

Effective Whitney Stratification of Algebraic Maps & Applications

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joint work with **Vidit Nanda**, *University of Oxford*

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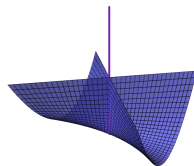
In \mathbb{C}^n , FOCM, 2022: <https://doi.org/10.1007/s10208-022-09574-8>

In \mathbb{R}^n , 2023, Arxiv: <http://arxiv.org/abs/2307.05427>

Faster alg. \mathbb{R}^n or \mathbb{C}^n , 2024 (with R. Mohr): <https://arxiv.org/abs/2406.17122>

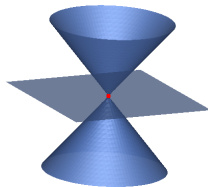
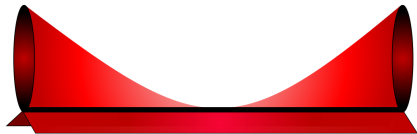
App. in physics, 2024 (with F. Tellander): <https://arxiv.org/abs/2402.14787>

August 29, 2024



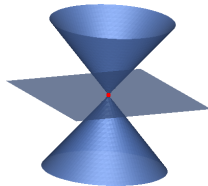
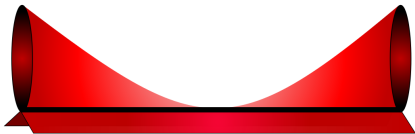
Stratifying Singular Varieties and Maps

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Spaces: separate into smooth manifolds which join in a nice way.

Maps: subdivide codomain so that the fibers over any point in a region look the same.

Stratifying Spaces

More precisely, for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we will consider algebraic varieties

$$X = \mathbb{V}(I_X) = \mathbb{V}(f_1, \dots, f_r) = \{p \in \mathbb{K}^n \mid f_1(p) = \dots = f_r(p) = 0\}.$$

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Or **full semi-algebraic sets**: $W = X \cap Z$,

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A *stratification* is a filtration, $X_\bullet, \emptyset = X_{-1} \subset X_0 \subset \dots \subset X_n = X$ of X s.t. $X = \cup_i X_i$ and s.t. each **strata**, which is a connected component of $\mathcal{M}_i := X_i - X_{i-1}$, is either empty or a smooth manifold of dimension i .

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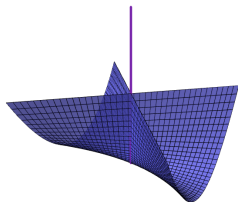
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Additionally: want decomposition $X = \coprod_i \mathcal{M}_i$ to be **equisingular**; i.e. neighbourhoods in X of distinct points in a connected comp. of \mathcal{M}_i are “similar”.

Whitney stratification, using **Whitney's Condition B**, accomplishes this & always exist & we have an algorithm to compute it.



Stratifying Maps

Map Stratification:

Let X, Y be algebraic varieties and $f : X \rightarrow Y$ an algebraic map. A **stratification of f** , is a Whitney stratification of X and Y so that for every stratum S of X there is a stratum R of Y such that $f(S) \subset R$, and the derivative $d(f|_S)$ of $f|_S : S \rightarrow R$ is surjective at each point of S .

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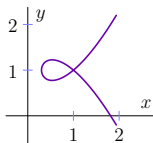
The 2023 & 2024 preprints extend this to **real varieties and full semi-algebraic sets** as well as introduce **new and more efficient methods to compute Whitney stratification** (the most difficult/expensive step).

Why Whitney Stratification is needed to Study Maps

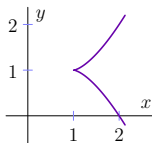
Suppose we wish to study the changes in topology of the curve in \mathbb{R}^2 defined by the parametric polynomial

$$f_z(x, y) = (y - 1)^2 - (x - z)(x - 1)^2$$

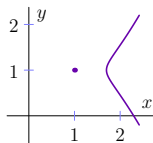
in variables x, y with parameter z .



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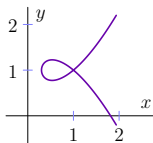
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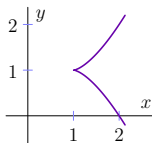
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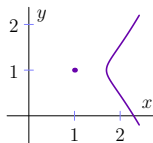
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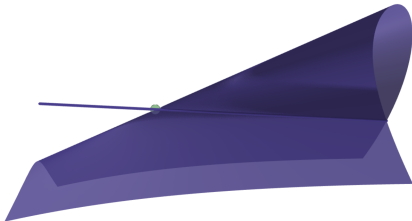


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Take $X = \mathbb{V}(f) \subset \mathbb{R}^3$. It is equivalent to ask when the fibers of the projection map $\pi : X \rightarrow \mathbb{R}_z$ change topology.

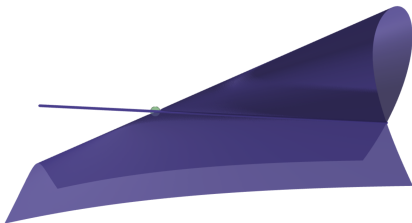
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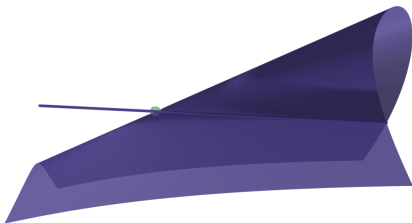


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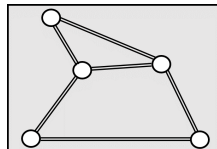
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Note: in this example the relevant map is a submersion at the point $(1, 1, 1)$, and this point is only detected via Whitney stratification.

Motivation: Singularities in Inverse Kinematics

Consider a **kinematic map** $f : C \rightarrow S$ where C is the configuration space and S the output state space (workspace), with f a polynomial map, S and C real varieties.

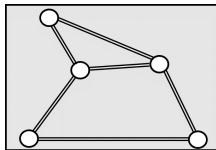
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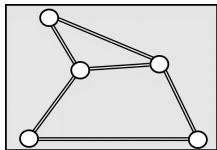


Note: if we stratify the map f , then the fibers $f^{-1}(q)$ and $f^{-1}(q')$ are topologically identical for any two points in a stratum N of S . We can then use topological criteria, e.g. [Leve, 2020, Homological invariants for classification of kinematic singularities], to **classify all singularities globally**.

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Road Map:

1. **Stratify f :** we can now do this (both in theory and in practice).
2. **Sample a point q in each strata** of S (theory exists, coding ongoing).
3. Compute $\dim(f^{-1}(q))$ & the **topology of $f^{-1}(q)$** (more later).

New Whitney Stratification Approach: Polar Varieties

Let $X \subset \mathbb{C}^n$ be an algebraic variety of dimension $d = \dim(X)$.

Consider a flag of general linear spaces $L_0 \subset \cdots \subset L_d$ each containing the origin with $\dim(L_i) = i + (n - \dim(X))$.

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The i^{th} (generic) *polar variety*

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Note we have $\delta(X, L_0) \subset \cdots \subset \delta(X, L_d) = X$.

Additionally each $\delta(X, L_i)$ is irreducible and has pure dimension.

New Algebraic Criterion: Polar

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Theorem (Helmer, Leykin and Nanda, 2024)

Let $A = \bigcup_i \delta(X, L_i)$, for i s.t. $\dim(Y \cap \delta(X, L_i)) < \dim(Y)$.

Then Whitney's *condition B* holds for the pair

$$(X_{\text{reg}}, Y_{\text{reg}} - A),$$

as long as subspaces L_i are sufficiently generic.

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Comp. to Our Other Methods (2022 paper, 2024 preprint)

Pros: Work in n rather than $2n$ variables (**significant pro!**).

Cons: Probabilistic, lose sparsity, statifications never minimal.

What Size of Examples Can We Handle?

To stratify a map $f : X \rightarrow Y$, $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ the main cost is comp. of Whitney stratification in $n + m$ variables, i.e. in \mathbb{R}^{n+m} .

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Two examples (to give an idea of the range) are below.

$$W_1 = \mathbb{V}(x_1 x_2^3 - 2x_1 x_2^2 x_4 + x_1 x_2 x_3 x_5, -x_7 x_6 x_8 x_9 - x_7 x_4 x_5) \subset \mathbb{R}^9.$$

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Using conormal methods finding a Whitney stratification of W_2 takes roughly 20 minutes, with our new polar variety algorithm it takes less than 1 second.

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Summary: conormal methods have been used to successfully tackle physics examples in \mathbb{R}^9 or \mathbb{R}^{10} , we hope the new algorithm will allow for this to be pushed a bit further.

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3. This gives us **equations for $f^{-1}(q)$** ; using these we can compute **$\dim(f^{-1}(q))$** symbolically (Gröbner basis) or via a numerical irreducible decomposition of $f^{-1}(q)$. **Gives $\dim(f^{-1}(q))$ in \mathbb{C}^n .**

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1. Use our algorithm to **compute a stratification** (X_\bullet, Y_\bullet) of f . Each strata in Y_\bullet is described as a set difference $Y_i - Y_{i-1}$.
2. **Sample a point q in each irreducible component of $Y_i - Y_{i-1}$** using either symbolic (Gröbner basis) methods or homotopy continuation (e.g. with Bertini, Homotopy.jl, etc.).
3. This gives us **equations for $f^{-1}(q)$** ; using these we can compute **$\dim(f^{-1}(q))$** symbolically (Gröbner basis) or via a numerical irreducible decomposition of $f^{-1}(q)$. **Gives $\dim(f^{-1}(q))$ in \mathbb{C}^n .**

Step 1: Our **WhitneyStatifications M2 Package**.

Steps 2 and 3: Standard tools available in M2, Sage, Julia, etc.

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Let I be an ideal in $\mathbb{R}[x_1, \dots, x_n]$ and $X = \mathbb{V}(I) \subset \mathbb{C}^n$ be an irreducible complex variety, $X_{\text{reg}} = X - X_{\text{Sing}}$ the set of smooth points and $\Re(X)$ the real solution set of the same prime ideal (same as the real points in X).

Theorem (Classical, e.g. Marshall, 2008)

If $\Re(X_{\text{reg}}) \neq \emptyset$ then $\dim(X) = \dim(\Re(X))$.

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Over \mathbb{C} our connected strata have the form $M = W - Z$ for W irreducible and M smooth over \mathbb{C} .

If M has a real point \Rightarrow real alg. set (defined by the same poly.) has the same dimension over \mathbb{R} or \mathbb{C} , otherwise it is empty.

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The algorithm of Harris, Hauenstein, and Szanto [HHS], (among others) can sample one smooth point q in each connected component of M if these exist, hence this gives us the dimension.

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The HHS algorithm is not yet implemented in a publicly available library, but shows promise on examples.

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Step 1 towards Goal: Give (reasonably) tight bounds on the Betti numbers for an arbitrary compact real variety (compactness isn't strictly required but we stick to this case for simplicity).

Bounding Betti Numbers of (Compact) Real Varieties

Let $X = \mathbb{V}(f_1, \dots, f_m) \subset \mathbb{R}^n$ be a compact real alg. variety. Let $\epsilon > 0$, a *tubular neighbourhood of X of radius ϵ* is

$$X_{\leq \epsilon} := \{x \in \mathbb{R}^n \mid f_1(x)^2 + \dots + f_m(x)^2 \leq \epsilon^2\}.$$

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Also let $p = f_1(x)^2 + \dots + f_m(x)^2$, $W = \mathbb{V}(p - \epsilon^2) \subset \mathbb{R}^{n+1}$, and the projection map $\pi_\epsilon : W \rightarrow Y = \mathbb{R}$ onto the last coordinate given by $\pi_\epsilon : (x_1, \dots, x_n, \epsilon) \mapsto \epsilon$.

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Theorem (Helmer, Nanda, 2024)

Let (W_\bullet, Y_\bullet) be a *stratification of π_ϵ* ; note $Y_0 \subset \mathbb{R}$ is finite. Then for all $\delta < \min(|\gamma| \mid \gamma \in (Y_0 - \{0\}))$, $X_{\leq \delta}$ is *homeomorphic to X* .

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Upshot: Using our stratification algorithm we can find a bound on thickening our variety X to a smooth manifold with boundary $X_{\leq \epsilon}$.

Bounding Betti Numbers of (Compact) Real Varieties

The *Roman surface*: $X = \mathbb{V}(f)$, $f = x^2y^2 + y^2z^2 + z^2x^2 - xyz$.

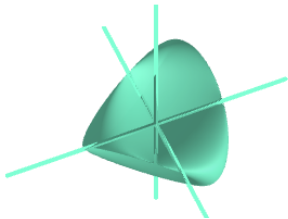


Figure 2: Roman Surface

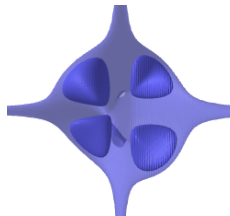


Figure 3: Thickened Boundary

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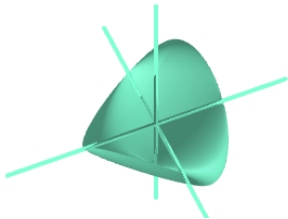


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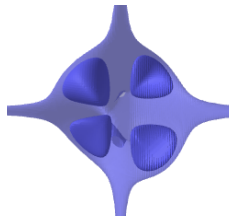


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Stratifying the projection $\pi : \mathbb{V}(f - \epsilon^2) \rightarrow \mathbb{R}_\epsilon$ we obtain that the critical value is $\epsilon = 1/256$.

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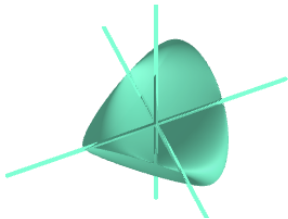


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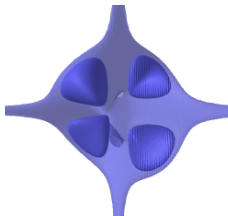


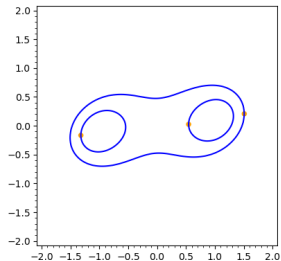
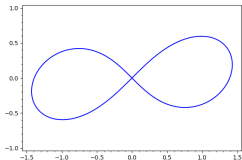
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Hence the thickening $X_{\leq \frac{1}{970}}$ is homeomorphic to X .

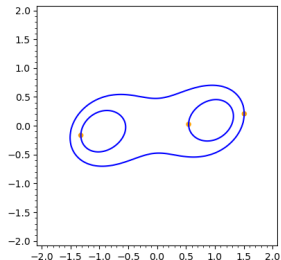
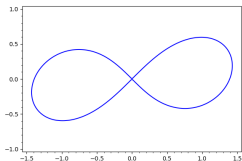
Its smooth boundary $W = \partial X_{\leq \frac{1}{970}}$ is above right.

Bounding Betti Numbers of (Compact) Real Varieties



First: Given a variety $X = \mathbb{V}(p)$ calculate thickening ϵ via stratification.

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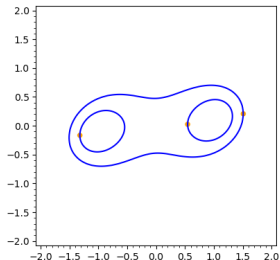
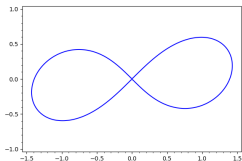
Let $W = \partial X_{\geq \epsilon} = \mathbb{V}(p - \epsilon^2)$ and let $\ell = \sum \lambda_i x_i$ be a general linear form.

Let K be the ideal generated by the 2×2 minors of

$$\begin{bmatrix} \nabla \ell \\ \nabla p \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \frac{\partial p}{\partial x_1} & \frac{\partial p}{\partial x_2} & \cdots & \frac{\partial p}{\partial x_n} \end{bmatrix}.$$

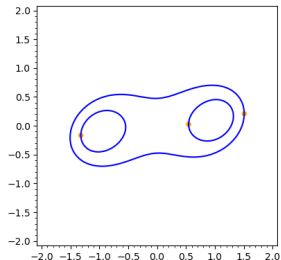
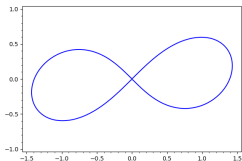
The variety $\text{crit} := \mathbb{V}(K + \langle p - \epsilon^2 \rangle)$ is finite and its points are the critical points of the Morse function $\ell : W \rightarrow \mathbb{R}$, $\ell : w \mapsto \ell(w)$.

Bounding Betti Numbers of (Compact) Real Varieties



For each $x \in \text{crit}$, set $u' = \nabla_x(\ell)$ and $u = \frac{u'}{\|u'\|}$ if $p(x + \nu u) < \epsilon$ for ν sufficiently small then x is a *type N critical point*; we compute its index from the appropriate Hessian.

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Bounding The Betti Numbers

The i^{th} Betti number of X , $b_i(X)$, is bounded by the number of type-N critical points of W of index i .

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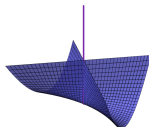
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Ongoing Work: leverage the above to compute the Betti numbers.

Package Docs: <https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/WhitneyStratifications/html/index.html>

Thank You!

Thank you for your attention!



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In \mathbb{C}^n , FOCM, 2022: <https://doi.org/10.1007/s10208-022-09574-8>

In \mathbb{R}^n , 2023, Arxiv: <http://arxiv.org/abs/2307.05427>

Faster alg. \mathbb{R}^n or \mathbb{C}^n , 2024 (with R. Mohr): <https://arxiv.org/abs/2406.17122>

App. in physics, 2024 (with F. Tellander): <https://arxiv.org/abs/2402.14787>

Code (M2 distributed): <https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/WhitneyStratifications/html/index.html>

Code (Most Recent): <http://martin-helmer.com/Software/WhitStrat/>