Mathematically Justified Computational Platform for Nonlinear Dynamics

Identification of Bifurcations via Cellular Sheaf Cohomology

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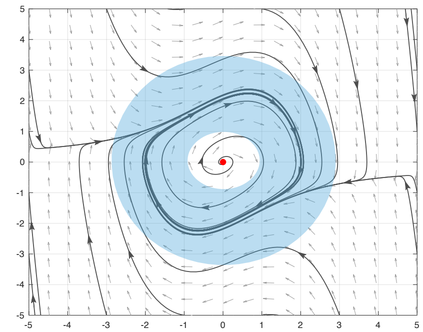
Plan:

- Global dynamics has an algebraic structure.
  - Attractors form an order-theoretic lattice.
  - This structure forms a sheaf over a parametrized family of systems.
  - Sheaf cohomology can detect bifurcation.
- Efficient, applicable computational frameworks can be built from combinatorial and topological tools.

$$\phi \colon X \times \mathbb{T}^+ \to X \text{ for } \mathbb{T} = \mathbb{R} \text{ or } \mathbb{Z}$$

$$\phi(x,0) = x \text{ and } \phi(x, s+t) = \phi(\phi(x,s), t) \text{ for all } x \in X \text{ and } t, s \geq 0.$$

Attractors

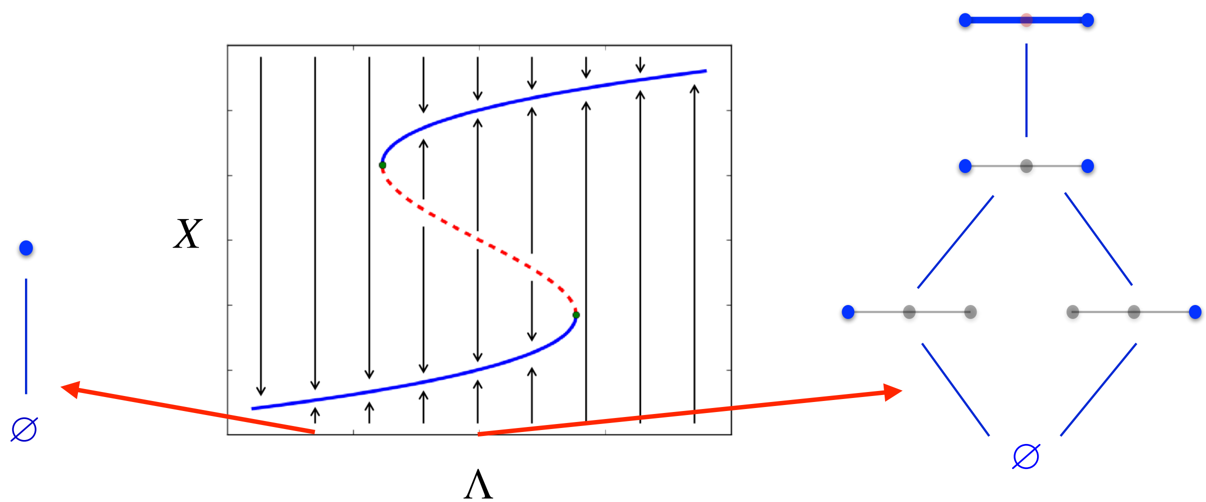


$U \subset X$ is an attracting neighborhood if $\phi(\mathrm{cl}\, U, t) \subset \mathrm{int}(U)$ for all $t \geq t_0 > 0$.

$A \subset X$ is an attractor if $A = \omega(U)$ for some attracting neighborhood.

The set of all attractors $\mathrm{Att}(\phi)$ is a bounded, distributive lattice:
$A \vee A' = A \cup A'$ and $A \wedge A' = \omega(A \cap A')$. (Kalies, Mischaikow, VanderVorst 2013)

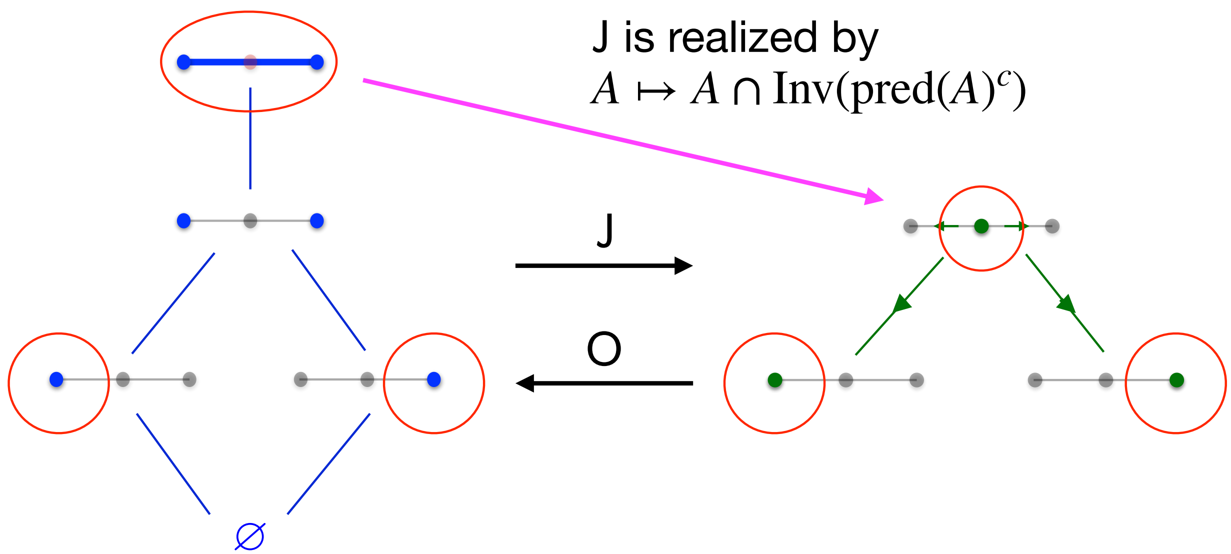
$\omega\colon (\mathrm{ANbhd}(\phi), \cup, \cap) \to (\mathrm{Att}(\phi), \vee, \wedge)$ is a surjective homomorphism. $\omega(U) = \bigcap_{t>0} \mathrm{cl}\,(\phi(U, [t, \infty)))$



Morse decompositions

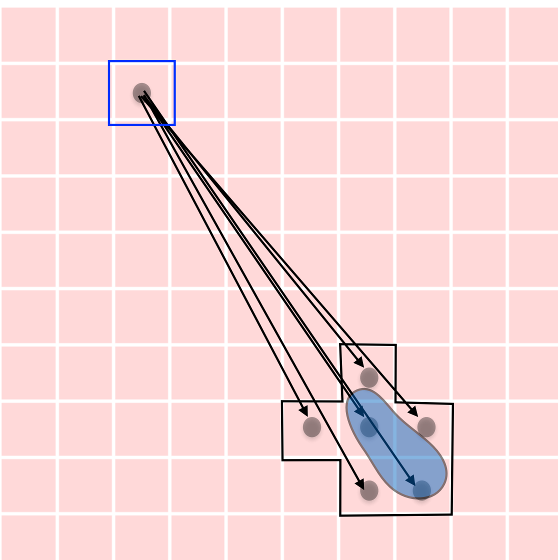
Via Birkhoff's theorem, a finite sublattice of attractors is dual to a poset of invariant sets called a Morse decomposition. (Kalies, Mischaikow, VanderVorst 2021)

All recurrent dynamics is contained in the Morse sets, ie. the system is gradient-like outside of the Morse sets. The connecting orbits respect the order on the Morse sets.



Combinatorial dynamics

- Multivalued map $\sim$ directed graph $\mathscr{F}\colon \mathscr{X}^{\mathrm{top}} \rightrightarrows \mathscr{X}^{\mathrm{top}}$ on a finite cell complex $\mathscr{X}$.
- $\mathscr{F}$ is an outer approximation of $f\colon X \to X$ on the realization $X = |\mathscr{X}|$ if



$$f(|\xi|) \subset \mathrm{int}\,|\mathscr{F}(\xi)| \text{ for all } \xi \in \mathscr{X}.$$

Recurrent components: poset RC of maximal subgraphs for which every vertex is reachable from every other vertex and contains at least one edge

For an outer approximation: the (chain-) recurrent set of f is contained in the realization $|\cup_{\mathrm{RC}} R| \iff f$ is gradient-like on $|\cup_{\mathrm{RC}} R|^{c}$.

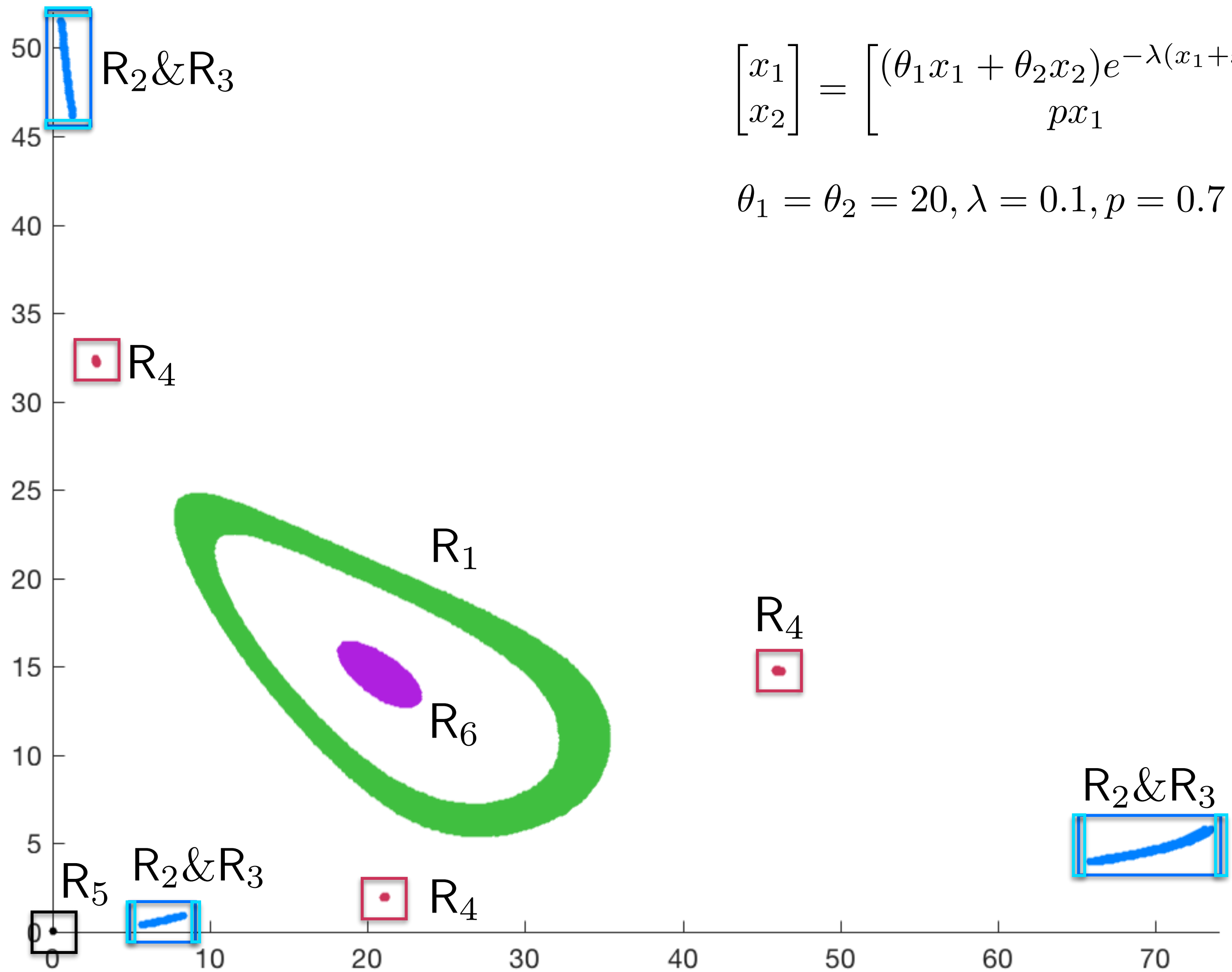
Combinatorial order theory (Kalies, Mischaikow and VanderVorst 2021)

- Morse graph: $\mathsf{MG}(\mathscr{F})$ is the Hasse diagram of RC.
- Attractor lattice: $\mathsf{Att}(\mathscr{F})$ is the finite, distributive lattice of downsets in $\mathscr{X}$ of RC. $\mathscr{A} \in \mathsf{Att}(\mathscr{F})$ iff $\mathscr{F}(\mathscr{A}) = \mathscr{A}$.

- **Translation to classical dynamics:** for an outer approximation $\mathscr{F}$ of f,
  - The maximal invariant sets inside the realizations of elements in $\mathsf{MG}(\mathscr{F})$ form a Morse decomposition for f.
  - The maximal invariant sets inside the realizations of elements in $\mathsf{Att}(\mathscr{F})$ form a finite sublattice of attractors for f.

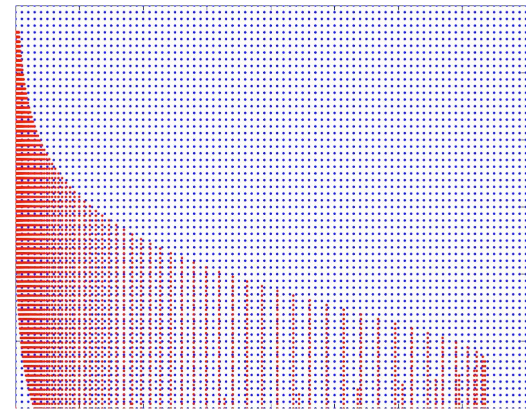
- **NOTE:** a multivalued map $\mathscr{F}$ is an outer approximation for an infinite family of maps.

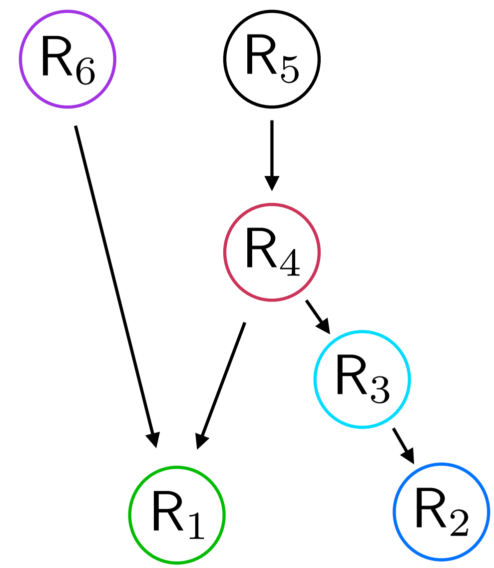
Example



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-\lambda(x_1+x_2)} \\ p x_1 \end{bmatrix}$$

$$\theta_1 = \theta_2 = 20, \lambda = 0.1, p = 0.7$$





Homological dynamics and the Conley index

- Extend $\mathscr{F} : \mathscr{X}^{\text{top}} \rightrightarrows \mathscr{X}^{\text{top}}$ to $\mathscr{F} : \mathscr{X} \rightrightarrows \mathscr{X}$ inductively and define

- The Conley index of the pair of attractors $\mathscr{A}_0 \subset \mathscr{A}_1$ is the shift equivalence class of $\mathscr{F}_* \colon H_*(\mathscr{A}_1, \mathscr{A}_0) \to H_*(\mathscr{A}_1, \mathscr{A}_0)$, which is well-defined if $\bar{H}_*(\xi) = 0$ for all $\xi \in \mathscr{X}$.

- This agrees with the classical definition when $\mathscr{F}$ is an outer approximation.

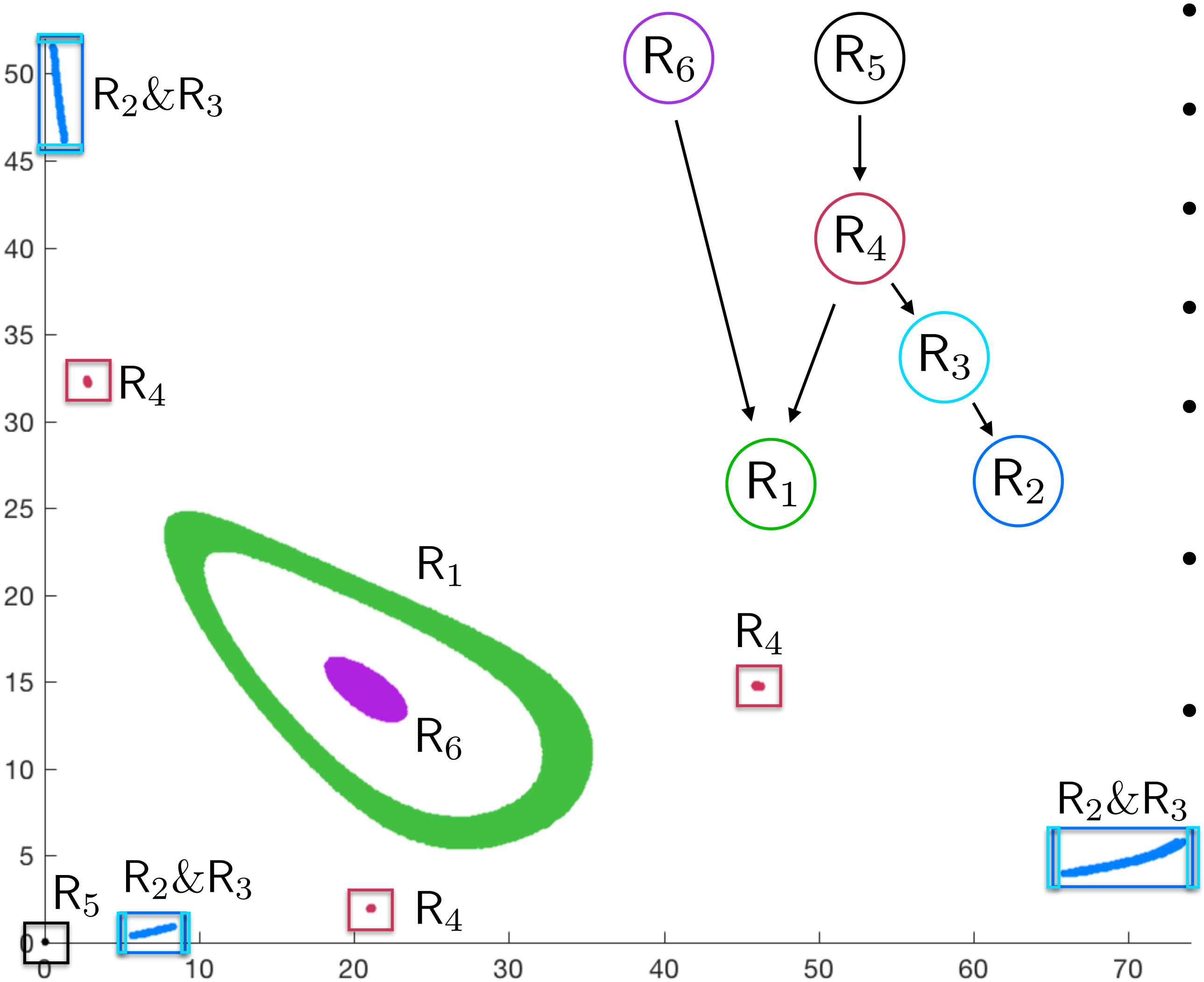
Homological Conley Index
+
something

$\Longrightarrow$

Classical Dynamical
Systems
Information

- Nontrivial Conley index $\Rightarrow$ existence of a nonempty, isolated invariant set.

- Algebraic topology in the form of homology is sufficiently powerful to allow us to obtain many qualitative results of interest for applications from purely *combinatorial* information.



- R_6 contains a fixed point
- R_5 has trivial index*
- R_4 contains a period-3 orbit
- R_3 has trivial index
- R_2 contains an attractor with chaotic dynamics
- R_1 contains a nontrivial attractor
- Multistability

Morse Decomposition

Conley-Morse Graph

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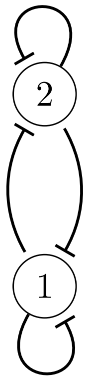
Dynamic Signatures Generated by Regulatory Networks

Mischaikow, Gameiro, et.al.

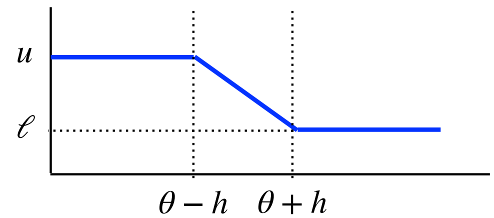


$$\dot{x}_1 = -\gamma_1 x_1 + r_{1,1}(x_1) r_{1,2}(x_2)$$
$$\dot{x}_2 = -\gamma_2 x_2 + r_{2,1}(x_1) r_{2,2}(x_2)$$

Nonlinearities?

Step

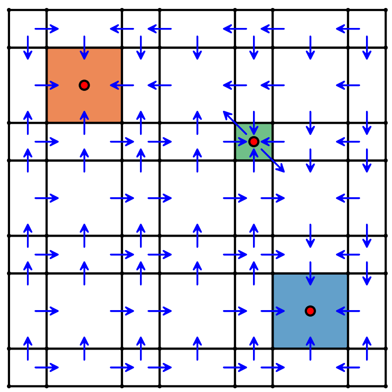
Sigmoid: Hill, tanh, arctan, …

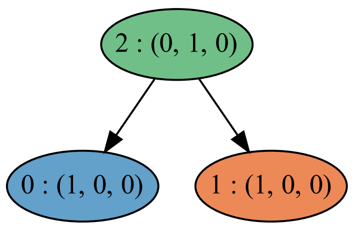


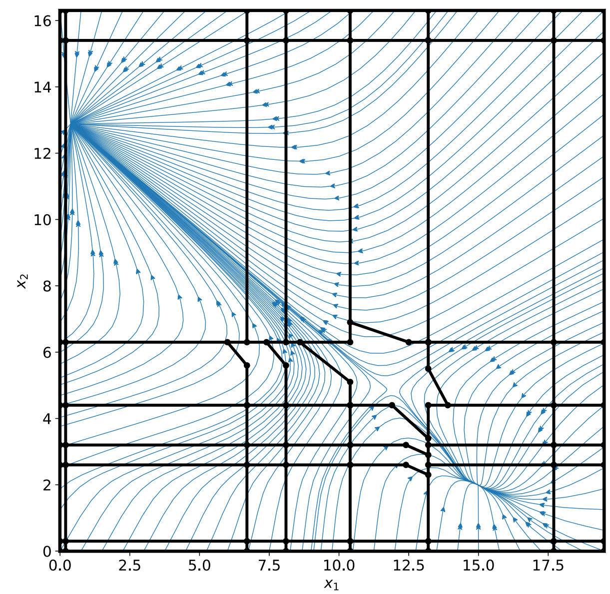
Ramp

$$r_{i,j}(x) = \begin{cases} \nu_{i,j,1}, & \text{if } x < \theta_{i,j} - h_{i,j} \\ L_{i,j}(x), & \text{if } \theta_{i,j} - h_{i,j} \le x \le \theta_{i,j} + h_{i,j} \\ \nu_{i,j,2}, & \text{if } x > \theta_{i,j} + h_{i,j} \end{cases}$$

$$L_{i,j}(x) = \frac{\nu_{i,j,2} - \nu_{i,j,1}}{2h_{i,j}}(x - \theta_{i,j}) + \frac{\nu_{i,j,1} + \nu_{i,j,2}}{2}$$







$\nu_{1,1,1} = 3.7, \ \nu_{1,1,2} = 1.4$

$\nu_{1,2,1} = 10.7, \ \nu_{1,2,2} = 0.1$

$\nu_{2,1,1} = 9.2, \ \nu_{2,1,2} = 0.2$

$\nu_{2,2,1} = 6.2, \ \nu_{2,2,2} = 1.4$

$\theta_{1,1} = 6.4, \ \theta_{1,2} = 5.6$

$\theta_{2,1} = 11.1, \ \theta_{2,2} = 1.8$

$h_{1,1} = 0.3, \ h_{1,2} = 0.35$

$h_{2,1} = 0.6, \ h_{2,2} = 0.3$

$\gamma_1 = 1, \ \gamma_2 = 1$

DSGRN creates a database of 1600 semi-algebraic sets that partition parameter space organized in a graph based on codimension-1 boundaries.

Parameter space: $(0,\infty)^{18}$

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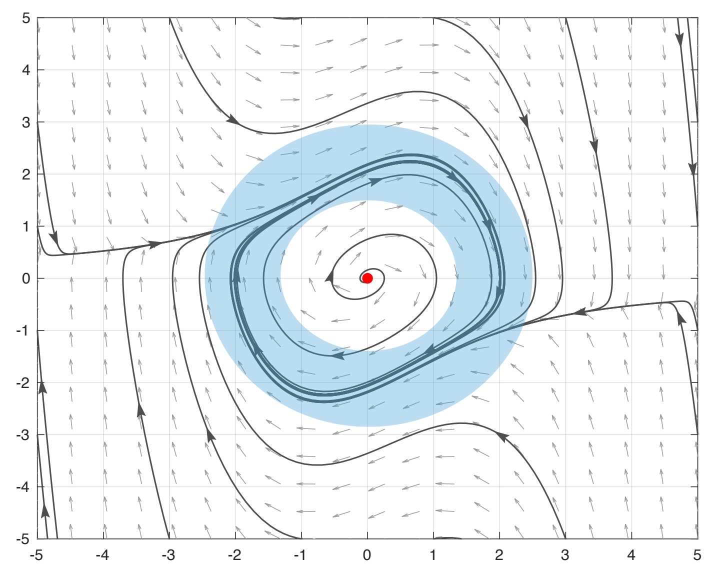
Sheaves

- Sheaves attach data to the topology of a space in a consistent way.
- A sheaf $\mathscr{S}$ over a topological space Λ assigns data $\mathscr{S}(U)$ to each open set $U \subset \Lambda$, and assigns to each inclusion of open sets $U \subset V$ a restriction map $\rho_{U\subset V}\colon \mathscr{S}(V) \to \mathscr{S}(U)$ such that $\rho_{U\subset U} = \mathrm{id}_U$ and $\rho_{U\subset V} \circ \rho_{V\subset W} = \rho_{U\subset W}$ for $U \subset V \subset W$.

- Why do attractors have a sheaf structure?
- Given (ϕ, A) there is an attracting neighborhood N for A, and attracting neighborhoods are robust, ie. N is an attracting neighborhood for all ψ near ϕ.
- **NOTE:** Attractors are not robust!

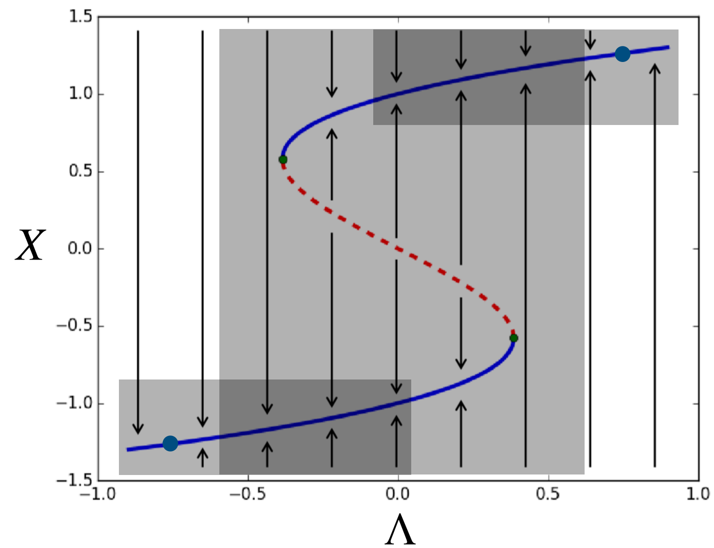


Attractor sheaves (Dowling, Kalies, VanderVorst 2023)

ANbhd, Att: $\mathbf{DS} \to \mathbf{BDLat}$ are functors.

ω: ANbhd $\to$ Att is a natural transformation.

Attractors form a $\mathbf{BDLat}$-valued sheaf.



$\Pi[\text{Att}] = \{(\phi, A) \mid A \in \text{Att}(\phi)\}$ with the appropriate topology is an etale space (⇨ sheaf).

$(\phi, A) \mapsto \phi$ is a local homeomorphism $\pi\colon \Pi[\text{Att}] \to \mathbf{DS}$.

For parametrized family ϕ_* over Λ, $\phi_*^{-1}\Pi[\text{Att}] = \{(\lambda, \phi_\lambda, A) \mid A \in \text{Att}(\phi_\lambda)\}$.

Conjugacy Invariance Theorem: Let X, Y be compact metric spaces. Let $\phi_*, \psi_*\colon \Lambda \to \mathbf{DS}$ be families of dynamical systems on X, Y respectively, parametrized over Λ. If ϕ_*, ψ_* are conjugate, then $\phi_*^{-1}\Pi[\text{Att}]$ and $\psi_*^{-1}\Pi[\text{Att}]$ are homeomorphic.

Attractor sheaf cohomology

Sheaf cohomology is a tool that characterizes obstructions for local sections to lift to global sections $\Rightarrow$ bifurcation!

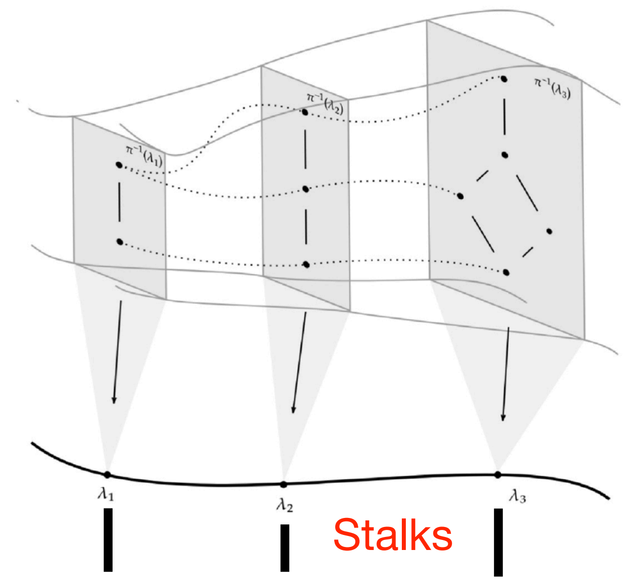
Three technical issues:

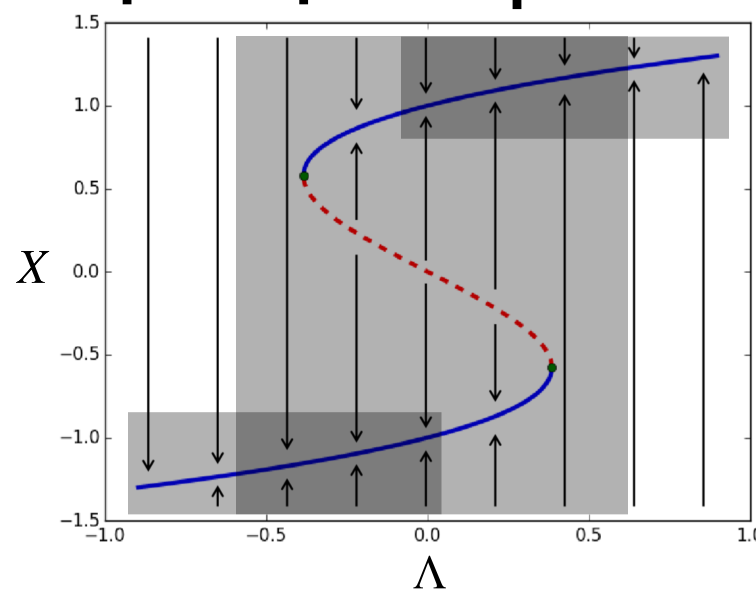
[1] Etale space to sheaf: $\Pi[\text{Att}] \Rightarrow \mathcal{S}^{\text{Att}}$.

A section is a continuous map $\sigma\colon U \to \Pi[\text{Att}]$ on an open set U such that $\pi \circ \sigma = \text{id}_U$.

$\mathcal{S}^{\text{Att}}(U) = \{\text{sections } \sigma\colon U \to \Pi[\text{Att}]\}$

Global sections: $\sigma\colon \Lambda \to \Pi[\text{Att}]$.





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[2] Sheaf cohomology requires a sheaf with values in an abelian category. **BDLat** is not abelian.

BDLat $\to$ **BoolAlg** $\to$ **BoolRing**

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[3] What is sheaf cohomology exactly?

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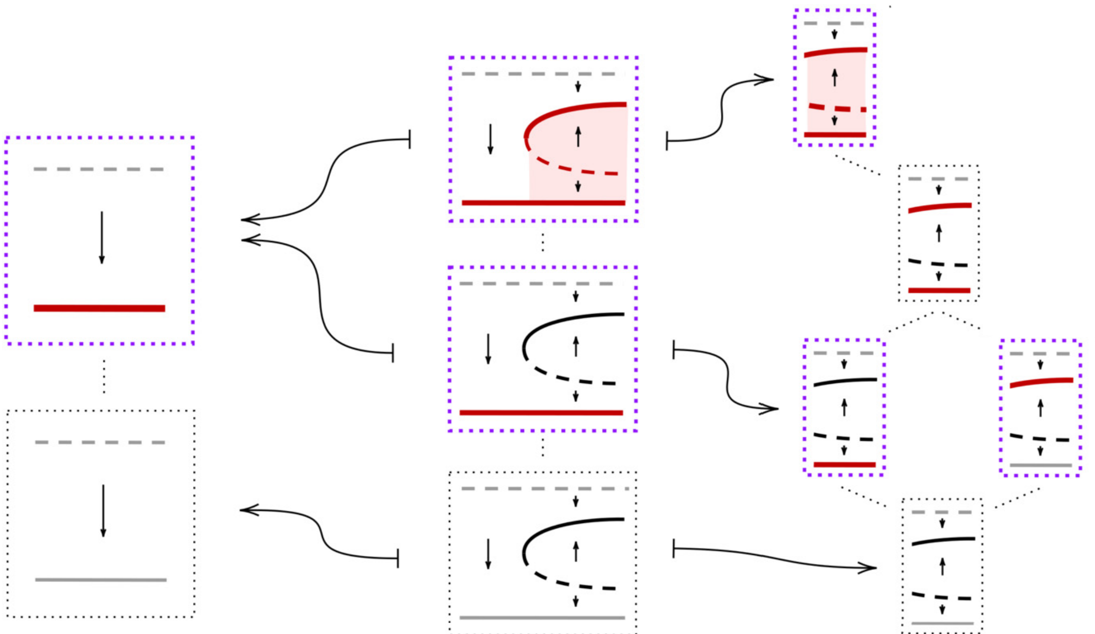
Cellular sheaves

- Given a cell complex $\mathscr{X}$, let $\mathsf{FP}(\mathscr{X})$ denote the face poset of $\mathscr{X}$. Then a cellular sheaf with values in a category C is a functor $\mathsf{F}\colon \mathsf{FP}(\mathscr{X}) \to \mathsf{C}$.
- For each cell $\sigma \in \mathscr{X}$, F assigns an object $\mathsf{F}_\sigma \in \mathsf{C}$, the stalk over σ.
- For pairs $\sigma \preceq \tau$, ie. σ is a face of τ, F gives restriction maps $\mathsf{F}_{\sigma \preceq \tau}\colon \mathsf{F}_\sigma \to \mathsf{F}_\tau$ so that for $\rho \preceq \sigma \preceq \tau$, $\mathsf{F}_{\sigma \preceq \sigma} = \mathrm{id}_{\mathsf{F}_\sigma}$ and $\mathsf{F}_{\sigma \preceq \tau} \circ \mathsf{F}_{\rho \preceq \sigma} = \mathsf{F}_{\rho \preceq \tau}$.
- Sheaf over topological space + (locally) finite collection of open sets $\Rightarrow$ cellular sheaf with the nerve as the cell complex.
- Cellular sheaf cohomology is computed using Cech cohomology with $\mathbb{Z}_2$ coefficients.
- (Ghrist and Riess 2022)
- Convergence theorems for attractor sheaves (Dowling 2023)

“Saddle-node”: $f(x) = x + 0.2(-x^3 + (4 + 0.5\lambda)(x^2 - x))$ **SN:** $\lambda = 0,\ x = 2$



$H^0(\mathsf{F}) = \mathbb{Z}_2$

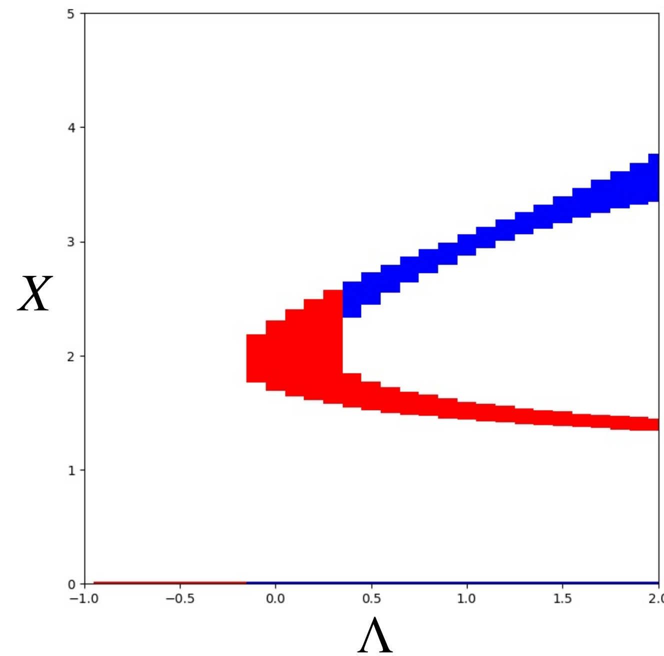
Local sections before bifurcation

$H^0(\mathsf{F}) = \mathbb{Z}_2^2$

Global sections

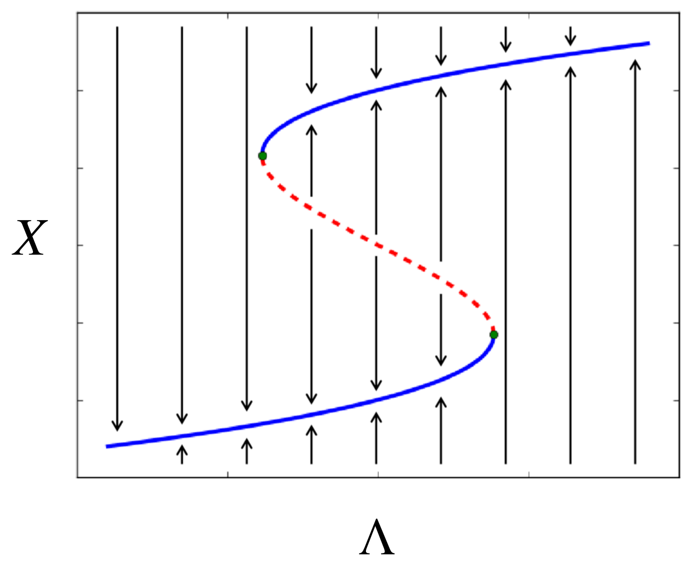
$H^0(\mathsf{F}) = \mathbb{Z}_2^3$

Local sections after bifurcation



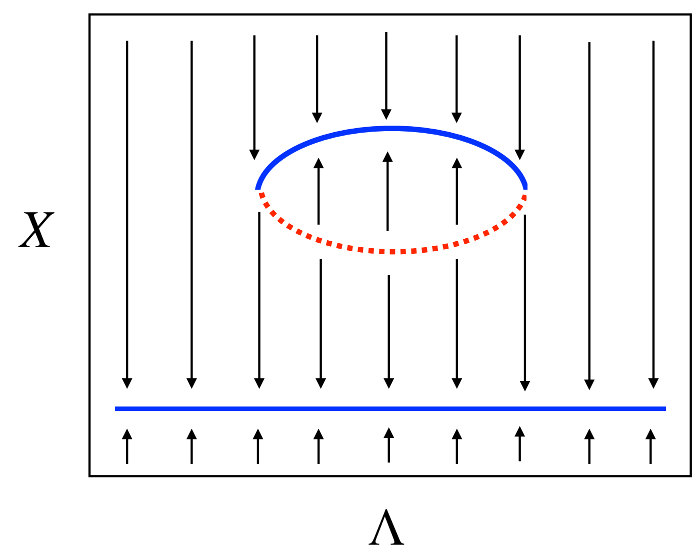
Cellular sheaf of outer approximations over the nerve of a uniform subdivision of parameter interval $[-1,2]$.

Monostable - Bistable - Monostable: hysteresis or isola?



$$H^0(\mathscr{S}^{\mathsf{Att}}) = \mathbb{Z}_2^1$$

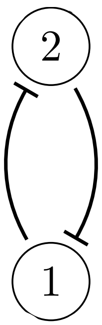
Global sections



$$H^0(\mathscr{S}^{\mathsf{Att}}) = \mathbb{Z}_2^2$$

Global sections

Cusp bifurcation in DSGRN

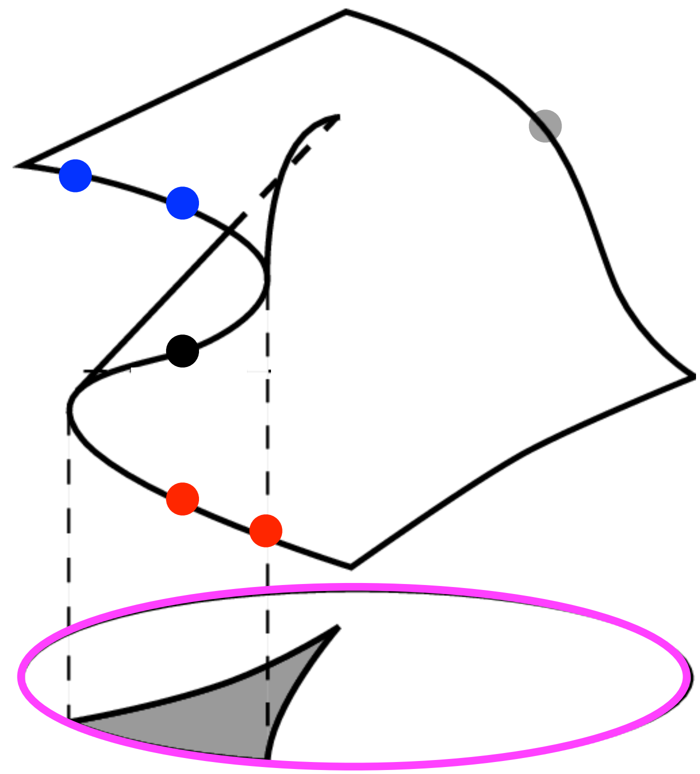


$$\dot{x}_1 = -\gamma_1 x_1 + \begin{cases} \ell_{12} + \delta_{12} & x_2 < \theta_{12} \\ \ell_{12} & x_2 > \theta_{12} \end{cases}$$

$$\dot{x}_2 = -\gamma_2 x_2 + \begin{cases} \ell_{21} + \delta_{21} & x_1 < \theta_{21} \\ \ell_{21} & x_1 > \theta_{21} \end{cases}$$

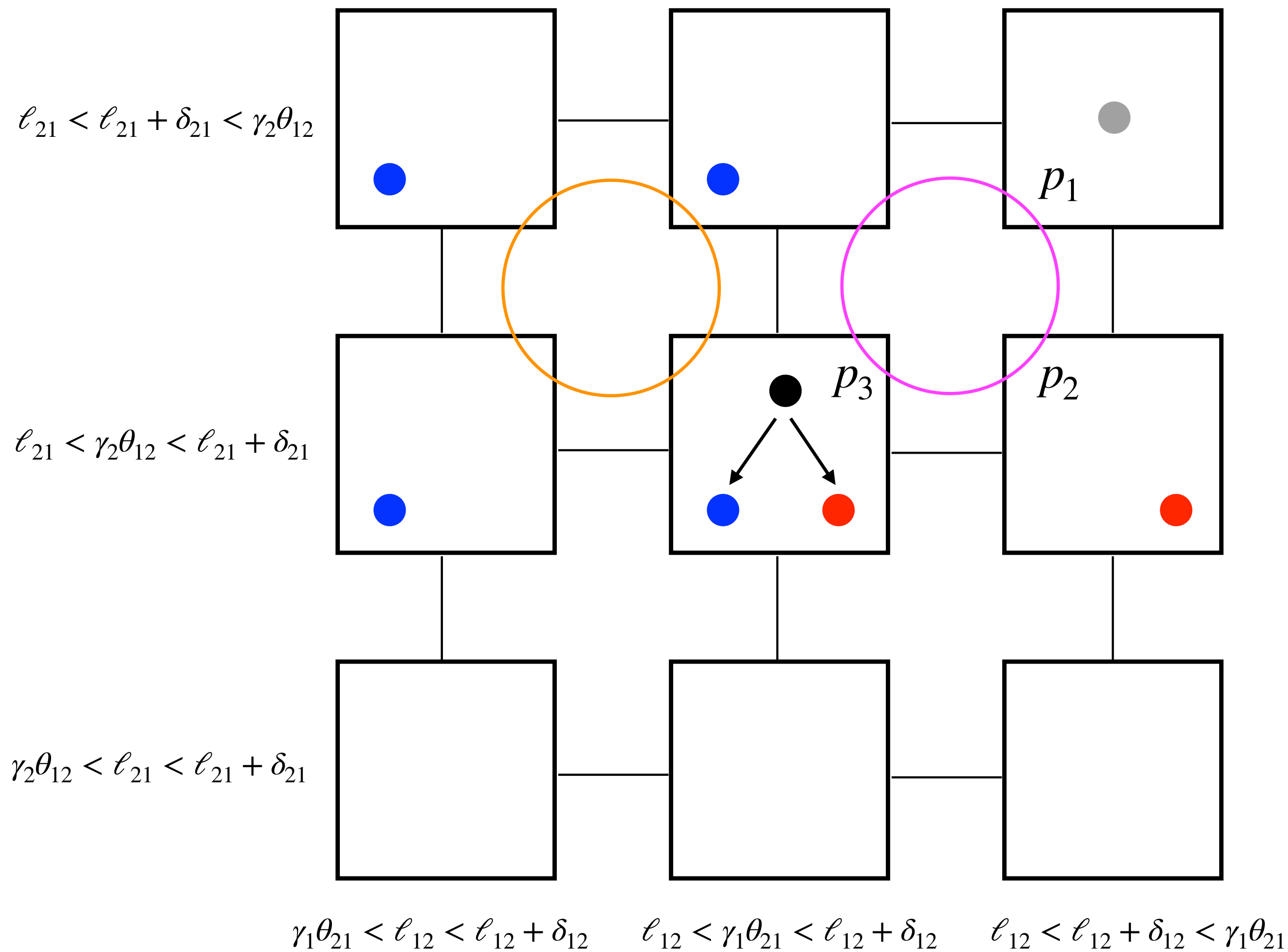
Parameter space is $(0,\infty)^8$,
which consists of 9 regions.

Use sheaf cohomology to check whether there is hysteresis on a closed curve, ie. Monostable-Bistable-Monstable along a single global section.

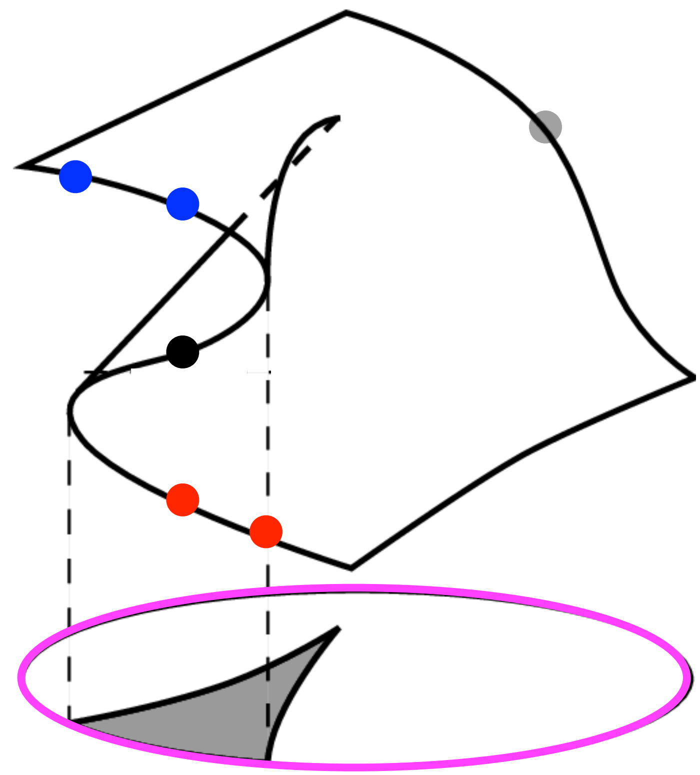


Codimension-2 bifurcation

9 parameter regions in $(0,\infty)^8$



Cusp bifurcation?



Rigorous verification (Lessard and Pugliese 2024)

Sigmoidal, Hill model with $d = 10$.

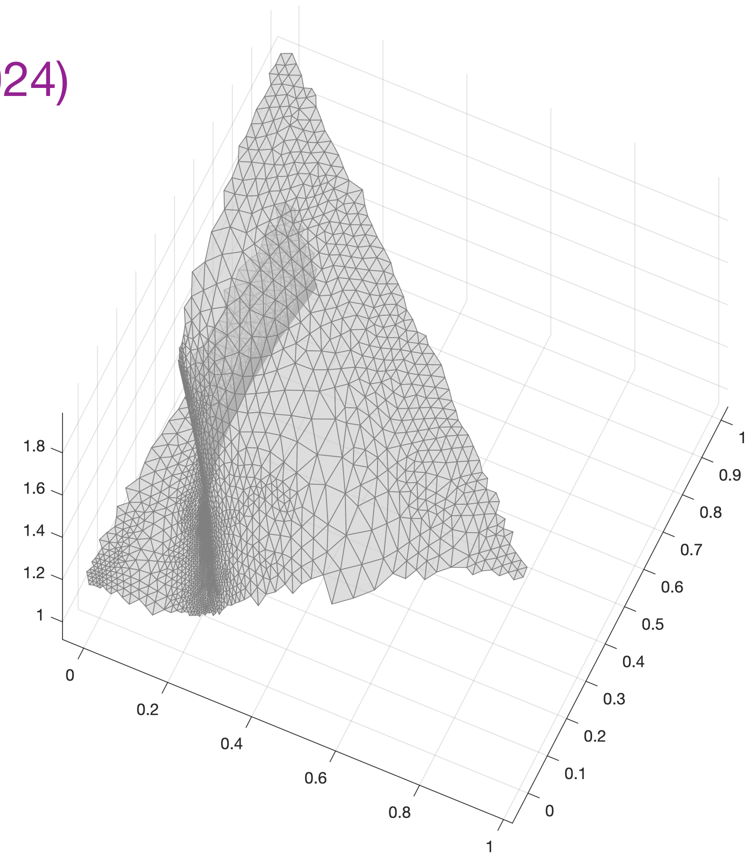
$$\dot{x}_1 = -\gamma_1 x_1 + \ell_{12} + \delta_{12}\frac{\theta_{12}^d}{\theta_{12}^d + x_2^d}$$

$$\dot{x}_2 = -\gamma_2 x_2 + \ell_{21} + \delta_{21}\frac{\theta_{21}^d}{\theta_{21}^d + x_1^d}$$

Reduce to a 2-dimensional subspace:

$$p = p_1 + s_1(p_2 - p_1) + s_2(p_3 - p_1) \quad 0 \le s_1 + s_2 \le 1$$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \ell_{12} \\ \ell_{21} \\ \delta_{12} \\ \delta_{21} \\ \theta_{12} \\ \theta_{21} \end{bmatrix} \quad p_1 = \begin{bmatrix} 1 \\ 0.9 \\ 1.1 \\ 1.1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad p_2 = \begin{bmatrix} 1 \\ 1.1 \\ 1.1 \\ 0.9 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad p_3 = \begin{bmatrix} 1 \\ 1.1 \\ 0.9 \\ 0.9 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Verified cusp bifurcation at approximately:
$s_1 = 0.176282044953037$ and
$s_2 = 0.177039860011836$

What next?

- Develop sheaf cohomology signatures that are searchable in the DSGRN database that are indicators of certain types of bifurcations

- Hysteresis - switching behavior in a network
  - Update a 2021 study of hysteresis in 3-node networks
  - Definition of hysteresis

- Higher codimension bifurcations
  - Swallowtail bifurcation (codimension-3)

- Relationship to connection matrices and transition matrices

Thank you!

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Robert VanderVorst, VU Amsterdam

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