Fixed Time Stability, Uniform Strong Dissipativity, and Stability of Feedback Systems

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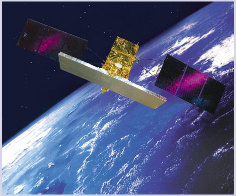
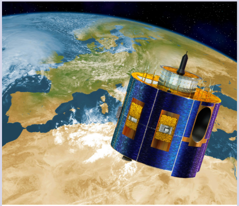
Outline

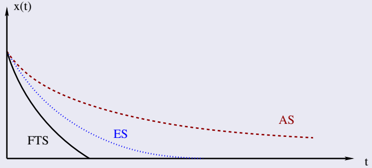
- New N & S Cond's for Fixed Time Stability
- Strong and Uniform Strong Dissipativity
  - Relation to Finite Time & Fixed Time Stability
- Kalman-Yakubovich-Popov (KYP) Conditions for S and US Dissipativity
  - S & US Passivity & Nonexpansivity
- Finite and Fixed Time Stability of FI
- Generalize Positivity and Small Gain Thms to Guarantee FT & FxT Stability
- Conclusion and Future Research





Motivation for Finite and Fixed Time Stability

- Finite and fixed time stability
  - Better robustness and disturbance rejection system properties
  - Optimal control without discontinuous dynamics (sliding mode)
    - Chattering due to system uncertainties or measurement imperfections
  - Finite and fixed time consensus, parallel formations, cyclic pursuit
    - Network systems and multiagent networks
- Upgraded speed of convergence



Motivation for Strong & Uniform Strong Dissipativity

- Dissipativity theory for dynamical system with C^1 flows addresses:
  - Robustness, disturbance rejection, stability of feedback system interconnections, optimality, and inverse optimality
- Strong and uniform strong dissipative systems can address:
  - Robustness, disturbance rejection, risk-sensitive control
  - Finite and fixed time stability of feedback interconnections
  - Finite and fixed time stabilization understood in physical terms
- Strong dissipativity = Dissipativity + Finite time stability
- Uniform Strong dissipativity = Dissipativity + Fixed time stability

Goals

- Strongly and uniformly strongly dissipative systems
  - New dissipation inequality: $\mathcal{G}_{\text{us-dissp}} \to \mathcal{G}_{\text{s-dissp}} \to \mathcal{G}_{\text{dissp}}$
  - Affine systems with quadratic supply rates
  - Extended Kalman-Yakubovitch-Popov conditions
- FT and FxT stability of S and US dissipative feedback systems
  - Storage functions for forward and feedback systems
  - Leading to closed-loop finite time and fixed time stability
  - Generalization of positivity & small gain thms for FT and FxT stabilization
  - Dynamic compensation

Finite Time and Fixed Time Stability

- Consider the nonlinear dynamical system $\mathcal{G}_{\text{closed}}$

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad t \geq t_0$$

  - $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is open, $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$, & $f(0) = 0$
  - Denote the solution of $\mathcal{G}_{\text{closed}}$ by $s : \overline{\mathbb{R}}_+ \times \mathcal{D} \to \mathbb{R}^n$

- The ZS $x(t) \equiv 0$ to $\mathcal{G}_{\text{closed}}$ is FTS if $\exists$ a STF T: $\mathcal{N} \setminus \{0\} \to (0, \infty)$
  - Finite time convergence ($\lim_{t \to T(x)} s^x(t) = 0$)
  - Lyapunov stability

- The ZS $x(t) \equiv 0$ to $\mathcal{G}_{\text{closed}}$ is fixed time stable if:
  - Finite time stability
  - Uniform boundedness of the settling time function ($T(x) \leq T_{\max}$)

Lyapunov Theorem for Finite Time Stability

- Assume $\exists$ C^0 function $V : \mathcal{D} \to \mathbb{R}$, $\alpha \in (0,1)$, $a > 0$, and a nbhd $\mathcal{M} \subseteq \mathcal{D}$ of the origin s.t.

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0, \quad x \in \mathcal{M} \backslash \{0\} \\ V'(x)f(x) &\le -a\,(V(x))^{\alpha}, \quad x \in \mathcal{M} \backslash \{0\} \end{aligned}$$

- Then the ZS $x(t) \equiv 0$ of $\mathcal{G}_{\rm closed}$ is finite time stable

- Moreover, $\exists$ a nbhd $\mathcal{N}$ of the origin & a STF $T : \mathcal{N} \to [0, \infty)$ s.t.

$$T(x_0) \le \frac{1}{a(1-\alpha)}\,(V(x_0))^{1-\alpha}, \quad x_0 \in \mathcal{N}$$

where $T(\cdot)$ is continuous on $\mathcal{N}$

Lyapunov Theorem for Fixed Time Stability

- Assume $\exists$ C^0 function $V : \mathcal{D} \to \mathbb{R}$, $\delta \in (0,1)$, $\theta > 1$ $a, b, c, k > 0$, & a nbhd $\mathcal{M} \subseteq \mathcal{D}$ of the origin s.t.

$$V(0) = 0$$

$$V(x) > 0, \quad x \in \mathcal{M}\backslash\{0\}$$

$$V'(x)f(x) \leq -\boxed{\left[aV^{\delta}(x) + bV^{\theta}(x)\right]^{k}} - \boxed{cV(x)}, \quad x \in \mathcal{M}\backslash\{0\}$$

- Then the ZS $x(t) \equiv 0$ of $\mathcal{G}_{\text{closed}}$ is fixed time stable
- Moreover, $\exists$ a nbhd $\mathcal{N}$ of the origin & a STF $T : \mathcal{N} \to [0, \infty)$ s.t.

$$T(x_0) \leq T_{\max} \triangleq \frac{1}{(1-\delta k)c} \ln\left[1 + \frac{c}{a^k}\left(\frac{a}{b}\right)^{\frac{1-\delta k}{\theta-\delta}}\right] + \frac{1}{(\theta k - 1)c} \ln\left[1 + \frac{c}{b^k}\left(\frac{b}{a}\right)^{\frac{\theta k-1}{\theta-\delta}}\right]$$

where $T(\cdot)$ is continuous on $\mathcal{N}$

Optimized Estimate of Settling Time Bound

- Proof follows by considering the comparison system

$$\dot{z}(t) = -[az^{\delta}(t) + bz^{\theta}(t)]^k - cz(t), \quad z(0) = z_0, \quad t \geq 0$$

- The ZS $z(t) \equiv 0$ is Lyapunov stable with LF $V(z) = z^2$
- Fixed time stability follows from

$$\begin{aligned}\lim_{z_0 \to \infty} T(z_0) &= \int_0^{\infty} \frac{\mathrm{d}z}{(az^{\delta} + bz^{\theta})^k + cz} \\ &\leq \frac{1}{(1-\delta k)c} \ln\left(1 + \frac{c}{a^k} r^{1-\delta k}\right) + \frac{1}{(\theta k - 1)c} \ln\left(1 + \frac{c}{b^k} r^{1-\theta k}\right) \triangleq g(r)\end{aligned}$$

- $\frac{\mathrm{d}g(r)}{\mathrm{d}r} = 0$ and $\frac{\mathrm{d}^2 g}{\mathrm{d}r^2} > 0 \Rightarrow g(r)$ attains its minimum at $r_{\min} = \left(\frac{a}{b}\right)^{1/(\theta - \delta)}$

$$T(z_0) \leq \frac{1}{(1-\delta k)c} \ln\left[1 + \frac{c}{a^k}\left(\frac{a}{b}\right)^{\frac{1-\delta k}{\theta - \delta}}\right] + \frac{1}{(\theta k - 1)c} \ln\left[1 + \frac{c}{b^k}\left(\frac{b}{a}\right)^{\frac{\theta k - 1}{\theta - \delta}}\right]$$

Remarks

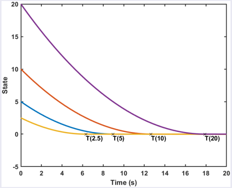
- Letting $c \to 0$ gives (Hu *et al.*, *Neural Networks*, 2017)

$$T_{\max,2} = \frac{1}{a^k}\left(\frac{a}{b}\right)^{\frac{1-\delta k}{\theta-\delta}}\left(\frac{1}{1-\delta k}+\frac{1}{\theta k-1}\right)$$

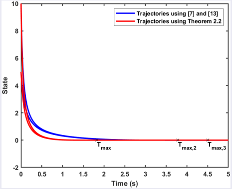
- Setting $r = 1$ and letting $c \to 0$ gives (Polyakov, *IEEE TAC*, 2012)

$$T_{\max,3} = \frac{1}{a^k(1-\delta k)}+\frac{1}{b^k(\theta k-1)}$$

- $T_{\max} < T_{\max,2} \leq T_{\max,3}$



$b = c = 0, a = 0.5, \delta k = 0.5$



$T_{\max}(c = 3, r = 0.397), T_{\max,2}(c = 0, r = 0.397), T_{\max,3}(c = 0, r = 1)$

Converse Lyapunov Theorem for FxT Stability

- Let $\mathcal{N} \subseteq \mathcal{D}$ be an open nbhd of the origin
- If $x(t) \equiv 0$ is FxTS and the STF $T(\cdot)$ is C^0 at $x = 0$
- Then there exists a C^0 function $V : \mathcal{N} \to \mathbb{R}$ & scalars $a, b, c, \delta, \theta,$ $k > 0$, $\delta k < 1$, $\theta k > 1$, s.t. $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{N}$, $x \neq 0$, &

$$\dot{V}(x) \leq -[aV^{\delta}(x) - bV^{\theta}(x)]^k - cV(x), \quad x \in \mathcal{N}$$

- $\dot{V}(x) \triangleq \lim_{h \to 0^+} \frac{1}{h} [V(s(h,x)) - V(x)]$
- $V(x) \triangleq \left(\frac{T(x)}{T_{\max}}\right)^{\frac{1}{1-\delta}}$
- $\dot{V}(x) = \frac{-1}{(1-\delta)T_{\max}} \left(\frac{T(x)}{T_{\max}}\right)^{\frac{\delta}{1-\delta}} \leq \frac{-1}{3(1-\delta)T_{\max}} \left[(V(x))^{\delta} + (V(x))^{\theta} + V(x)\right]$

Open Dynamical System

- Consider the open nonlinear dynamical system $\mathcal{G}$

$$\begin{aligned}\dot{x}(t) &= F(x(t), u(t)), \quad x(t_0) = x_0, \quad t \geq t_0 \\ y(t) &= H(x(t), u(t))\end{aligned}$$

  - $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ open with $0 \in \mathcal{D}$
  - $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$ and $y(t) \in Y \subseteq \mathbb{R}^l$
  - $F : \mathcal{D} \times U \to \mathbb{R}^n$ and $H : \mathcal{D} \times U \to Y$ are C^0 in x and u
  - $\mathcal{U}$ and $\mathcal{Y}$ define input and output spaces
  - $\mathcal{G}$ has at least one equilibrium so that wlog $F(0,0) = 0$ & $H(0,0) = 0$

Dissipativity, Strong Dissipativity, & Uniform Strong Dissipativity

- $\mathcal{G}$ is dissipative w.r.t $r(u, y)$ iff $\exists$ a C^0 SF $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ s.t. $V_\mathrm{s}(\cdot)$ is NND and

$$V_\mathrm{s}(x(t)) \leq V_\mathrm{s}(x(t_0)) + \int_{t_0}^{t} r(u(s), y(s))\mathrm{d}s, \qquad t \geq t_0$$

- $\mathcal{G}$ is S dissipative w.r.t. $r(u, y)$ iff $\exists$ a C^0 SF $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ & scalars $\alpha \in (0, 1)$, $a > 0$ s.t. $V_\mathrm{s}(\cdot)$ is NND &

$$V_\mathrm{s}(x(t)) + \boxed{a \int_{t_0}^{t} [V_\mathrm{s}(x(s))]^{\alpha}\, \mathrm{d}s} \leq V_\mathrm{s}(x(t_0)) + \int_{t_0}^{t} r(u(s), y(s))\mathrm{d}s, \qquad t \geq t_0$$

- $\mathcal{G}$ is US dissipative w.r.t. $r(u, y)$ iff $\exists$ a C^0 SF $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ & scalars $a, b, c, \delta, \theta, k > 0$ s.t. $\delta k < 1$, & $\theta k > 1$, $V_\mathrm{s}(\cdot)$ is NND, &

$$V_\mathrm{s}(x(t)) + \boxed{\int_{t_0}^{t} \left[\left[aV_\mathrm{s}^{\delta}(x(s)) + bV_\mathrm{s}^{\theta}(x(s))\right]^{k} + cV_\mathrm{s}(x(s))\right] \mathrm{d}s} \leq V_\mathrm{s}(x(t_0)) + \int_{t_0}^{t} r(u(s), y(s))\mathrm{d}s$$

Observations

- If $V_{\rm s}$ is C^1, then dissipativity implies

$$\dot{V}_{\rm s}(x,u) \le r(u, H(x,u)), \quad x \in \mathcal{D}, \quad u \in U$$

  - $\dot{V}_{\rm s}(x,u) = \frac{\mathrm{d}}{\mathrm{d}t} V_{\rm s}(s(t,x,u))\big|_{t=0}$ is the total derivative of $V_{\rm s}$ along $s(t,x,u)$

- If $V_{\rm s}$ is C^1, then S and US dissipativity imply

$$\dot{V}_{\rm s}(x,u) + \boxed{aV_{\rm s}^{\alpha}(x)} \le r(u, H(x,u)), \quad x \in \mathcal{D}, \quad u \in U$$

$$\dot{V}_{\rm s}(x,u) + \boxed{[aV_{\rm s}^{\delta}(x) + bV_{\rm s}^{\theta}(x)]^k + cV_{\rm s}(x)} \le r(u, H(x,u)), \quad x \in \mathcal{D}, \quad u \in U$$

- For a closed system (i.e., $u(t) \equiv 0$ & $y(t) \equiv 0$) and $V_{\rm s}(x) > 0$, $x \in \mathcal{D}$

$$\dot{V}_{\rm s}(x,0) \le \boxed{-aV_{\rm s}^{\alpha}(x)}, \quad x \in \mathcal{D}, \quad \text{(FT stability)}$$

$$\dot{V}_{\rm s}(x,0) \le \boxed{-[aV_{\rm s}^{\delta}(x) + bV_{\rm s}^{\theta}(x)]^k - cV_{\rm s}(x)}, \quad x \in \mathcal{D} \quad \text{(FxT stability)}$$

where $\dot{V}_{\rm s}(x,0) = \frac{\mathrm{d}}{\mathrm{d}t} V_{\rm s}(s(t,x,0))\big|_{t=0} = V_{\rm s}'(x)F(x,0)$

Nonlinear Affine Dynamical Systems

- Consider the nonlinear dynamical system $\mathcal{G}$

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0$$
$$y(t) = h(x(t)) + J(x(t))u(t)$$

- $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are C^0 mappings

- $u(\cdot) \in \mathcal{U}$ satisfies the required properties for the existence and uniqueness of solutions

- $\mathcal{G}$ has at least one equilibrium so that wlog $f(0) = 0$ & $h(0) = 0$

- Assume every storage function $V_{\mathrm{s}}(\cdot)$ for $\mathcal{G}$ is C^1

Extended Kalman-Yakubovich-Popov Conditions

- $\mathcal{G}$ is S dissipative w.r.t. $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ iff $\exists$ fun's V_{s}, ℓ, & $\mathcal{W}$, & $a > 0$, $\alpha \in (0, 1)$, s.t $V_{\mathrm{s}}(\cdot)$ is C^1, nnd, &

$$0 = V_{\mathrm{s}}'(x)f(x) + \boxed{aV^{\alpha}(x)} - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x)$$
$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x)$$
$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$$

- $\mathcal{G}$ is US dissipative w.r.t. $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ iff $\exists$ fun's V_{s}, ℓ, & $\mathcal{W}$, & $a, b, c, \delta, \theta, k > 0$, $\delta k < 1$, $\theta k > 1$, s.t. $V_{\mathrm{s}}(\cdot)$ is C^1, nnd, &

$$0 = V_{\mathrm{s}}'(x)f(x) + \boxed{\left[aV^{\delta}(x) + bV^{\theta}(x)\right]^k + cV(x)} - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x)$$
$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x)$$
$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$$

Uniform Strong Positive Real and Bounded Real Lemma

- $r(u,y) = 2u^{\mathrm{T}}y$ and $r(u,y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y,\ \gamma > 0$
- $\mathcal{G}$ is US passive iff $\exists$ fun's V_{s}, ℓ, & $\mathcal{W}$, & $a, b, c, \delta, \theta, k > 0$, $\delta k < 1$, $\theta k > 1$, s.t. $V_{\mathrm{s}}(\cdot)$ is C^1, nnd, &

$$0 = V_{\mathrm{s}}'(x)f(x) + \boxed{\left[aV^{\delta}(x) + bV^{\theta}(x)\right]^k + cV(x)} - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x)$$
$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x)$$
$$0 = J(x) + J^{\mathrm{T}}(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$$

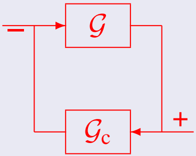
- $\mathcal{G}$ is US nonexpansive iff $\exists$ fun's V_{s}, ℓ, & $\mathcal{W}$, & $a, b, c, \delta, \theta, k > 0$, $\delta k < 1$, $\theta k > 1$, s.t. $V_{\mathrm{s}}(\cdot)$ is C^1, nnd, &

$$0 = V_{\mathrm{s}}'(x)f(x) + \boxed{\left[aV^{\delta}(x) + bV^{\theta}(x)\right]^k + cV(x)} + h^{\mathrm{T}}(x)h(x) + \ell^{\mathrm{T}}(x)\ell(x)$$
$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) + h^{\mathrm{T}}(x)J(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x)$$
$$0 = \gamma^2 I_m - J^{\mathrm{T}}(x)J(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$$

Stability of Feedback Dynamical Systems

- Consider the dynamical system $\mathcal{G}$ with nonlinear feedback system $\mathcal{G}_c$

$$\dot{x}_c(t) = f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t), \quad x_c(0) = x_{c0}, \quad t \geq 0$$
$$y_c(t) = h_c(u_c(t), x_c(t)) + J_c(u_c(t), x_c(t))u_c(t)$$



- Plant ($\mathcal{G}$) order n, compensator ($\mathcal{G}_c$) order n_c with $n_c \leq n$
- Assume feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed
  - $\det[I_m + J_c(y, x_c)J(x)] \neq 0$ for all y, x, and x_c

Finite Time Stabilization

- If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are strongly dissipative w.r.t. supply rates $r(u,y)$ & $r_{\rm c}(u_{\rm c},y_{\rm c})$, with SFs $V_{\rm s}(\cdot)$ & $V_{\rm sc}(\cdot)$
- And $\exists$ a scalar $\sigma > 0$ s.t.

$$\boxed{r(u,y) + \sigma r_{\rm c}(u_{\rm c},y_{\rm c}) \leq 0,\ u_{\rm c} = y,\ \text{and } y_{\rm c} = -u}$$

Then:

- The NFI of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is FTS
- There exists a C^0 STF T: $\mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} \to [0,\infty)$ s.t

$$T(x_0,x_{\rm c0}) \leq \frac{1}{\tilde{c}(1-\tilde{\alpha})}\left(V_{\rm s}(x_0) + V_{\rm sc}(x_{\rm c0})\right)^{1-\tilde{\alpha}}, \quad (x_0,x_{\rm c0}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}$$

  - $\tilde{\alpha} \triangleq \max\{\alpha,\alpha_{\rm c}\} \in (0,1)$
  - $\tilde{c} \triangleq \frac{1}{V(x_0,x_{\rm c0})^{\tilde{\alpha}}}\min\{cV(x_0,x_{\rm c0})^{\alpha}, \sigma^{1-\alpha_{\rm c}}c_{\rm c}V(x_0,x_{\rm c0})^{\alpha_{\rm c}}\} > 0$

Fixed Time Stabilization

- If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are uniformly strongly dissipative w.r.t. supply rates $r(u,y)$ & $r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}})$, with SFs $V_{\mathrm{s}}(\cdot)$ & $V_{\mathrm{sc}}(\cdot)$
- And $\exists$ a scalar $\sigma > 0$ s.t.

$$\boxed{r(u,y) + \sigma r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}) \leq 0, \ u_{\mathrm{c}} = y, \ \text{and } y_{\mathrm{c}} = -u}$$

Then:

- The NFI of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is FxTS
- There exists a C^0 STF $T\colon \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}} \to [0,\infty)$ s.t

$$T(x_0, x_{\mathrm{c}0}) \leq \frac{1}{(1-\tilde{\delta})\tilde{c}} \ln\left(1 + \frac{\tilde{c}}{\tilde{a}}\right) + \frac{1}{(\hat{\theta}-1)\tilde{c}} \ln\left(1 + \frac{2^{\tilde{\theta}-1}\tilde{c}}{\tilde{b}}\right)$$

- $\tilde{a} > 0, \tilde{b} > 0, \tilde{c} > 0, \tilde{\delta} \in (0,1), \tilde{\theta} > 1$, and $\hat{\theta} > 1$

Specialization to Quadratic Supply Rates

- Let $r(u,y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$, $r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}) = y_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}}S_{\mathrm{c}}u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}u_{\mathrm{c}}$
- Assume $\exists\ \sigma > 0$ s.t.

$$\hat{Q} = \begin{bmatrix} Q + \sigma R_{\mathrm{c}} & -S + \sigma S_{\mathrm{c}}^{\mathrm{T}} \\ -S^{\mathrm{T}} + \sigma S_{\mathrm{c}} & R + \sigma Q_{\mathrm{c}} \end{bmatrix} \leq 0$$

- If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are uniformly strongly dissipative w.r.t. $r(u,y)$ and $r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}})$, then the NFI of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is fixed time stable
- US passivity: $r(u,y) = 2u^{\mathrm{T}}y$ & $r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}) = 2u_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}}$
- US nonexpansivity: $r(u,y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$ & $r_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}) = \gamma_{\mathrm{c}}^2 u_{\mathrm{c}}^{\mathrm{T}}u_{\mathrm{c}} - y_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}}$, $\gamma\gamma_{\mathrm{c}} \leq 1$
- Generalizes positivity and small gain theorems to guarantee FxTS

Fixed Time Stabilization of a Rigid Satellite

- Consider the single DoF satellite

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0$$
$$\dot{x}_2(t) = u(t), \quad x_2(0) = x_{20}$$

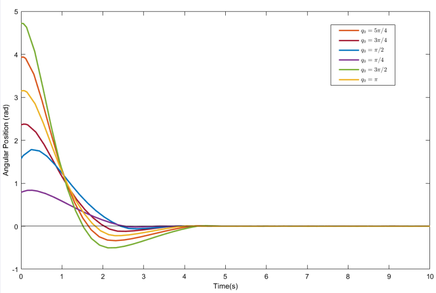
- Feedback control law $(\alpha \in (0,1))$

$$u(x_1,x_2) = -\mathrm{sign}(x_1)|x_1|^{\frac{\alpha}{2-\alpha}} - \mathrm{sign}(x_2)|x_2|^{\alpha} - \mathrm{sign}(x_1)|x_1|^{\frac{4-3\alpha}{2-\alpha}} - \mathrm{sign}(x_2)|x_2|^{\frac{4-3\alpha}{3-2\alpha}}$$

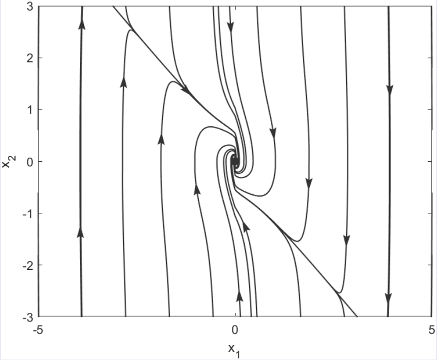
- Lyapunov function that shows FxT stability

$$V(x_1,x_2) = \frac{k_1(2-\alpha)}{3-\alpha}|x_1|^{\frac{3-\alpha}{2-\alpha}} + k_2 x_1 x_2 + \frac{1}{3-\alpha}|x_2|^{3-\alpha} + \frac{k_1'(2-\alpha)}{(3-2\alpha)(3-\alpha)}|x_1|^{\frac{(3-2\alpha)(3-\alpha)}{2-\alpha}}$$
$$+ \frac{k_2'}{(3-2\alpha)}\mathrm{sign}(x_1)|x_1|^{3-2\alpha}x_2 + \frac{k_2'(3-2\alpha)}{(3-\alpha)(3-2\alpha)-(2-\alpha)}\mathrm{sign}(x_2)|x_2|^{3-\alpha-\frac{2-\alpha}{3-2\alpha}}x_1$$

Controlled Satellite Simulations



- Controlled angular positions with $p_0 = \pi/6$ rad/s



- Phase portrait shows trajectories converge to a positively invariant terminal sliding mode in FxT

Dynamic Compensation Example

- Consider the first-order nonlinear dynamical system $\mathcal{G}$

$$\dot{x}(t) = -\mathrm{sign}(x)|x(t)|^{1/3} - \mathrm{sign}(x)|x(t)|^2 + u(t), \quad x(0) = x_0, \quad t \geq 0$$
$$y(t) = x(t)$$

- With candidate storage function $V_{\mathrm{s}}(x) = x^2$

$$\dot{V}_{\mathrm{s}}(x, u) = -2V_{\mathrm{s}}(x)^{0.67} - 2V_{\mathrm{s}}(x)^{1.5} + 2yu \quad (\mathcal{G} \text{ is US passive})$$

- Design a first-order dynamic compensator $\mathcal{G}_{\mathrm{c}}$

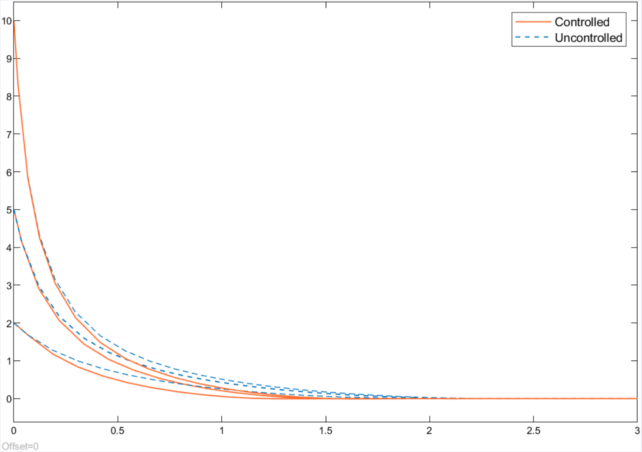
$$\dot{x}_{\mathrm{c}}(t) = -\mathrm{sign}(x_{\mathrm{c}})|x_{\mathrm{c}}(t)|^{1/2} - 4\mathrm{sign}(x_{\mathrm{c}})|x_{\mathrm{c}}(t)|^2 + u_{\mathrm{c}}(t), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad t \geq 0$$
$$y_{\mathrm{c}}(t) = x_{\mathrm{c}}(t)$$

- With candidate storage function $V_{\mathrm{sc}}(x) = x_{\mathrm{c}}^2$

$$\dot{V}_{\mathrm{sc}}(x, u) = -2V_{\mathrm{sc}}(x_{\mathrm{c}})^{0.75} - 8V_{\mathrm{sc}}(x_{\mathrm{c}})^{1.5} + 2y_{\mathrm{c}}u_{\mathrm{c}} \quad (\mathcal{G}_{\mathrm{c}} \text{ is US passive})$$

- The NFI of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is fixed time stable

Simulations



Controlled versus uncontrolled state trajectories

Conclusion and Future Research

- Extended dissipativity theory to S and US dissipativity
  - Connections to finite time and fixed time stability
  - New KYP conditions for quadratic supply rates
- Developed FT and FxT stability results for NL feedback systems
- Connect S and US dissipativity theory and optimal and inverse optimal FxT stabilization using HJB theory
  - Time-optimal control problem
  - C^0 Lyapunov functions $\rightarrow$ viscosity solutions of HJB equations
- Discrete-time and hybrid extensions