Topological Methods for Assured Transitions in Hybrid Systems

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AFOSR Dynamical Systems and Control Theory Review

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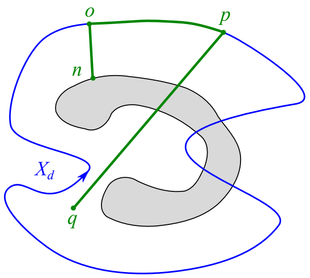
Brief Progress Report

- Theoretical work in Y2 focused mostly on developing global aspects of applying topological transition guarantees (this talk, joint work with Yu Wang):
  - *"Topologically-Aware Planning Under Linear Temporal Logic Constraints"*
    (submission to DAM imminent, [1])
- Applications were developed and submitted, with Sage C. Edwards as lead:
  - *"Multi-Agent Localization Using Geometric Constraints on Relative Distance Measurements in the Presence of Intermittent State Feedback"*
    (from Y1, submitted to IJRR, [2])
  - *"Occluded Target Surveillance: A Topological Perspective on Intermittent Target Tracking with Lyapunov-based Deep Neural Networks"*
    (from Y2, submitted to TRO, [3])
- Sage C. Edwards graduated in May 2024 and moved to AFRL-RW, Autonomy Group.
- Work from Y1 with Federico M. Zegers completed:
  - *"Event-Triggered Multi-Agent System Rendezvous with Graph Maintenance in Varied Hybrid Formulations: A Comparative Study"*
    (published in TAC, [4])
- New PhD student, Yixuan Wang (finally!!) hired in May 2024.

Recap: Topological Transition Guarantees

- **Overarching goal:** Facilitate symbolic planning in realistic conditions, where:
  - transition boundaries are geometrically and topologically complex;
  - state feedback is incomplete/uncertain, but state error growth rate bounds are known.
- **Motivating Example: Relay-Explorer Problems [5, 6, 7, 8, 9].**
  - State feedback only available in $\mathcal{F}$ (grey).
  - Plan to track X_d for as long as possible (legs $\overline{no}$ and $\overline{op}$); then return to $\mathcal{F}$ (leg $\overline{pq}$) to regulate your state; replan & repeat indefinitely.
  - Lyapunov-based design guarantees rapid regulation of the state while $x \in \mathcal{F}$, and known error growth bounds while $x \in \mathcal{F}^{\complement}$.
  - Growth rate of error bounds determines when to head back into $\mathcal{F}$, provided we know how to pick q, given p.

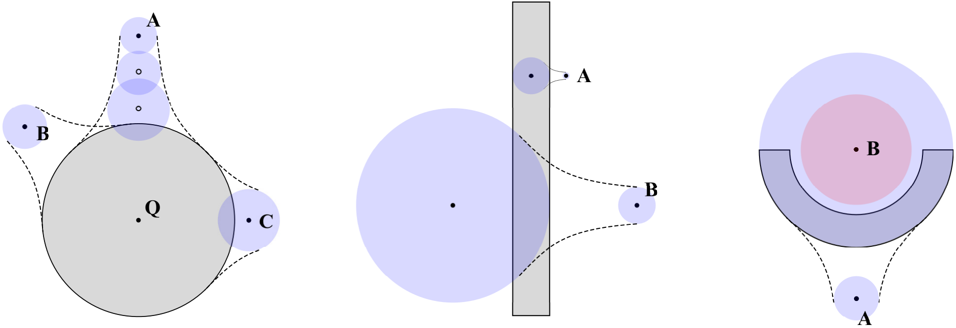


A single RE planning cycle.

Recap: Topological Transition Guarantees

Key Observations [10]:

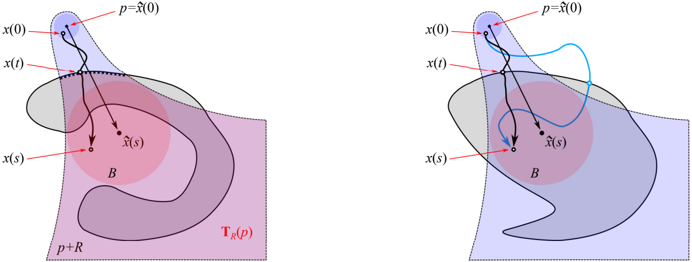
- *Symmetric shapes of $\mathcal{F}$ are too easy to handle.* Scarcity of symmetries generates regions where aiming to fit your error ball inside $\mathcal{F}$ gets you lost.



Recap: Topological Transition Guarantees

Key Observations [10]:

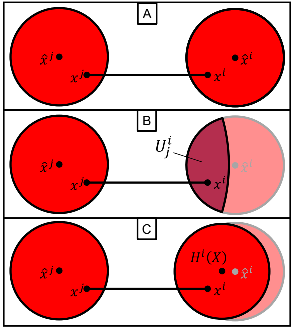
- *Guaranteeing a transition into $\mathcal{F}$ is about relative connectivity (AKA 0-homology of pairs).* Given a plan $\hat{x}$, R contains the error envelope until time t, and B_t is the error ball at time t, saying that any path from B_0 (centered at $\hat{x}(0) = p$) to B_s (centered at $\hat{x}(s) = q$) must intersect $\mathcal{F}$ is equivalent to $\partial\mathcal{F}$ separating B_0 from B_s.



General TTG (left) improves on inscribed-ball approach (right) because the latter takes no account of the error envelope.

Recap: Topological Transition Guarantees

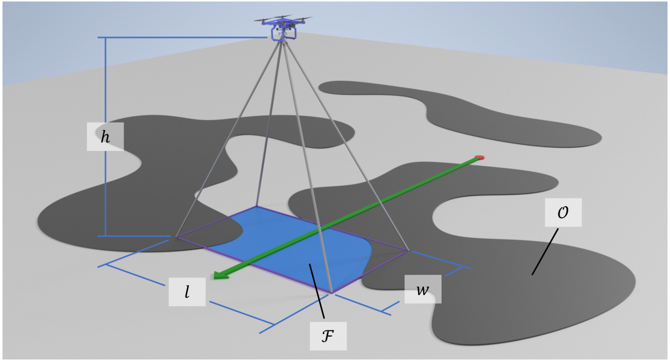
- *Multiple relay-explorers can assist each other by communicating relative range information with bounded error [2].*

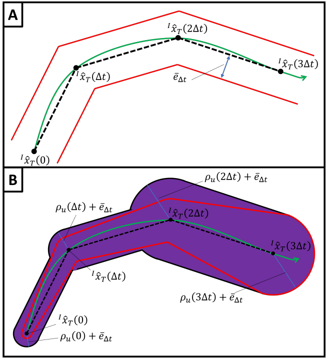


An agent updates its position estimate using ranging information from a collaborator and minimal circumscribed ball computation. General position arrangements of agents lead to significant reductions in estimation error bounds, but also to hybrid dynamics due to discontinuities in state estimates.

Recap: Topological Transition Guarantees

- *An agent with a camera can track a target through an occluded environment, provided a decent predictor [3], by controlling camera pose.*





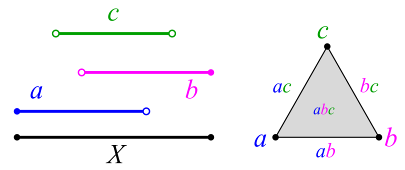
Symbolic Planning/Navigation in Topological Spaces

- **Symbolic constraints complicate TTGs.** If $\{\mathcal{F}_\alpha\}_{\alpha\in\mathrm{AP}}$ are symbols, $\mathcal{F}_\alpha \subset X$, then
  - $\mathcal{F}_\alpha$ may not be contained in a single chart
  - X may not even be a manifold
  - transition into $\mathcal{F}_\alpha$ may be required to satisfy a Boolean expression in the $\mathcal{F}_\beta$, $\beta \neq \alpha$
- **A common fix:** • Select the symbols $\mathcal{F}_\alpha$ to be pairwise disjoint convex sets;
  • Ensure that $X \setminus \bigcup_{\alpha\in\mathrm{AP}} \mathcal{F}_\alpha$ is path connected.
- **Another fix, e.g. [11]:** Tile X with convex polytopes, make *them* your symbols.
- **Instead, topology may offer a similar, but systematic paradigm:**
  - Present X as the union of contractible sub-spaces refining the set of atomic propositions AP
  - Replace AP with this set, rewrite any constraints in new AP
  - Plan "high-level" paths satisfying the constraints in the nerve of this cover
  - Consider ways of realizing the planned paths in the space and compute associated TTG

The Nerve Simplicial Complex (SCX) and Good Covers

Let $(X, \mathscr{T})$ be a nice[2] topological space.

- *An indexed cover* is a map $\mathbb{U} : \mathrm{AP} \to \mathscr{T}$ such that[3] $X = \bigcup_{\alpha \in \mathrm{AP}} \mathbb{U}(\alpha)$.
- A subset $\sigma \subset \mathrm{AP}$ is $\mathbb{U}$-*consistent*, if $\widetilde{\mathbb{U}}(\sigma) \triangleq \bigcap_{\alpha \in \sigma} \mathbb{U}(\alpha)$ is non-empty.
- *The Nerve* $N(\mathbb{U})$ is the SCX of all $\mathbb{U}$-consistent sets $\sigma \subset \mathrm{AP}$.



[2]e.g., $(X, \mathscr{T})$ is completely regular, *II*-countable, connected, and locally contractible.
[3]AP may be infinite, in which case $\mathbb{U}$ must be locally finite.

The Nerve Simplicial Complex ($\textsc{scx}$) and Good Covers

- The *geometric realization* of a $\textsc{scx}$ K on a vertex set V is

$$\left|K\right| = \bigcup_{\sigma \in K} \Delta^{\sigma}, \tag{1}$$

where

$$\Delta^{\sigma} \triangleq \left\{ \xi \in \mathbb{R}^{V}_{\geq 0} \colon \sum_{v \in V} \xi(v) = 1, \xi(V \setminus \sigma) = \{0\} \right\}, \tag{2}$$

taken with the weak topology.

Theorem (Nerve Lemma)

X is homotopy-equivalent to the geometric realization of $N(\mathbb{U})$ if every $\widetilde{\mathbb{U}}(\sigma)$, $\sigma \in N(\mathbb{U})$ is contractible. An open cover with this property is called a good cover.

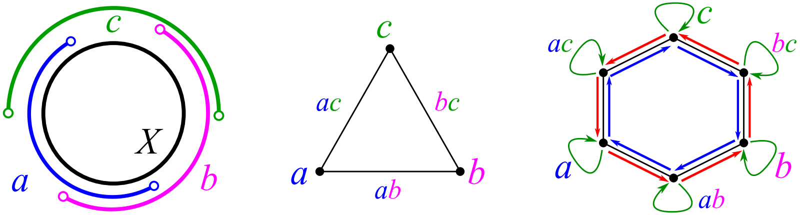
The *optimistic* Planning Pipeline

Example: LTL-based planning on a transition system (TS) over the circle.

(1) Find a good cover $\mathbb{U}$;

(2) Construct $\mathrm{sd}(N(\mathbb{U}))$;

(3) Form a TS over $\mathrm{sd}(N(\mathbb{U}))$;



A good cover of a circle with cw/ccw rotations converted into a non-deterministic TS over the 1-skeleton of $\mathrm{sd}(N(\mathbb{U}))$.

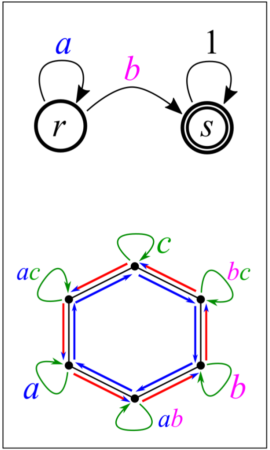
The *optimistic* Planning Pipeline

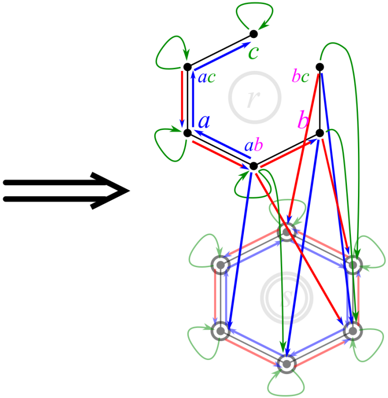
Example: LTL-based planning on a transition system (TS) over the circle.

(4) Pick a Büchi automaton;

(5) Form the product TS;

(6) Obtain a solution path.

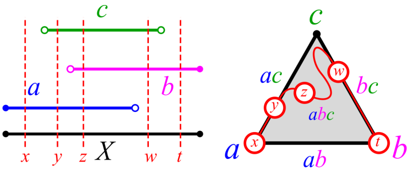




(†) Other control paradigms are applicable, e.g. reactive strategies [12].

Challenges of Nerve-Based Planning

- **First Challenge:** Not all $\sigma \in N(\mathbb{U})$ are witnessed by a point of X.



- $\{c\}$ is not realized;
- $\{a, b\}$ is not realized.

Definition (Realizability)

Define a map $\varsigma : X \to \mathbb{U}$ by $\varsigma(x) \triangleq \{\alpha \in \mathrm{AP} \colon x \in \mathbb{U}(\alpha)\} \in N(\mathbb{U})$.
A simplex $\sigma \in N(\mathbb{U})$ is $\mathbb{U}$-*realized*, if $\sigma \in \varsigma(X)$.

- $\sigma \in N(\mathbb{U})$ is realized $\iff \bigcap_{\alpha\in\sigma} \mathbb{U}(\alpha) \setminus \bigcup_{\beta\in\mathrm{AP}\setminus\sigma} \mathbb{U}(\beta) \neq \varnothing$.
- Unrealized simplices are one obstruction to planning using $N(\mathbb{U})$.

$\rightsquigarrow$ *Not every path in $|N(\mathbb{U})|$ is realized by a path in X*

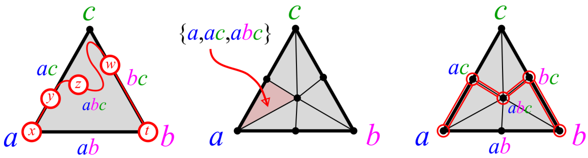
$\rightsquigarrow$ *Deforming a bad plan to a realized one may violate task constraints.*

Challenges of Nerve-Based Planning

- Deleting the unrealized simplices of $N(\mathbb{U})$ produces no meaningful model of X!

Recall: if K is a SCX, then $\mathrm{sd}(K)$ is the SCX of all $T \subset K$ that are $(\subseteq)$-chains.[2]

$\rightsquigarrow$ *n-simplices of $\mathrm{sd}(N(\mathbb{U}))$ are increasing maps $T : [n+1] \to N(\mathbb{U})$, naturally and consistently orienting $\mathrm{sd}(N(\mathbb{U}))$*



There is no way to access a from abc except via ac, so the red simplex of $\mathrm{sd}(N(\mathbb{U}))$ in the center should not be deemed realizable, yielding a "reduced nerve" as in the diagram on the right (red).

- What is a good definition of the "reduced nerve"?
- Why is $\mathrm{sd}(N(\mathbb{U}))$ a natural point of departure?
- Will homotopy type be preserved under deletion of unrealized simplices?

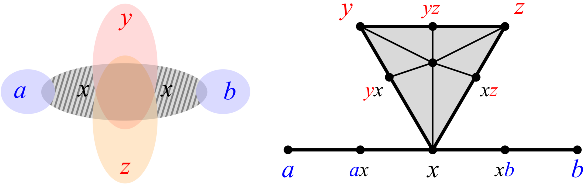
[2] $T \in \mathrm{sd}(K)$ iff, for all $\sigma, \tau \in T$ one has $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

Challenges of Nerve-Based Planning

- **Second Challenge:** No path correspondence between X and $\mathrm{sd}(N(\mathbb{U}))$, recall [13].

Definition (Tame Path)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. A continuous path $c : J \to X$ is $\mathbb{U}$-tame if $\varsigma \circ c$ is piecewise constant with jump points not accumulating in J.



A disconnected witness set causing trouble.

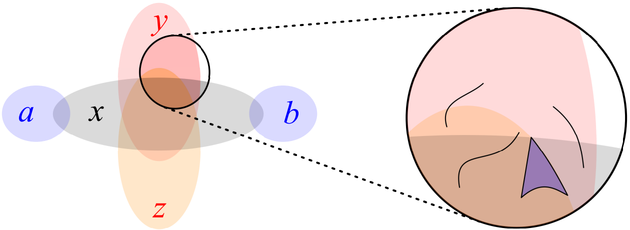
- **Example:** Here, a topological disk X is the union of convex open regions labeled a, b, y, z, x, consituting a good cover $\mathbb{U}$ of X (left).

  An edge-path in $\mathrm{sd}(N(\mathbb{U}))$ from $\{a\}$ to $\{b\}$ via $\{x\}$ exists (right), while avoiding any simplices containing y or z, but there is no tame path in X avoiding the region $y \cup z$.

The Reduced Nerve and the Connectivity Trisp

Definition ($\mathbb{U}$-small singular simplex)

Let $T \subset 2^{\mathrm{AP}}$. Then, T is a $\mathbb{U}$-*small* singular T-simplex if there exists a continuous map $g : \Delta^T \to X$ such that $(\varsigma \circ g)(\Delta^S) = S$ for all $S \subseteq T$. Let $\mathbf{C}_{\mathbb{U}}(T)$ be the space of all $\mathbb{U}$-small singular T-simplices, and $\mathbf{C}^0_{\mathbb{U}}(T)$ be the set of path components of $\mathbf{C}_{\mathbb{U}}(T)$.



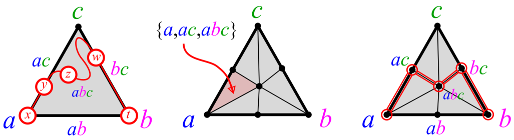
Some $\mathbb{U}$-small singular T-simplices with, from left to right, $T = \{y, yz\}$, $\{yz, xyz\}$, $\{y, yz, xyz\}$, and $\{y, xy\}$. The $\{yz, xyz\}$-simplex is in the same path component of $\mathbf{C}_{\mathbb{U}}(\{yz, xyz\})$ as a 1-face of the $\{y, yz, xyz\}$-simplex.

The Reduced Nerve and the Connectivity Trisp

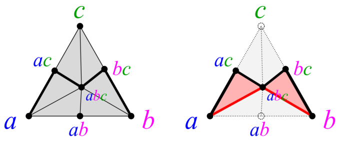
Lemma (Definition of the Reduced Nerve)

The set of all $T \subseteq 2^{\mathrm{AP}}$ with $\mathbf{C}_{\mathbb{U}}(T) \neq \varnothing$ is a sub-complex, $N_{red}(\mathbb{U})$, of $\mathrm{sd}(N(\mathbb{U}))$.

- **recall the example of the first challenge...**



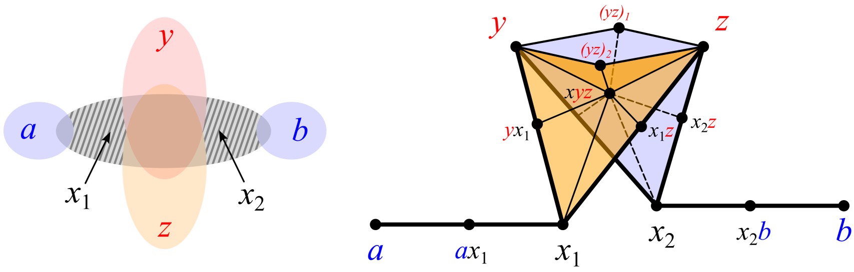
There is no way to access a from abc in X except via ac, so $\{a, ac, abc\} \notin N_{red}(\mathbb{U})$.



Deleting the unrealized vertices of $\mathrm{sd}(N(\mathbb{U}))$ (right) does *not* produce $N_{red}(\mathbb{U})$ (bold, left).

The Reduced Nerve and the Connectivity Trisp

- Elements of $\mathbf{C}^0_{\mathbb{U}}(T)$ may be regarded as abstract $(|T|-1)$-simplices, giving rise to a triangulated space $\widetilde{\mathbf{R}}_{\mathbb{U}}$—the *connectivity trisp*—encoding how they are glued together in X.



The trisp $\widetilde{\mathbf{R}}_{\mathbb{U}}$ resolving the example of the second challenge.

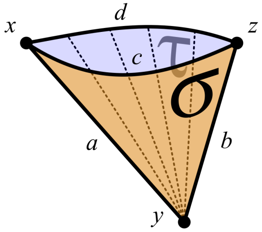
The Reduced Nerve and the Connectivity Trisp

Triangulated Spaces (Trisps [14]).

- **Simplex Category $\mathbf{\Delta}$.**
  - $\mathrm{Ob}\mathbf{\Delta}$ is all the non-negative integers;
  - $\mathbf{\Delta}(m, n)$ is all the increasing maps[3] $\alpha : [m+1] \to [n+1]$.
- **Trisp [Gluing Data]:** any co-functor $\Gamma : \mathbf{\Delta} \to \mathbf{Set}$.
  - $\Gamma(n)$ is the set of simplices of dimension n;
  - $\Gamma(\alpha) : \Gamma(n) \to \Gamma(m)$ lists which m simplex is the α-face of which n simplex.

$$\begin{aligned} \Gamma(0) &:= \{x, y, z\} \\ \Gamma(1) &:= \{a, b, c, d\} \\ \Gamma(2) &:= \{\sigma, \tau\} \end{aligned}$$

σ and τ share three vertices $(0, 1, 2)$ and the edges a, b, but not c or d.



[3] We use the notation $[n] = \{1, \ldots, n\}$.

The Reduced Nerve and the Connectivity Trisp

- **Trisp Geometric Realization** is the quotient

$$\left|\Gamma\right| \triangleq \coprod_{n\geq 0} \Gamma(n) \times \Delta^{[n+1]} \Big/ (\sim_\Gamma), \tag{3}$$

where the equivalence relation $(\sim_\Gamma)$ is generated by all expressions of the form

$$\left(\sigma, \left|\alpha\right|(\xi)\right) \sim_\Gamma (\Gamma(\alpha)\sigma, \xi) \tag{4}$$

for $\alpha \in \mathbf{\Delta}(m, n)$, $\xi \in \Delta^{[m+1]}$, and $\sigma \in \Gamma(n)$, where $\left|\alpha\right| : \Delta^{[m+1]} \to \Delta^{[n+1]}$,

$$\left|\alpha\right|(\xi) \triangleq \sum_{i\in[m+1]} \xi(i) e_{\alpha(i)} \tag{5}$$

is the geometric α-face of the standard n-simplex.

The Reduced Nerve and the Connectivity Trisp

Definition ($N_{red}(\mathbb{U})$ as a Trisp)

The gluing data for $N_{red}(\mathbb{U})$ are the trisp $\mathbf{R}_{\mathbb{U}}$ defined as follows:

- For $n \in \mathrm{Ob}\,\mathbf{\Delta}$, $\mathbf{R}_{\mathbb{U}}(n)$ is the set of all $T \in N_{red}(\mathbb{U})$ with $\left|T\right| = n+1$;
- For $\alpha \in \mathbf{\Delta}(m, n)$, $\mathbf{R}_{\mathbb{U}}(\alpha) : \mathbf{R}_{\mathbb{U}}(n) \to \mathbf{R}_{\mathbb{U}}(m)$ is given by

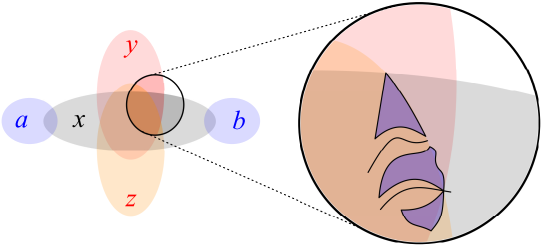
$$\mathbf{R}_{\mathbb{U}}(\alpha)T \triangleq T \circ \alpha, \tag{6}$$

  where $T \in N_{red}(\mathbb{U})$ is regarded as an increasing map $[n+1] \to N(\mathbb{U})$.

The Reduced Nerve and the Connectivity Trisp

Putting $\mathbb{U}$-small T-simplices together:

- Each $g \in \mathbf{C}_{\mathbb{U}}(T)$, $|T| = n + 1$ contributes a point $[g] \in \mathbf{C}^0_{\mathbb{U}}(T)$;
- Restricting g to an α-face of Δ^T for any $\alpha \in \mathbf{\Delta}(m, n)$ yields a point of $\mathbf{C}^0_{\mathbb{U}}(T \circ \alpha)$.



Representatives of 2-simplices in $\widetilde{\mathbf{R}}_{\mathbb{U}}$ with the edges joining them.

Definition (Connectivity Trisp)

For each $n, m \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbf{\Delta}(m, n)$, define $\widetilde{\mathbf{R}}_{\mathbb{U}}(n)$ and $\widetilde{\mathbf{R}}_{\mathbb{U}}(\alpha) : \widetilde{\mathbf{R}}_{\mathbb{U}}(m) \to \widetilde{\mathbf{R}}_{\mathbb{U}}(n)$ as

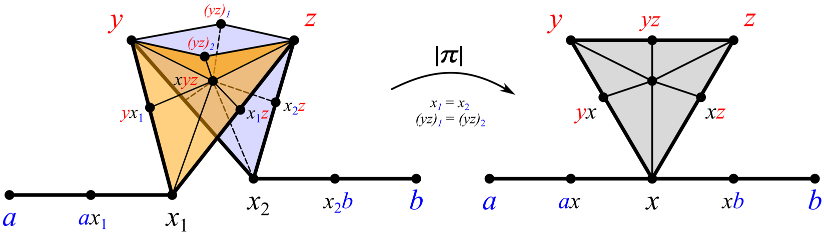
$$\widetilde{\mathbf{R}}_{\mathbb{U}}(n) \triangleq \bigcup_{T \in N_{red}(\mathbb{U}), |T| = n+1} \mathbf{C}^0_{\mathbb{U}}(T), \quad \widetilde{\mathbf{R}}_{\mathbb{U}}(\alpha)[g] \triangleq [g \circ |T \circ \alpha \hookrightarrow T|], \tag{7}$$

for all $g \in \mathbf{C}_{\mathbb{U}}(T)$ with $|T| = m + 1$.

The Reduced Nerve and the Connectivity Trisp

Theorem (Path Correspondence)

Every $\mathbb{U}$-tame path in X induces an edge-path in $\widetilde{\mathbf{R}}_{\mathbb{U}}$, and hence in $N_{red}(\mathbb{U})$. Conversely, any edge-path in $\widetilde{\mathbf{R}}_{\mathbb{U}}$ is induced by a $\mathbb{U}$-tame path in X.



The *canonical covering map* $\{\pi(n) : \widetilde{\mathbf{R}}_{\mathbb{U}}(n) \to \mathbf{R}_{\mathbb{U}}(n)\}_{n \in \mathrm{Ob}\,\mathbf{\Delta}}$ given by

$$[g] \mapsto T \;\Leftrightarrow\; g \in \mathbf{C}^0_{\mathbb{U}}(T) \tag{8}$$

is a dimension-preserving natural transformation, giving rise to a surjective PL map

$$\left|\pi\right| : \left|\widetilde{\mathbf{R}}_{\mathbb{U}}\right| \to \left|N_{red}(\mathbb{U})\right|. \tag{9}$$

The Reduced Nerve and the Connectivity Trisp

Definition

$\mathbb{U}$ has the *path-lifiting property* if for every pair of vertices $v \in N_{red}(\mathbb{U})$ and $\tilde{v} \in \pi^{-1}(v)$ and edge-path γ in $N_{red}(\mathbb{U})$ emanating from v there exists an edge-path $\tilde{\gamma} \in \widetilde{\mathbf{R}}_{\mathbb{U}}$ emanating from $\tilde{v}$ such that $\pi \circ \tilde{\gamma} = \gamma$.

Corollary (Path Correspondence Criterion)

For an open cover $\mathbb{U} : \mathrm{AP} \to \mathscr{T}$, the following are equivalent:

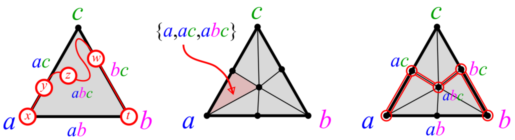
1. $\mathbb{U}$ *has the path-lifting property;*
2. *For every $x \in X$ and every edge-path γ in $N_{red}(\mathbb{U})$, there exists a tame path c in X emanating from x and inducing γ.*

Corollary

If $\varsigma^{-1}(\sigma)$ is path-connected for all $\mathbb{U}$-realized $\sigma \in N(\mathbb{U})$, the path correspondence holds for $\mathbb{U}$.

The Reduced Nerve: is it a deformation retract of $\mathrm{sd}(N(\mathbb{U}))$?

We want a theorem of the form: "Let $\mathbb{U}$ be a good cover. *If $\mathbb{U}$ satisfies additional conditions,* then $N_{red}(\mathbb{U})$ is a deformation retract (DR) of $\mathrm{sd}(N(\mathbb{U}))$."



The unrealized simplices $\{c\}$ and $\{a, b\}$ seem to lie "on the boundary" of $N(\mathbb{U})$.

Indeed, there are some hints in this direction:

Lemma (unrealized part is sparse)

Maximal simplices of $N(\mathbb{U})$ are $\mathbb{U}$-realized. $\square$

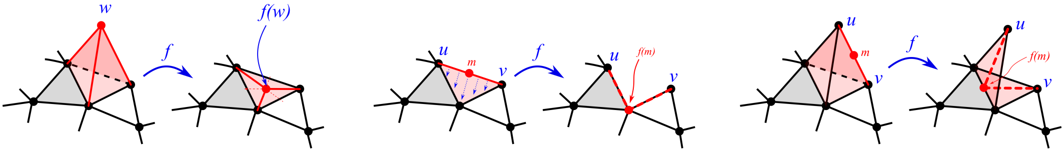
Lemma (actually, the following statement is FALSE:)

If $\sigma \in N(\mathbb{U})$ is unrealized, then σ is contained in a unique maximal simplex (σ is a free face).

The Reduced Nerve: is it a deformation retract of $\mathrm{sd}(N(\mathbb{U}))$?

Recap: Free simplices and simplicial collapses.

- A simplex σ of a complex K is *free* if it is contained in only one maximal simplex.
- The deletion $\mathrm{del}_K(\sigma)$ is obtained by removing all simplices of K containing σ.
- If σ is free, the $\mathrm{del}_K(\sigma)$ is said to have been obtained from K by (one) simplicial collapse, and is an SDR of K.

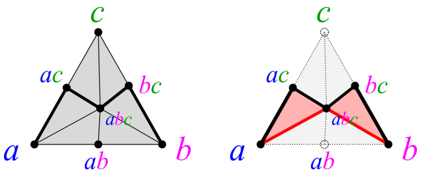


A vertex collapse (left), elementary collapse (center), and edge collapse (right) and the correponding retractions.

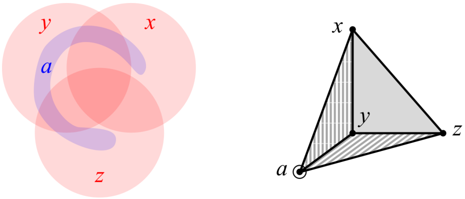
- $\mathrm{st}^\circ_K(\sigma) \triangleq \{\tau \in K \colon \sigma \subseteq \tau\}$
- $\mathrm{st}_K(\sigma) \triangleq \{\tau \in K \colon \tau \cup \sigma \in K\}$
- $\mathrm{lk}_K(\sigma) \triangleq \{\tau \in \mathrm{st}_K(\sigma) \colon \tau \cap \sigma = \varnothing\}$

The Reduced Nerve: is it a deformation retract of $\mathrm{sd}(N(\mathbb{U}))$?

- Removing the two unrealized vertices of $\mathrm{sd}(N(\mathbb{U}))$ in our running example:



- Collapsing the [only] unrealized simplex is impossible:



Only $\{a\}$ is unrealized in $N(\mathbb{U})$, but it is not free.

Current Results: Removing Unrealized Simplices of $N(\mathbb{U})$

Since simplicial collapses won't work for us...

Definition

A simplex σ of a scx K is said to be removable, if $\mathrm{lk}_K(\sigma)$ is contractible.

- If $\sigma \in K$ is removable then the deletion $\mathrm{del}_K(\sigma)$ is an SDR of K (see, e.g., [15])
- If L is obtainable from K by a sequence of deletions of removable simplices, we say L is obtained from K by generalized collapse.

Lemma (unrealized is removable)

Suppose $\sigma \in N(\mathbb{U})$ is an unrealized simplex of a good cover $\mathbb{U}$. Then:

1. *σ is a removable simplex of $N(\mathbb{U})$;*
2. *σ is a removable vertex of $\mathrm{sd}(N(\mathbb{U}))$.*

Current Results: Removing Unrealized Simplices of $N(\mathbb{U})$

Proof sketch: Let $K \triangleq N(\mathbb{U})$, $N \triangleq \mathrm{sd}(K)$, $L \triangleq \mathrm{lk}_K(\sigma)$, and $\mathfrak{S}^\sigma \triangleq \{A\colon A \subsetneq \sigma\}$.

Removability of σ in K:

- If σ is unrealized, then the $\widetilde{\mathbb{U}}(\sigma \cup \{\alpha\})$, $\alpha \in \mathrm{AP} \setminus \sigma$ form a good cover of $\widetilde{\mathbb{U}}(\sigma)$;
- The nerve of this cover is L, which is therefore contractible, by the nerve lemma.

Removability of $\{\sigma\}$ in $N = \mathrm{sd}(K)$:

- The link $\mathrm{lk}_N(\{\sigma\})$ is simplicially isomorphic to the join $\mathfrak{S}^\sigma * \mathrm{sd}(L)$.
- Since L is contractible, so is $\mathrm{lk}_N(\{\sigma\})$. □

Corollary (A single DR removes all unrealized simplices!!)

The subcomplex $N[0] \triangleq \bigcap_{\sigma \in K \text{ not realized}} \mathrm{del}_N(\{\sigma\})$ *is an SDR of* N.

Proof sketch: Remove all the unrealized vertices $\{\sigma\}$ of dimension 0 in one fell swoop, then the ones of dimension 1, and so forth... □

Current Efforts: Step-by-Step Retraction to $N_{red}(\mathbb{U})$

We would like to continue removing unrealized simplices inductively, e.g.:

- Set $N[d] \triangleq \bigcap_{T \in N \text{ unrealized of } \dim \leq d} \mathrm{del}_N(T)$, then $N_{red}(\mathbb{U}) = N[d]$ for eventually all d.
- Plan: Proceed to show $N[d+1]$ is a generalized collapse of $N[d]$ for all $d \geq 0$.
- Challenge: The link of a simplex $T = \{\sigma_0 \subset \ldots \subset \sigma_d\} \in N$ is

$$\mathrm{lk}_N(T) \cong \mathfrak{S}^{\sigma_0} * \mathfrak{S}^{\sigma_1 \setminus \sigma_0} * \ldots * \mathfrak{S}^{\sigma_d \setminus \sigma_{d-1}} * \mathrm{sd}(\mathrm{lk}_K(\sigma_d)).$$

  Can we argue that $\mathrm{lk}_{N[d-1]}(T)$ is contractible?
- Note: $N[1] = N_{red}(\mathbb{U})$ if $N_{red}(\mathbb{U})$ happens to be a flag complex!
  - A complex K over V satisfies the *flag condition*, if $\mathfrak{S}^{\sigma} \subset K$ implies $\sigma \in K$ for all $\sigma \subseteq V$.
- When would this be true? When, for each $T = \{\sigma_1 \subset \ldots \subset \sigma_{d+1}\} \in N$ with all the σ_i realized, the subspace $\bigcup_{i=0}^{d} \varsigma^{-1}(\sigma_i)$ of X is contractible.
- WHAT IS THE HOMOTOPY TYPE OF $\left|\widetilde{\mathbf{R}}_{\mathbb{U}}\right|$? (And when?)

Thank You!

References

[1] D. P. Guralnik, Y. Wang, and W. E. Dixon, “Topologically-Aware Planning Under Temporal Constraints,” *(in preparation)*, 2024.

[2] S. M. Edwards, D. P. Guralnik, A. Isaly, and W. E. Dixon, “Multi-Agent Localization Using Geometric Constraints on Relative Distance Measurements in the Presence of Intermittent State Feedback,” *submitted to IJRR*, 2024.

[3] S. C. Edwards, O. Patil, D. P. Guralnik, and W. E. Dixon, “Occluded Target Surveillance: A Topological Perspective on Intermittent Target Tracking with Lyapunov-based Deep Neural Networks,” *submitted to TRO*, 2024.

[4] F. M. Zegers, D. P. Guralnik, and W. E. Dixon, “Event-Triggered Multi-Agent System Rendezvous with Graph Maintenance in Varied Hybrid Formulations: A Comparative Study,” *IEEE Trans. Autom. Control*, 2024.

[5] H.-Y. Chen, Z. I. Bell, P. Deptula, and W. E. Dixon, “A switched systems framework for path following with intermittent state feedback,” *IEEE Control Syst. Lett.*, vol. 2, pp. 749–754, Oct. 2018.

[6] F. Zegers, H.-Y. Chen, P. Deptula, and W. E. Dixon, “Distributed coordination of a multi-agent system with intermittent communication: A switched systems approach,” in *Proc. ASME Dyn. Syst. Control Conf.*, 2018.

[7] R. Sun, C. Harris, Z. Bell, and W. E. Dixon, “Relay-explorer approach for multi-agent exploration of an unknown environment with intermittent communication,” in *Proc. IEEE Conf. Decis. Control*, pp. 5218–5223, 2020.

[8] R. Sun, Z. Bell, F. Zegers, and W. E. Dixon, “A switched systems approach to unknown environment exploration with intermittent state feedback for nonholonomic systems,” in *Proc. Am. Control Conf.*, pp. 5275–5280, 2020.

[9] D. Le, H.-Y. Chen, A. R. Teel, and W. E. Dixon, “Path following with stable and unstable modes subject to time-varying dwell-time conditions,” in *IFAC World Congr.*, 2020.

[10] S. Edwards, D. Le, D. Guralnik, and W. E. Dixon, “A Topologically Inspired Path-Following Method with Intermittent State Feedback,” *IEEE Robot. Automat. Lett.*, vol. 6, no. 3, pp. 4449–4456, 2021.

[11] G. E. Fainekos, A. Girard, H. Kress-Gazit, and G. J. Pappas, “Temporal logic motion planning for dynamic robots,” *Automatica*, vol. 45, pp. 343–352, Feb. 2009.

[12] M. Erdmann, “On the topology of discrete strategies,” *Intl. J. Robot. Res.*, vol. 29, no. 7, pp. 855–896, 2010.

[13] G. Pappas, G. Lafferriere, and S. Sastry, “Hierarchically consistent control systems,” *IEEE Transactions on Automatic Control*, vol. 45, pp. 1144–1160, June 2000.

[14] D. Kozlov, *Combinatorial algebraic topology*, vol. 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[15] J. A. Barmak, *Algebraic Topology of Finite Topological Spaces and Applications*, vol. 2032 of *Lecture Notes in Mathematics*. Springer-Verlag, 2011.

[16] M. A. Shtanko and M. I. Shtogrin, “On the embedding of cubic manifolds and complexes in a cubic lattice,” *Russian Mathematical Surveys*, vol. 47, pp. 267–268, Feb 1992.

The Nerve vs. 2^{AP}: the Shtan'ko-Shtogrin map [16]

- The geometric realization $\left|N(\mathbb{U})\right|$ of the nerve is constructed in $\mathbb{R}^{\mathrm{AP}}$, as a union of geometric simplices spanned by the e_α, $\alpha \in \mathrm{AP}$

$$\left|N(\mathbb{U})\right| \triangleq \bigcup_{\sigma \in N(\mathbb{U})} \dot{\Delta}^\sigma, \quad \dot{\Delta}^\sigma \triangleq \left\{\sum_{\alpha\in\sigma} t_\alpha e_\alpha \in \mathbb{R}^{\mathrm{AP}} \colon \sum_{\alpha\in\sigma} t_\alpha = 1, (\forall_{\alpha\in\sigma})(t_\alpha > 0)\right\}$$

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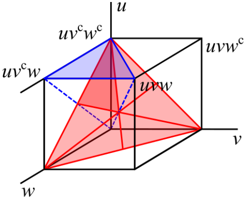
- The nerve is mapped *homeomorphically* into the positive boundary of the unit cube:

$$\Box^{\mathrm{AP}} \triangleq [0,1]^{\mathrm{AP}} \subset \mathbb{R}^{\mathrm{AP}}, \quad \Box_+^{\mathrm{AP}} \triangleq \{\xi \in \Box^{\mathrm{AP}} \colon \exists_\alpha\, \xi(\alpha) = 1\},$$



$$c \colon \begin{cases} \Delta^{\mathrm{AP}} & \to & \Box_+^{\mathrm{AP}} \\ \xi & \mapsto & \dfrac{\xi}{\|\xi\|_\infty}, \end{cases}$$

realizing the natural map of $N(\mathbb{U})$ into $\mathbf{2}^{\mathrm{AP}}$.

$\rightsquigarrow$ *each d-simplex is made of $(d+1)$ d-cubes meeting in its barycenter and creating a 'corner'*