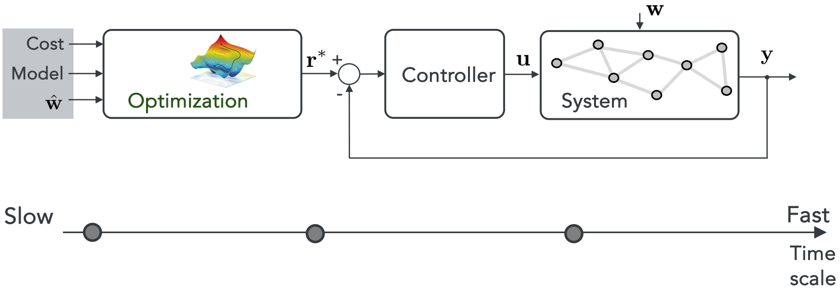


Optimization-based control
for safety

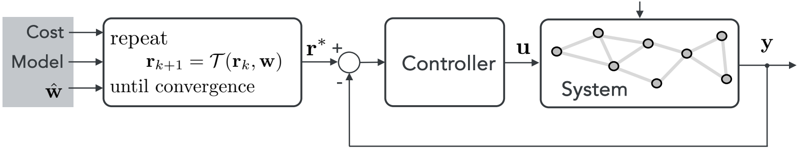
Goal: safe online feedback optimization for dynamical systems

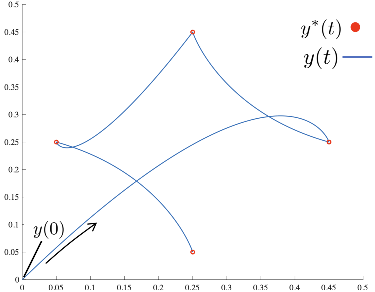
Online Feedback Optimization

Classical architecture

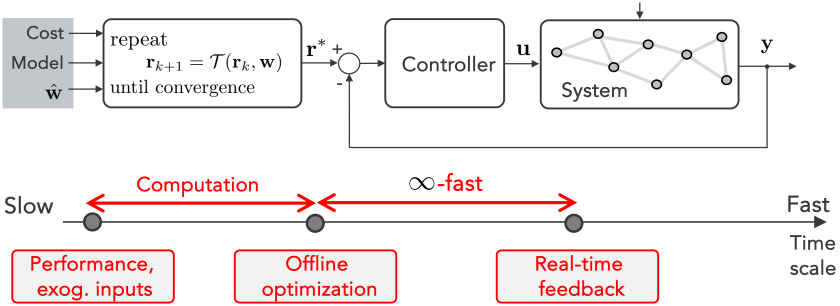


Classical architecture





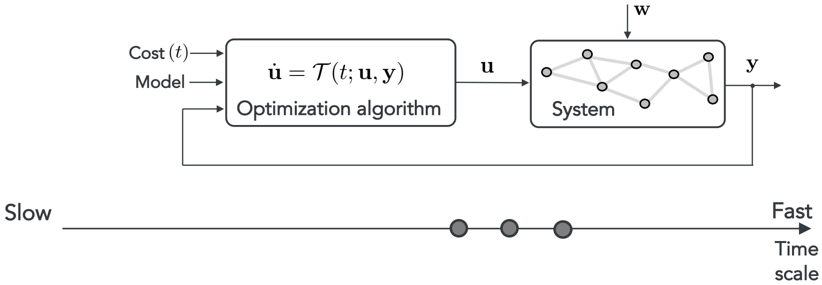
Classical operation with time-scale separation



Infeasible or inefficient in modern systems:

- What if the system conditions or the environment change *faster* than the solution time of the optimization problem?
- Infeasible pervasive metering to acquire exogenous inputs or disturbances

Feedback optimization

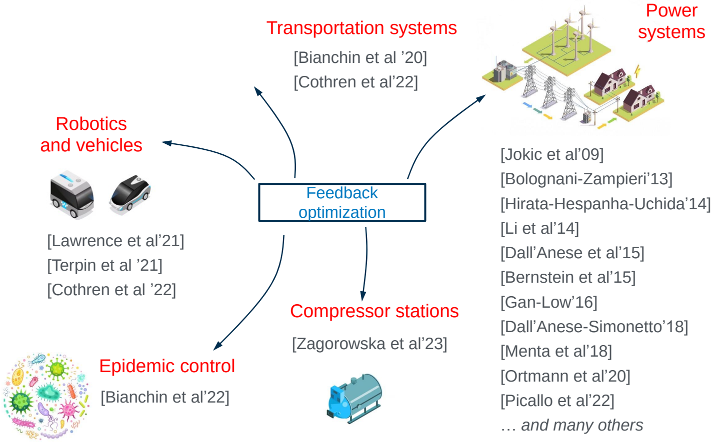


Online feedback optimization: Optimization algorithms as feedback controllers

Design philosophy: utilize measurements from the system to drive the system to solutions of an optimization problem

Features: Does not require measuring exogenous inputs; compresses the time-scales between optimization and control

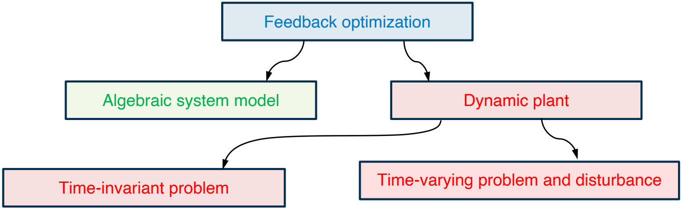
Current gaps



OFO applied in many areas, but:

- lacks of safety guarantees
- has models with practical limitations

Current gaps: limited classes of problems

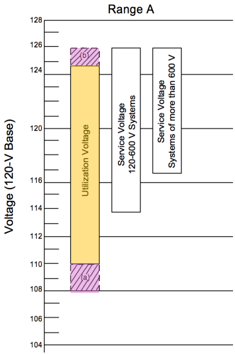


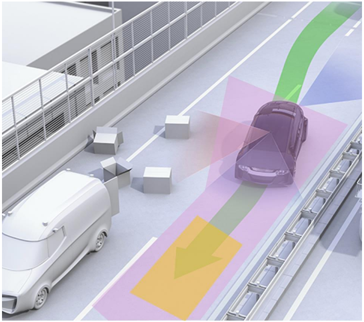
[Jokic et al'09]
[Brunner et al'12]
[Hirata-Hespanha-Uchida'14]
[Li et al'14]
[Menta et al'18]
[Lawrence et al'18]
[Hauswirth-Bolognani-Hug-Dorfler '19]
[Lawrence-Simpson Porco-Mallada '21]
[Simpson Porco '22]
… *and others*

[Colombino-Dall'Anese-Bernstein'18]
[Zheng-Simpson Porco-E. Mallada '20]
[Bianchin-Poveda-Dall'Anese '20]
[Bianchin et al '21]
[Belgioioso at al '21]
[Cothren-Bianchin-Dall'Anese '22]
… *and others*

Current limitation: consider unconstrained optimization problems or with constraints on the inputs. How about constraint on the system state or output?

Current gaps: no transient guarantees





Power systems [Li-Chen-Zhao-Low'14], [Dall'Anese-Simonetto'16], ...:

$$\{\text{Utilization } \omega, V\} \subseteq \{\text{Critical } \omega, V\}$$

Autonomous vehicles [Terpin-Fricker-Perez-de Badyn-Dorfler'23]:

$$\{\text{Rendevouz or way-point}\} \subseteq \{\text{Collision-free}\}$$

OFO for constrained optimization problems

System model

Plant model:

$$\dot{x} = f(x, u, w), \qquad x(t_0) = x_0$$

- $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^n$, with $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^{n_u}$, and $\mathcal{W} \subseteq \mathbb{R}^{n_w}$, is continuously differentiable and Lipschitz-continuous
- $u \in \mathcal{U}_c \subset \mathcal{U}$: control input
- $w \in \mathcal{W}_c \subset \mathcal{W}$: unknown disturbance

(As. 1). $\exists$ a unique continuously differentiable function $h : \mathcal{U} \times \mathcal{W} \to \mathcal{X}$ such that $f(h(\bar{u}, \bar{w}), \bar{u}, \bar{w}) = 0$ for any fixed $\bar{u} \in \mathcal{U}$ and $\bar{w} \in \mathcal{W}$. Moreover, the Jacobian $J_h(u) := \frac{\partial h(u)}{\partial u}$ is locally Lipschitz.

Definition: $\mathcal{X}_{\text{eq}} := h(\mathcal{U}_c \times \mathcal{W}_c)$

Definition: $r > 0$ largest constant such that $\mathcal{X}_r := \mathcal{X}_{\text{eq}} + \mathcal{B}_n(\mathbb{0}_n, r) \subseteq \mathcal{X}$.

System model

Plant model:

$$\dot{x} = f(x, u, w), \qquad x(t_0) = x_0$$

- $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^n$, with $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^{n_u}$, and $\mathcal{W} \subseteq \mathbb{R}^{n_w}$, is continuously differentiable and Lipschitz-continuous
- $u \in \mathcal{U}_c \subset \mathcal{U}$: control input
- $w \in \mathcal{W}_c \subset \mathcal{W}$: unknown disturbance

(As. 2). $\exists\ a, k > 0$ such that

$$\|x(t) - h(\bar{u}, \bar{w})\| \leq k\|x_0 - h(\bar{u}, \bar{w})\|e^{-a(t-t_0)}, \quad \forall\ t \geq t_0$$

for any fixed $\bar{u} \in \mathcal{U}_c$ and $\bar{w} \in \mathcal{W}_c$, for some $t_0 \geq 0$, and for every initial condition $x_0 \in \mathcal{X}_0 := \mathcal{X}_{\text{eq}} + \mathcal{B}_n(\mathbb{0}_n, r_0)$, $r_0 < r/k - \text{diam}(\mathcal{X}_{eq})$,

$\Longrightarrow$ Lyapunov function via Converse Theorems [Hauswirth et al'21], ...

Optimization problem

Equilibrium selection problem:

$$\min_{u\in\mathbb{R}^{n_u}} \quad \phi(u)+\psi\left(h(u,w)\right)$$
$$\text{s.t. } \ell\left(h(u,w)\right)\le 0, \quad \gamma(u)\le 0$$

- $\phi:\mathbb{R}^{n_u}\to\mathbb{R}$, $\psi:\mathbb{R}^n\to\mathbb{R}$, and $\ell:\mathbb{R}^n\to\mathbb{R}^p$ have a locally Lipschitz continuous gradient (Jacobian), $\gamma(u):\mathbb{R}^{n_u}\to\mathbb{R}^m$ continuously differentiable.
- $\gamma(u)\le 0$: constraints on the input
- $\ell\left(x\right)\le 0$: constraint on the *state of the system* $x=h(u,w)$

(As. 3). For any $i\in[m]$ and any $u\in\mathcal{U}_c:=\{u:\gamma(u)\le 0\}$, it holds that $\nabla\gamma_i(u)\neq 0$ if $\gamma_i(u)=0$.

Optimization problem

Equilibrium selection problem:

$$\min_{u \in \mathbb{R}^{n_u}} \quad \phi(u) + \psi\left(h(u,w)\right)$$
$$\text{s.t. } \ell\left(h(u,w)\right) \leq 0, \quad \gamma(u) \leq 0$$

- $\phi : \mathbb{R}^{n_u} \to \mathbb{R}$, $\psi : \mathbb{R}^{n} \to \mathbb{R}$, and $\ell : \mathbb{R}^{n} \to \mathbb{R}^{p}$ have a locally Lipschitz continuous gradient (Jacobian), $\gamma(u) : \mathbb{R}^{n_u} \to \mathbb{R}^{m}$ continuously differentiable.
- $\gamma(u) \leq 0$: constraints on the input
- $\ell\left(x\right) \leq 0$: constraint on the *state of the system* $x = h(u,w)$

(As. 4) Let $u^* \in \mathcal{U}_c$ be a local minimizer and an isolated KKT. The following hold:
(i) Strict complementarity condition and the LICQ hold at u^*.
(ii) $u \mapsto \gamma(u)$, $u \mapsto \phi(u)$, $u \mapsto \psi(h(u,w))$, and $u \mapsto \ell(h(u,w))$ are twice continuously differentiable over some open neighborhood of u^* and their Hessian matrices are positive semi-definite at u^*.
(iii) $\nabla^2 \phi(u^*)$ is positive definite.

Optimization problem

Equilibrium selection problem:

$$\min_{u \in \mathbb{R}^{n_u}} \quad \phi(u) + \psi\left(h(u, w)\right)$$
$$\text{s.t. } \ell\left(h(u, w)\right) \leq 0, \quad \gamma(u) \leq 0$$

- $\phi : \mathbb{R}^{n_u} \to \mathbb{R}$, $\psi : \mathbb{R}^{n} \to \mathbb{R}$, and $\ell : \mathbb{R}^{n} \to \mathbb{R}^{p}$ have a locally Lipschitz continuous gradient (Jacobian)
- $\gamma(u) \leq 0$: constraints on the input
- $\ell\left(x\right) \leq 0$: constraint on the *state of the system* $x = h(u, w)$

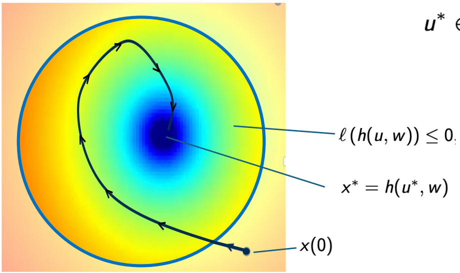
Q: Why these "complicated" assumptions?

Re: provide a minimal set of assumptions that allows one to consider nonconvex problems while still deriving strong stability guarantees (as shown next)

Note: Assumptions satisfied for existing setups based on convex programs

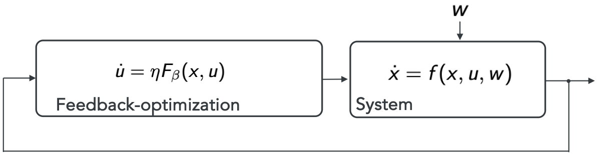
Problem revisited

Design a feedback controller to regulate inputs and states of the system $\dot{x} = f(x, u, w)$ to a minimizer u^* and the optimal state $x^* = h(u^*, w)$ without requiring knowledge of w, while respecting input constraints at all times.



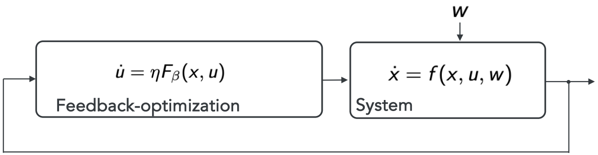
$$u^* \in \arg\min_{u \in \mathbb{R}^{n_u}} \quad \phi(u) + \psi(h(u,w))$$
$$\text{s.t. } \ell(h(u,w)) \leq 0, \quad \gamma(u) \leq 0$$

Controller design



Design: gradient flow + control barrier function + online feedback optimization

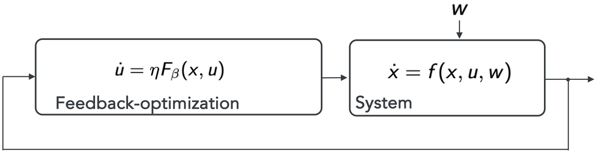
Controller design



Design: gradient flow + control barrier function

$$
\begin{aligned}
\dot{u} &= \eta F_\beta(x, u) \\
F_\beta(x, u) &:= \arg\min_\theta \|\theta + \nabla\phi(u) + J_h(u)^\top \nabla\psi(h(u, w))\|_2^2 \\
&\qquad \text{s.t. } \frac{\partial \ell}{\partial x}(h(u, w)) J_h(u)\theta \le -\beta \ell(h(u, w)) \\
&\qquad\qquad \frac{\partial \gamma}{\partial u}(u)\theta \le -\beta\gamma(u)
\end{aligned}
$$

Controller design



Design: gradient flow + online feedback optimization + control barrier function

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&\dot{u} = \eta F_\beta(x,u) \\
&F_\beta(x,u) := \arg\min_\theta \|\theta + \nabla\phi(u) + J_h(u)^\top \nabla\psi(x)\|_2^2 \\
&\qquad \text{s.t. } \frac{\partial \ell}{\partial x}(x) J_h(u)\theta \leq -\beta\ell(x) \\
&\qquad\qquad \frac{\partial \gamma}{\partial u}(u)\theta \leq -\beta\gamma(u)
\end{aligned}
$$

(As. 5) Problem is feasible and satisfies the Mangasarian-Fromovitz Constraint Qualification and the constant-rank condition.

Intermediate results

Closed-loop system:

$$\dot{z} = F(z, w), \qquad z := (x, u), \qquad F(z, w) := \begin{bmatrix} f(x, u, w) \\ \eta F_\beta(x, u) \end{bmatrix} \tag{CS}$$

Proposition. (CS) renders the set $\mathcal{U}_c := \{u : \gamma(u) \leq 0\}$ forward invariant.

Proposition. Under the current assumptions, it follows that:
(i) for any $u \in \mathcal{U}_c$, $x \mapsto F_\beta(x, u)$ is locally Lipschitz continuous;
(ii) For any compact subset $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ and any $x \in \tilde{\mathcal{X}}$, $u \mapsto F_\beta(x, u)$ is locally Lipschitz continuous.

Proposition. $\exists\ \lambda^*$ such that (u^*, λ^*) is a KKT point for the optimization problem if and only if $(h(u^*, w), u^*)$ is an equilibrium for (CS).

Main result

Closed-loop system:

$$\dot{z} = F(z, w), \qquad z := (x, u), \qquad F(z, w) := \begin{bmatrix} f(x, u, w) \\ \eta F_\beta(x, u) \end{bmatrix} \tag{CS}$$

Theorem *(Local exponential stability)*. Consider the system (CS) and let $(x(t), u(t))$, $t \geq t_0$, be the unique trajectory. Let $\tilde{z} = (x - x^*, u - u^*)$, and assume that $r_0 > \sqrt{d_2/d_1}\,\mathrm{diam}(\mathcal{X}_{eq})$. Then, $\exists\ \eta^* > 0$ such that for each $0 < \eta < \eta^*$, $\exists\ \varrho > 0$ and $a > 0$ such that

$$\|\tilde{z}(t)\| \leq \varrho \,\|\tilde{z}(t_0)\|\, e^{-\frac{1}{2}a(t-t_0)}, \ \forall\ t \geq t_0,$$

for any $(x(t_0), u(t_0))$ such that $\mathrm{dist}(x(t_0), \mathcal{X}_{eq}) \leq \sqrt{\frac{d_1}{d_2}}\, r_0 - \mathrm{diam}(\mathcal{X}_{eq})$ and $\|u(t_0) - u^*\| \leq \frac{e_1}{L}(1 - s)$.

*detailed expressions for η^*, ϱ, a available (but not presented here)

**free parameter $s \in (0, 1)$ affects a and the size of the set of initial conditions

Main result

Closed-loop system:

$$\dot{z} = F(z, w), \qquad z := (x, u), \qquad F(z, w) := \begin{bmatrix} f(x, u, w) \\ \eta F_\beta(x, u) \end{bmatrix} \tag{CS}$$

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for any $(x(t_0), u(t_0))$ such that $\mathrm{dist}(x(t_0), \mathcal{X}_{eq}) \leq \sqrt{\frac{d_1}{d_2}}\, r_0 - \mathrm{diam}(\mathcal{X}_{eq})$ and $\|u(t_0) - u^*\| \leq \frac{e_1}{L}(1 - s)$.

Obs. 1: $\eta < \eta^*$ implies that the controller is "sufficiently slower" that the plant

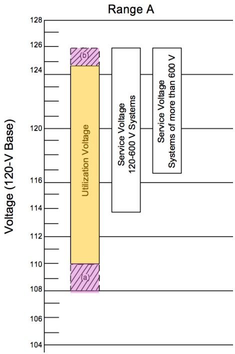
Obs. 2: local practical exponential stability if F_β is inexact

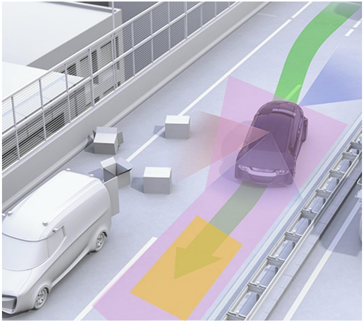
Safety in OFO
(preliminary results, ongoing)

Setup

Feasible and safe sets:

- Feasible set: $\mathcal{F} := \{z \in \mathbb{R}^{n_z} : \ell(z) \leq 0\}$
- Safe set: $\mathcal{S} := \{z \in \mathbb{R}^{n_z} : \xi(z) \geq 0\}$, $\mathcal{F} \subseteq \mathcal{S}$





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Focus on control-affine systems: $\dot{z} = F(z) + G(z)v$

If $v = 0$, $\dot{z} = F(z)$ is the nominal system (NS)

For OFO:

$$f(x,u) = \tilde{f}(x) + g(x)u, \qquad F(z) = \begin{bmatrix} \tilde{f}(x) + g(x)u \\ \eta F_\beta(x,u) \end{bmatrix}, \qquad G(z) = \begin{bmatrix} g(x) \\ 0 \end{bmatrix}$$

Setup

Feasible and safe sets:

- Feasible set: $\mathcal{F} := \{z \in \mathbb{R}^{n_z} : \ell(z) \leq 0\}$
- Safe set: $\mathcal{S} := \{z \in \mathbb{R}^{n_z} : \xi(z) \geq 0\}$, $\mathcal{F} \subseteq \mathcal{S}$

Focus on control-affine systems: $\dot{z} = F(z) + G(z)v$

If $v = 0$, $\dot{z} = F(z)$ is the nominal system (NS)

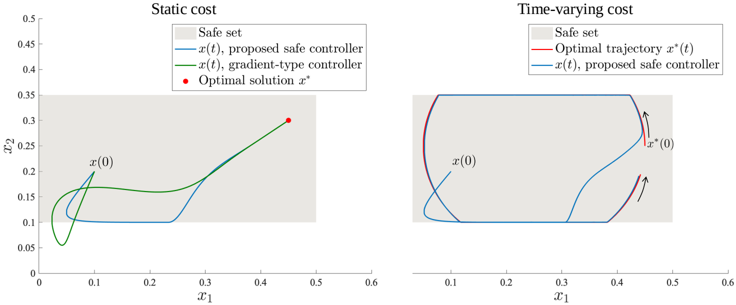
Definition (CBF). Let $\xi : \mathbb{R}^{n_z} \to \mathbb{R}^n$ be a continuously differentiable function such that $\mathcal{S} = \{z \in \mathbb{R}^{n_z} : \xi(z) \geq 0\}$, and $\nabla\xi(z) \neq 0$ for all $z \in \partial\mathcal{S}$. The function ξ is a CBF of the set $\mathcal{S}$ for the system (NS) if $\exists$ an extended class $\mathcal{K}_\infty$ function α such that for each $x \in \mathcal{S}$, $\exists\ \theta \in \mathbb{R}^m$ satisfying

$$\nabla\xi(z)^\top(F(z) + G(z)\theta) + \alpha(\xi(z)) \geq 0.$$

Adding a safety layer

System with *safety filter*:

$$\begin{aligned} \dot{z} &= F(z) + G(z)v(z) \\ v(z) &:= \arg\min_{\theta \in \mathbb{R}^{n_u}} \|\theta\|^2 \\ &\quad \text{s.t. } \nabla\xi(z)^T(F(z) + G(z)\theta) \geq -\alpha(\xi(z)). \end{aligned} \tag{FS}$$



Adding a safety layer

System with *safety filter*:

$$\dot{z} = F(z) + G(z)v(z) \tag{FS1}$$

$$v(z) := \arg\min_{\theta \in \mathbb{R}^{n_u}} \|\theta\|^2 \tag{FS2}$$

$$\text{s.t. } \nabla\xi(z)^T (F(z) + G(z)\theta) \geq -\alpha(\xi(z)).$$

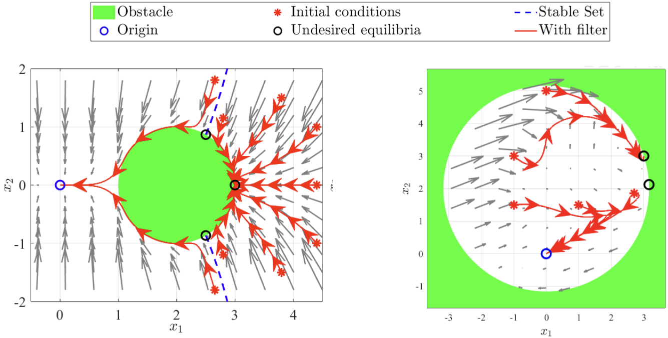
(As. 1). $\xi : \mathbb{R}^{n_z} \to \mathbb{R}$ twice continuously differentiable; $\alpha : \mathbb{R} \to \mathbb{R}$ continuously differentiable extended class-$\mathcal{K}_\infty$ function.

(As. 2). For all $z \in \mathcal{S}$, problem (FS2) satisfies Slater's condition.

Note: for multiple constraints (i.e., obstacles), impose MFCQ + constant-rank

Emergence of undesirable equilibria

Challenge: emergence of undesirable equilibria [Reis-Aguilar-Tabuada'21], [Tan-Dimarogonas'24], ...



Q: Can we find a pair (ξ, α) that minimizes the number of undesirable equilibria?

Q: In general, how does the pair (ξ, α) affect the dynamics?

Equilibria do not depend on ξ

Proposition. Let $\xi : \mathbb{R}^{n_z} \to \mathbb{R}$ be a CBF of $\mathcal{S}$ for $\dot{z} = F(z)$, and suppose that (ξ, α) satisfies Assumptions 1 and 2. Then,

- A point $z_0 \in \text{Int}(\mathcal{S})$ is an equilibrium of the nominal system (NS) if and only if z_0 is an equilibrium of the filtered system (FS), in which case there exists a neighborhood N_{z_0} of z_0 such that $v(z) = 0$, for all $z \in N_{z_0}$.
- If a point $z_0 \in \partial\mathcal{S}$ is an equilibrium of (NS), it is also an equilibrium of (FS).

* In $\text{Int}(\mathcal{S})$, the equilibria of (NS) and (FS) coincide, for any pair (ξ, α).

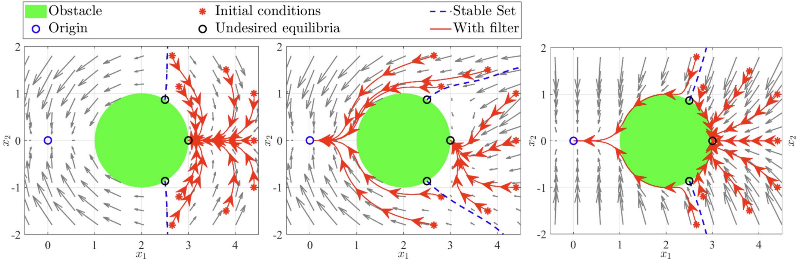
** The safety filter does not affect the equilibria of the (NS) in $\partial\mathcal{S}$

*** Undesirable equilibria, if any, are in $\partial\mathcal{S}$

Equilibria do not depend on (ξ, α)

Proposition. Let $\xi_1 : \mathbb{R}^{n_z} \to \mathbb{R}$ and $\xi_2 : \mathbb{R}^{n_z} \to \mathbb{R}$ be CBFs of $\mathcal{S}$ for $\dot{z} = F(z)$, and suppose that (ξ_1, α_1) and (ξ_2, α_2) satisfy Assumptions 1 and 2. Then,

- $z_0 \in \partial\mathcal{S}$ is an equilibrium of (FS) under the pair (ξ_1, α_1) if and only if it is an equilibrium of (FS) under the pair (ξ_2, α_2).
- $v_{(\xi_1,\alpha_1)}(z_0) = v_{(\xi_2,\alpha_2)}(z_0)$ for all $z_0 \in \partial\mathcal{S}$.

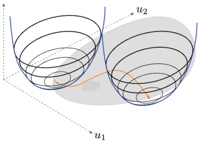


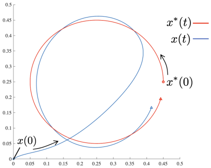
Note: additional results stating that *stability properties are independent* of (ξ, α)

Next steps

- Time-varying problems (equilibrium tracking)

$$\min_{u \in \mathbb{R}^{n_u}} \quad \phi(u, \theta_\phi(t)) + \psi\left(h(u, w(t), \theta_\psi(t))\right)$$
$$\text{s.t. } \ell\left(h(u, w(t))\right) \leq 0, \quad \gamma(u, \theta_\gamma(t)) \leq 0$$





- Learning-based implementations of the OFO method

- Continue the analysis of OFO + safety filters

- Development and analysis of safe OFO methods

Y. Chen, L. Cothren, J. Cortes, and E. Dall'Anese, "Online Regulation of Dynamical Systems to Solutions of Constrained Optimization Problems," *IEEE Control Systems Letters*, 2023 (and ACC 2024).

Y. Chen, P. Mestres, E. Dall'Anese, and J. Cortes, "Characterization of the Dynamical Properties of Safety Filters for Linear Planar Systems," *IEEE Conference on Decision and Control*, Dec. 2024.

Y. Chen, P. Mestres, J. Cortes, and E. Dall'Anese, "Equilibria Do Not Depend on the Control Barrier Function in Safe Optimization-based Control," to be submitted to *Automatica*.

Thank you!

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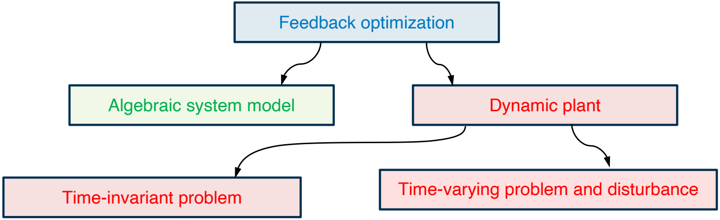
Backup slides

Roots

Feedback optimization

- Algebraic system model
  - Time-invariant problem
    Static cost, constant disturbance
    - [Low-Lapsley’99]
    - [Bolognani-Zampieri’13]
    - [Bolognani et al’14]
    - [Dall’Anese et al’15]
    - [Bernstein et al’15]
    - [Gan-Low’16]
    - [Hauswirth et al’16]
    - [Tang et al’17]
    - [Picallo et al’22]
    - … *and many others*
  - Time-varying (aka online) problem
    Time-varying cost and disturbance
    - [Dall'Anese-Simonetto-Dhople’16]
    - [Dall’Anese-Simonetto’18]
    - [Zhou et al’17]
    - [Bernstein-Dall'Anese’19]
    - … *and others*
- Dynamic plant

Roots



[Jokic et al'09]
[Brunner et al'12]
[Hirata-Hespanha-Uchida'14]
[Li et al'14]
[Menta et al'18]
[Lawrence et al'18]
[Hauswirth-Bolognani-Hug-Dorfler '19]
[Lawrence-Simpson Porco-Mallada '21]
[Simpson Porco '22]
… *and others*

[Colombino-Dall'Anese-Bernstein'18]
[Zheng-Simpson Porco-E. Mallada '20]
[Bianchin-Poveda-Dall'Anese '20]
[Bianchin et al '21]
[Belgioioso at al '21]
[Cothren-Bianchin-Dall'Anese '22]
… *and others*

Converse theorems

[Hauswirth et al'21] Assumptions 1-2 $\implies$ $\exists\ W : \mathcal{X}_0 \times \mathcal{U} \to \mathbb{R}$ that satisfies

$$
\begin{aligned}
& d_1 \|x - h(u,w)\|^2 \leq W(x,u) \leq d_2 \|x - h(u,w)\|^2, \\
& \frac{\partial W}{\partial x} f(x,u,w) \leq -d_3 \|x - h(u,w)\|^2, \\
& \left\| \frac{\partial W}{\partial x} \right\| \leq d_4 \|x - h(u,w)\|, \quad \left\| \frac{\partial W}{\partial u} \right\| \leq d_5 \|x - h(u,w)\|
\end{aligned}
$$