Homogenisation for Stochastic Advection by Lie Transport (SALT)

Ruiao Hu

Department of Mathematics, Imperial College London

Joint work with Theo Diamantakis and James-Michael Leahy

AFOSR Annual Review 2024

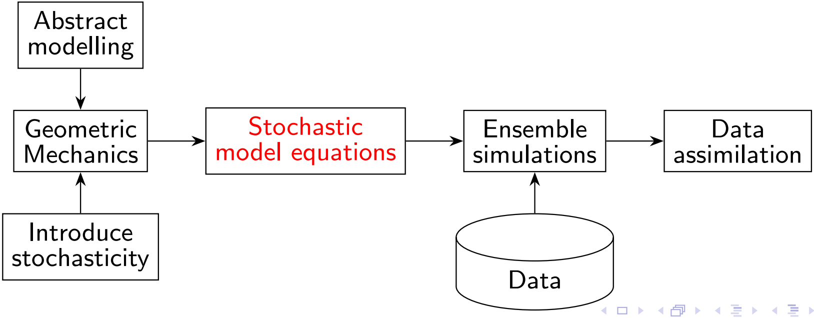
Imperial College London

August 26, 2024

Motivation: Multi-scale, multi-physics systems

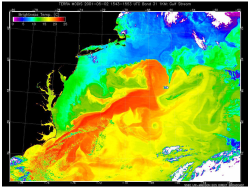
- Classical & quantum interactions in rapid molecular processes.
- Dynamics of complex fluids in, e.g., liquid crystals and superfluids.
- Turbulence in fluid dynamics.
- Atmosphere & ocean interactions in the climate systems.
- Control and response.

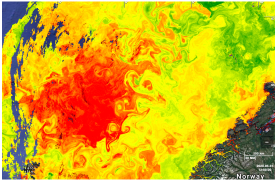
Versatile geometric stochastic modelling paradigm

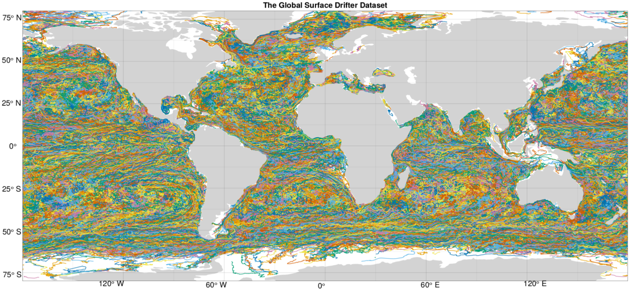


Motivation: Turbulent ocean dynamics

Observational data:







Ansatz: Stochastic dynamics of Lagrangian particle trajectory,

$$\mathrm{d}x_t := u_t(x_t)\,\mathrm{d}t + \sum_i \xi_i(x_t) \circ \mathrm{d}W_t^i \,.$$

What will we talk about

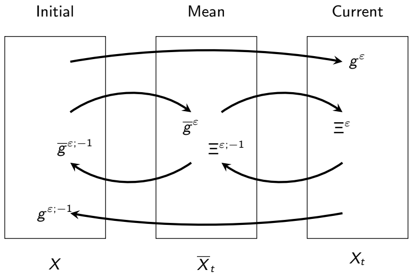
1. Homogenisation of Lagrangian trajectories
2. Stochastic variational closures
3. Example

Lagrangian decomposition of flow maps

Goal: Derive the Eulerian velocity decomposition

$$\mathrm{d}X_t = U_t(X_t)\,\mathrm{d}t + \sum_{k=1}^{K} \xi_k(X_t) \circ \mathrm{d}W_t^k \,,$$

from the homogenisation of a multi-scale dynamical system.



- Lagrangian trajectories factorise

$$g_t^\varepsilon = \Xi_t^\varepsilon \circ \overline{g}_t^\varepsilon \in \mathrm{Diff}(\mathbb{T}^d)$$

- Coordinates defined by

$$\overline{X}_t = \overline{g}_t^\varepsilon(X)\,, \quad X_t = \Xi_t^\varepsilon(\overline{X}_t)\,.$$

- Eulerian velocity field

$$\begin{aligned}\widetilde{U}_t^\varepsilon &=: \dot{g}_t^\varepsilon g_t^{\varepsilon;-1} \\ &:= \dot{\Xi}_t^\varepsilon \Xi_t^{\varepsilon;-1} + \Xi_{t*}^\varepsilon \dot{\overline{g}}_t^\varepsilon \overline{g}_t^{\varepsilon;-1}\,.\end{aligned}$$

Homogenisation of fast map Ξ^ε

Let Ξ^ε be the flow map of Y_t^ε, $\Xi_t^\varepsilon(X) = Y_t(X)$, with the evolution equations

$$\mathrm{d}Y_t^\varepsilon = \sum_{k=1}^{K} \xi_k(Y_t^\varepsilon)\mathrm{d}\mathbf{W}_t^{\varepsilon;k}, \quad Y_0^\varepsilon = X \in \mathbb{T}^d,$$

$$\text{where} \quad \mathbf{W}^\varepsilon = (W^\varepsilon, \mathbb{W}^\varepsilon), \quad W_t^\varepsilon = \varepsilon \int_0^{t\varepsilon^{-2}} \lambda_s^\varepsilon \mathrm{d}s, \quad \mathbb{W}_{st}^\varepsilon = \int_s^t (W_u^\varepsilon - W_s^\varepsilon) \otimes \mathrm{d}W_u^\varepsilon.$$

Assumption (Fast map assumptions [KM17; Che+19])

- *Weak invariance principle holds such that* $\mathbf{W}^\varepsilon \to_{\mathbb{P}} \mathbf{W}^\Gamma = (W, \mathbb{W}^\Gamma)$ *where* $\mathbb{P}$ *is an invariant measure of* λ *dynamics.*

Proposition (Homogenisation of fast map)

Assume that above assumption and smoothness criterion of ξ_k *holds. For all* $X \in \mathbb{T}^d$, $Y^\varepsilon \to_{\mathbb{P}} Y$ *as* $\varepsilon \downarrow 0$ *satisfying*

$$\mathrm{d}Y_t = \sum_{k=1}^{K} \xi_k(Y_t)\mathrm{d}\mathbf{W}_t^\Gamma = \sum_{k=1}^{K} \xi_k(Y_t) \circ \mathrm{d}W_t^k + \frac{1}{2}\sum_{k,l=1}^{K} \Gamma^{kl}\left[\xi_k(X_t), \xi_l(X_t)\right] \mathrm{d}t. \quad (1)$$

Thus, $\Xi^\varepsilon \to_{\mathbb{P}} \Xi$ *where* Ξ *is the flow map of* Y_t.

Homogenisation of composite map $\Xi^\varepsilon \circ \overline{g}^\varepsilon$

Assumption (Limit of slow map)

- *There exists* $\Psi_t(\cdot, \mathbf{W}^\varepsilon) = \overline{g}^\varepsilon_t$ *the flow map of* $\overline{g}^\varepsilon$.
- $\overline{g}^\varepsilon \to_{\mathbb{P}} \overline{g}$ *converges as* $\varepsilon \downarrow 0$ *and* $\exists\, \overline{u}_t : \Omega \to C^1([0,T], \mathfrak{X})$, *such that,*

$$\mathrm{d}\overline{g}_t(X) = \overline{u}_t(\overline{g}_t(X))\mathrm{d}t, \quad \overline{g}_0(X) = X\,. \tag{2}$$

Theorem (Homogenisation of composite map [DHL24])

Let the assumptions on the fast map Ξ^ε *and mean map* $\overline{g}^\varepsilon$ *hold. Then, the composite map* $g^\varepsilon = \Xi^\varepsilon \circ \bar{g}^\varepsilon$ *converges* $g^\varepsilon \to_{\mathbb{P}} g = \Xi \circ \bar{g}$ *as* $\varepsilon \downarrow 0$. *Moreover, for all* $t > 0$ *and* $X \in \mathbb{T}^d$,

$$\begin{aligned}
\mathrm{d}g_t(X) &= \Xi_{t*}\overline{u}_t(g_t(X))\,\mathrm{d}t + \sum_{k=1}^{K} \xi_k(g_t(X))\mathrm{d}\mathbf{W}^\Gamma_t\,, \\
&= \left(u_t(g_t(X)) + \sum_{k,l=1}^{K} \Gamma^{kl}\left[\xi_k(g_t(X))\,,\, \xi_l(g_t(X))\right] \right)\mathrm{d}t + \sum_{k=1}^{K} \xi_k(g_t(X)) \circ \mathrm{d}W^k_t\,.
\end{aligned} \tag{3}$$

What will we talk about

1. Homogenisation of Lagrangian trajectories
2. Stochastic variational closures
3. Example

Variational closures

Goal: from the homogenised flow maps

$$\mathrm{d}\Xi_t \Xi_t^{-1} = \sum_{k,l=1}^{K} \Gamma^{kl} [\xi_k \, , \, \xi_l] \, \mathrm{d}t + \sum_{k=1}^{K} \xi_k \circ \mathrm{d}W_t^k \, , \qquad \mathrm{d}\overline{g}_t \overline{g}_t^{-1} = \overline{u}_t \, \mathrm{d}t \, , \quad g = \Xi \circ \overline{g} \, ,$$

$$\mathrm{d}g_t g_t^{-1} = \left(u_t + \sum_{k,l=1}^{K} \Gamma^{kl} [\xi_k \, , \, \xi_l] \right) \mathrm{d}t + \sum_{k=1}^{K} \xi_k \circ \mathrm{d}W_t^k \, , \quad \text{where} \quad u_t := \Xi_{t*} \overline{u}_t \, ,$$

derive equations of motion for the velocities u and $\overline{u}$ using physical principles.

Additional physics

- Advected quantities $a_t = a_0 g_t^{-1} \in V^*$.
- g_t acts on a_0 via pullback, $a_0 g_t = g_t^* a_0$.
- The Lie-derivative is

$$\mathcal{L}_u a := \frac{d}{dt}\Big|_{t=0} \exp{(tu)}^* a \, .$$

- Define $\diamond : V \times V^* \to \mathfrak{X}^*(\mathcal{D})$ by

$$\langle -b \diamond a \, , \, u \rangle_{\mathfrak{X}(\mathcal{D}) \times \mathfrak{X}^*(\mathcal{D})} = \langle \mathcal{L}_u a \, , \, b \rangle_{V \times V^*} \, .$$

- Define adjoint action $\mathrm{ad}_u \, v = -[u, v]$.
- Define the coadjoint action by

$$\langle \mathrm{ad}_u \, v \, , \, m \rangle = \langle v \, , \, \mathrm{ad}_u^* \, m \rangle = \langle v \, , \, \mathcal{L}_u m \rangle \, ,$$

Euler-Poincaré variational principle

$$0 = \delta S(g_t) = \delta \int_{t_0}^{t_1} \ell(U_t, a_t) \, .$$

Defining variations

A variation of the homogenised composite map $g_t = \Xi_t \circ \overline{g}_t$ is constructed by

$$\overline{g}_{\epsilon,t} = e_{\epsilon,t} \circ \overline{g}_t \,, \quad g_{\epsilon,t} = \Xi_t \circ \overline{g}_{\epsilon,t} \,,$$

where $e_\epsilon : \Omega \to C^1([0,T]; \mathrm{Diff}(\mathcal{D}))$ be the solution of the random ODE

$$\frac{d}{dt} e_{\epsilon,t} = \epsilon \dot{v}_t e_{\epsilon,t} \,, \quad e_{\epsilon,0} = Id \,, \quad v : \Omega \to C^1([0,T]; \mathfrak{X}(\mathcal{D})) \,, \quad v_0 = v_T = 0 \,.$$

The perturbed diffeomorphism $g_{\epsilon,t}$ gives

$$\overline{u}_{\epsilon,t} := \epsilon \frac{d}{dt} v_t + \mathrm{Ad}_{e_{\epsilon,t}} \overline{u}_t \,, \quad \overline{a}_{\epsilon,t} := a_0 \overline{g}_{\epsilon,t}^{-1} \,, \quad u_{\epsilon,t} := \mathrm{Ad}_{\Xi_t} \overline{u}_{\epsilon,t} \,, \quad a_{\epsilon,t} := a_0 g_{\epsilon,t}^{-1} \,.$$

Proposition (Variation induced by $e_{\epsilon,t}$)

$e_{\epsilon,t}$ introduces the following variations of u_t and a_t

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \overline{u}_{\epsilon,t} = \frac{d}{dt} v_t + \mathrm{ad}_{v_t} \overline{u}_t \,, \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} u_{\epsilon,t} = \mathrm{Ad}_{\Xi_t} \left(\frac{d}{dt} v_t + \mathrm{ad}_{v_t} \mathrm{Ad}_{\Xi_t^{-1}} u_t \right) \,,$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \overline{a}_{\epsilon,t} = -\mathcal{L}_{v_t} \overline{a}_t \,. \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_{\epsilon,t} = -\left(\mathcal{L}_{v_t} a_t \Xi_t\right) \Xi_t^{-1} \,.$$

Modified SALT closure I.

Modelling choices (SALT modelling choice)

- *Given a Lagrangian functional* $\ell : \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$, *we assume it is a function of the drift velocity of the homogenised Lagrangian particle dynamics* $\mathrm{d}g_t g_t^{-1}$.

$$\ell = \ell(u_t, a_t)\,.$$

Proposition (SALT Euler-Poincaré equations)

The variational principle

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S(g_{\epsilon,t}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^T \ell(u_{\epsilon,t}, a_{\epsilon,t})\,\mathrm{d}t\,,$$

yields the stochastic Euler-Poincaré equations

$$\begin{cases} \mathrm{d}\frac{\delta \ell}{\delta u_t} + \mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u_t}\,\mathrm{d}t + \sum_{k=1}^K \mathrm{ad}^*_{\xi_k}\frac{\delta \ell}{\delta u_t}\circ \mathrm{d}W_t^k + \frac{1}{2}\sum_{k,l=1}^K \mathrm{ad}^*_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}\frac{\delta \ell}{\delta u_t}\,\mathrm{d}t = \frac{\delta \ell}{\delta a_t}\diamond a_t\,\mathrm{d}t\,, \\ \mathrm{d}a_t + \mathcal{L}_{u_t}a_t\,\mathrm{d}t + \sum_{k=1}^K \mathcal{L}_{\xi_k}a_t\circ \mathrm{d}W_t^k + \frac{1}{2}\sum_{k,l=1}^K \mathcal{L}_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}a_t\,\mathrm{d}t = 0\,. \end{cases}$$

Previously derived in e.g., [Hol15].

Modified SALT closure II.

Noting that $u_t = \mathrm{Ad}_{\Xi_t} \overline{u}_t$ and $a_t = \overline{a}_t \Xi_t$, we can define a random, time dependent Lagrangian $\ell^{\Xi} : \Omega \times [0, T] \times \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}$ by

$$\ell^{\Xi}(\overline{u}, \overline{a}) := \ell(\mathrm{Ad}_{\Xi} \overline{u}, \overline{a}\Xi) \quad \forall \overline{u} \in \mathfrak{X}(\mathcal{D}), \quad \forall \overline{a} \in V^* .$$

Proposition

The variational principle

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S^{\Xi}(\overline{g}_{\epsilon,t}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^T \ell^{\Xi}(\overline{u}_{\epsilon,t}, \overline{a}_{\epsilon,t})\, \mathrm{d}t \,,$$

yields the random coefficient Euler-Poincaré equations

$$\begin{cases} \mathrm{d}\frac{\delta \ell^{\Xi}}{\delta \overline{u}_t} + \mathcal{L}_{\overline{u}} \frac{\delta \ell^{\Xi}}{\delta \overline{u}_t}\, \mathrm{d}t = \frac{\delta \ell^{\Xi}}{\delta \overline{a}_t} \diamond \overline{a}_t \, \mathrm{d}t \,, \\ \mathrm{d}\overline{a}_t + \mathcal{L}_{\overline{u}_t} \overline{a}_t \, \mathrm{d}t = 0 \,, \end{cases}$$

Equivalence of Euler-Poincaré equations

Proposition (Equivalence of Euler-Poincaré equations)

Let $a, \overline{a}, u, \overline{u}, \ell, \ell^{\Xi}$ and Ξ be defined as before. Then, the random coefficient Euler-Poincaré equations

$$\begin{cases} \mathrm{d}\frac{\delta \ell^{\Xi}}{\delta \overline{u}_t} + \mathrm{ad}^*_{\overline{u}_t} \frac{\delta \ell^{\Xi}}{\delta \overline{u}_t}\,\mathrm{d}t = \frac{\delta \ell^{\Xi}}{\delta \overline{a}_t} \diamond \overline{a}_t\,\mathrm{d}t\,, \\ \mathrm{d}\overline{a}_t + \mathcal{L}_{\overline{u}_t}\overline{a}_t\,\mathrm{d}t = 0\,, \end{cases}$$

and the stochastic Euler-Poincaré equations are equivalent

$$\begin{cases} \mathrm{d}\frac{\delta \ell}{\delta u_t} + \mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u_t}\,\mathrm{d}t + \sum_{k=1}^{K} \mathrm{ad}^*_{\xi_k}\frac{\delta \ell}{\delta u_t} \circ \mathrm{d}W^k_t + \frac{1}{2}\sum_{k,l=1}^{K} \mathrm{ad}^*_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}\frac{\delta \ell}{\delta u_t}\,\mathrm{d}t = \frac{\delta \ell}{\delta a_t} \diamond a_t\,\mathrm{d}t\,, \\ \mathrm{d}a_t + \mathcal{L}_{u_t}a_t\,\mathrm{d}t + \sum_{k=1}^{K}\mathcal{L}_{\xi_k}a_t \circ \mathrm{d}W^k_t + \frac{1}{2}\sum_{k,l=1}^{K}\mathcal{L}_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}a_t\,\mathrm{d}t = 0\,. \end{cases}$$

Deterministic closure

Assumption (Alternative limit of slow map)

- $\overline{g}^{\varepsilon} \to_{\mathbb{P}} \overline{g}$ *converges as* $\varepsilon \downarrow 0$ *and* $\exists$ *a non-random* $\overline{u}_t \in \mathfrak{X}$, *such that,*

$$\partial_t \overline{g}_t(X) = \overline{u}_t(\overline{g}_t(X)), \quad \overline{g}_0(X) = X\,.$$

From ℓ, we may define,

$$L : [0, T] \times \mathfrak{X}(\mathcal{D}) \times V^* \to \mathbb{R}, \quad L(t, \overline{u}, \overline{a}) := \mathbb{E}\left[\ell(\mathrm{Ad}_{\Xi}\, \overline{u}, \overline{a}\Xi)\right]\,.$$

Proposition

The variational principle

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S^{\Xi}(\overline{g}_{\epsilon,t}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^T L(t, \overline{u}_{\epsilon,t}, \overline{a}_{\epsilon,t})\,\mathrm{d}t\,,$$

yields the deterministic Euler-Poincaré equations

$$\begin{cases} \partial_t \frac{\delta L}{\delta \overline{u}_t} + \mathcal{L}_{\overline{u}_t} \frac{\delta L}{\delta \overline{u}_t} = \frac{\delta L}{\delta \overline{a}_t} \diamond \overline{a}_t \\ \partial_t \overline{a}_t + \mathcal{L}_{\overline{u}_t} \overline{a}_t = 0\,, \end{cases}$$

What will we talk about

Example: Euler's equation I.

For Euler's equation, the Lagrangian is given by

$$\ell(\Xi_{t*}\overline{u}, \Xi_{t*}\overline{D}) = \int_{\mathcal{D}} \frac{1}{2}\mathbf{g}\left(\Xi_{t*}\overline{u}, \Xi_{t*}\overline{u}\right)\Xi_{t*}\overline{D} = \frac{1}{2}\int_{\mathcal{D}} (\Xi_t^*\mathbf{g})(\overline{u}, \overline{u})\Xi_{t*}\overline{D}\,,$$

where we constrain the mean density $\overline{D} = d\overline{V} \in \mathrm{Den}(\mathcal{D})$. The corresponding random coefficient equations are

$$\begin{cases} \left(\mathrm{d} + \mathcal{L}_{\overline{u}\,\mathrm{d}t}\right)\Xi_t^*\left(\Xi_{t*}\overline{u}\right)^\flat = \mathbf{d}\left(\frac{1}{2}\Xi_t^*\left|\Xi_{t*}\overline{u}\right|^2\mathrm{d}t - \mathrm{d}p\right)\,, \\ \left(\mathrm{d} + \mathcal{L}_{\overline{u}\,\mathrm{d}t}\right)\overline{D} = 0\,. \end{cases}$$

Evaluating $\overline{D} = d\overline{V}$, we have the incompressibility condition $\mathcal{L}_{\overline{u}}(d\overline{V}) = 0$.

The equivalent stochastic equations in terms of $u = \Xi_{t*}\overline{u}$ and $D = \Xi_{t*}\overline{D}$ are

$$\begin{cases} \mathrm{d}u^\flat + \mathcal{L}_u u^\flat\,\mathrm{d}t + \sum_{k=1}^K \mathcal{L}_{\xi_k}u^\flat \circ \mathrm{d}W_t^k + \frac{1}{2}\sum_{k,l=1}^K \mathcal{L}_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}u^\flat\,\mathrm{d}t = \mathbf{d}\left(\frac{1}{2}\left|u\right|^2\mathrm{d}t - \Xi_{t*}\mathrm{d}p\right)\,dt\,, \\ \mathrm{d}D + \mathcal{L}_u D\,\mathrm{d}t + \sum_{k=1}^K \mathcal{L}_{\xi_k}D \circ \mathrm{d}W_t^k + \frac{1}{2}\sum_{k,l=1}^K \mathcal{L}_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}D\,\mathrm{d}t = 0\,. \end{cases}$$

Evaluating $\overline{D} = d\overline{V}$, we have the incompressibility condition of u and ξ_i when $\Xi_t \in \mathrm{SDiff}(\mathcal{D})$,

$$\mathcal{L}_u dV = \mathcal{L}_{\xi_i}dV = 0\,.$$

Example: Euler's equation II.

Kelvin circulation dynamics

For a given initial material loop c_0, one has conservation of the circulation integral of $\Xi_t^*(\Xi_{t*}\overline{u}_t)^\flat$ and $u^\flat$,

$$
\begin{aligned}
\mathrm{d}\oint_{\overline{g}_t c_0} \Xi_t^*(\Xi_{t*}\overline{u}_t)^\flat &= \oint_{\overline{g}_t c_0} \mathbf{d}\left(\frac{1}{2}\Xi_t^*\mathbf{g}(\Xi_{t*}\overline{u}, \Xi_{t*}\overline{u})\,\mathrm{d}t - \mathrm{d}p\right) = 0\,,\\
\mathrm{d}\oint_{g_t c_0} u_t^\flat &= \oint_{g_t c_0} \mathbf{d}\left(\frac{1}{2}\mathbf{g}(u_t, u_t)\,\mathrm{d}t - \Xi_{t*}\mathrm{d}p\right)\, dt = 0\;.
\end{aligned}
$$

Vorticity dynamics

Let $\omega_t = \mathbf{d}u_t^\flat = \mathbf{d}(\Xi_{t*}\overline{u}_t)^\flat \in \Lambda^2(\mathcal{D})$ be the vorticity of the drift velocity one-form and let the vorticity associated with mean velocity one-form be $\overline{\omega}_t = \mathbf{d}\Xi_t^* u_t^\flat = \Xi_t^*\omega_t$. The vorticity dynamics are

$$
\begin{aligned}
&(\mathrm{d} + \mathcal{L}_{\overline{u}_t\,\mathrm{d}t})\,\overline{\omega}_t = 0\,,\\
&\mathrm{d}\omega_t + \mathcal{L}_{u_t}\omega_t\,\mathrm{d}t + \sum_{k=1}^{K}\mathcal{L}_{\xi_k}\omega_t \circ \mathrm{d}W_t^i + \frac{1}{2}\sum_{k,l=1}^{K}\mathcal{L}_{\Gamma^{kl}[\xi_k\,,\,\xi_l]}\omega_t\,\mathrm{d}t = 0\,.
\end{aligned}
$$

Concluding remarks

What have we seen?

- Homogenisation of fast + slow decomposition of Lagrangian trajectory.
- Two equivalent forms of the SALT Euler-Poincaré equations.
- The example of Euler's fluid equation is given.

What's next?

- Generalise to arbitrary Lie groups.
- Consider the same treatment for Rough advection by Lie Transport (RALT) where Ξ_t is the solution to the RDE,

$$\mathrm{d}\Xi_t \Xi_t^{-1} = \sum_{k=1}^{K} \xi_k \mathrm{d}\mathbf{Z}_t^k \,.$$

- How can SALT/RALT be extended to geometric control theory?

References

[Che+19] Ilya Chevyrev et al. "Multiscale systems, homogenization, and rough paths". In: *Probability and Analysis in Interacting Physical Systems: In Honor of SRS Varadhan, Berlin, August, 2016*. Springer. 2019, pp. 17–48.

[DHL24] Theo Diamantakis, Ruiao Hu, and James-Michael Leahy. "Variational closures for composite homogenised fluid flows". 2024.

[Hol15] Darryl D. Holm. "Variational principles for stochastic fluid dynamics". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 471 (2176 2015). ISSN: 14712946. DOI: 10.1098/rspa.2014.0963.

[KM17] David Kelly and Ian Melbourne. "Deterministic homogenization for fast–slow systems with chaotic noise". In: *Journal of Functional Analysis* 272.10 (2017), pp. 4063–4102.