Boundary Stabilization of a Bending and Twisting Beam by LQG

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Introduction

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Later we shall consider a nonlinear extension of this linear model where the bending of the beam increases its torsional rigidity and the torsion of the beam increases its bending rigidity.

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In particular, we use two measurements to estimate the full state, the vertical and angular velocity at the tip of the beam.

Then we use the estimate of the full state in place of the full state in the LQR feedback. This form of dynamic compensation is called Linear Quadratic Gaussian (LQG).

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My current and future project is to couple the model of the beam with a model of the aerodynamic forces that a wing generates using the state space models of the classical Wagner or Theodorsen theory.

If time permits I will discuss the results to date.

Model

Let the y axis be the axis of rotation of the beam and suppose it is attached to its support at $y = 0$ and its free end is at $y = L$.

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Let $h(y, t)$ be the vertical deflection of beam at location y and time t and let $\alpha(y, t)$ be the angle of rotation of the beam around the y axis at location y and time t.

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According to Bisplinghoff, Ashley and Halfman, equations (3-155) and (3-156), the free vibrations of a uniform beam are governed by the two inertially coupled linear PDEs.

Model

$$\begin{bmatrix} m & -S_y \\ -S_y & I_y \end{bmatrix} \begin{bmatrix} \frac{\partial^2 h}{\partial t^2}(y,t) \\ \frac{\partial^2 \alpha}{\partial t^2}(y,t) \end{bmatrix} = \begin{bmatrix} -EI\frac{\partial^4 h}{\partial y^4}(y,t) \\ GJ\frac{\partial^2 \alpha}{\partial y^2}(y,t) \end{bmatrix}$$

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where

L	half span	$15m$
L_c	chord	$1m$
EA	elastic y-axis	$0.5m$
CG	center of gravity	$0.5m$
m	mass per unit span	$0.75kg/m$
EI	bending rigidity	$2 * 10^4 n\ m^2$
GJ	torsion rigidity	$10^4 N\ m^2$
S_y	static moment per unit span	$0.025kg$
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Constants from Hossein Modaress-Aval et al, 2019.

Boundary Conditions

The bending boundary conditions at the free end of the beam are

$$\frac{\partial^2 h}{\partial y^2}(L,t) = 0, \qquad \frac{\partial^3 h}{\partial y^3}(L,t) = 0$$

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and at the fixed end of the beam we assume that there is an actuator that can deliver a bending moment

$$h(0,t) = 0, \qquad \frac{\partial^2 h}{\partial y^2}(0,t) = B_1 u_1(t)$$

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The torsion boundary condition at the free end of the beam is

$$\frac{\partial \alpha}{\partial y}(L,t) \;=\; 0$$

and at the fixed end of the beam we assume that there is an actuator that can deliver a torque

$$\frac{\partial \alpha}{\partial y}(0,t) \;=\; B_2 u_2(t)$$

First Order System

We wish to express the dynamics as a first order system so we introduce a four vector valued variable

$$z(y,t) \;=\; \left[\; h(y,t) \quad \frac{\partial h}{\partial t}(y,t) \quad \alpha(y,t) \quad \frac{\partial \alpha}{\partial t}(y,t) \;\right]'$$

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then the model becomes

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where

$$D = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -EI\frac{\partial^4}{\partial y^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & GJ\frac{\partial^2}{\partial y^2} & 0 \end{array}\right]$$

$$M = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -S_y \\ 0 & 0 & 1 & 0 \\ 0 & -S_y & 0 & I_y \end{array}\right]$$

First Order System

Notice M is symmetric and invertible if $mI_y - S_y^2 \neq 0$

$$M^{-1} = \frac{1}{mI_y - S_y^2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & I_y & 0 & S_y \\ 0 & 0 & 1 & 0 \\ 0 & S_y & 0 & m \end{bmatrix}$$

Linear Quadratic Regulator

First we seek a full state feedback law of the form

$$u(t) = \int_0^L K(y)z(y,t)\,dy$$

to stabilize the bending and torsion oscillations so we set up an Linear Quadratic Regulator (LQR).

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For any given initial condition $z(y, 0)$ we seek to minimize by choice of $u(t)$ the quantity

$$\int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)Q(y_1,y_2)z(y_2,t)\;dA + u^T(t)Ru(t)\;dt$$

where $\mathcal{S}$ is the square $[0, L]^2$ and $dA = dy_1 dy_2$.

Linear Quadratic Regulator

Let $P(y_1, y_2)$ be a continuous 4×4 nonnegative definite matrix valued function that is symmetric in its arguments, $P(y_1, y_2) = P(y_2, y_1)$ and satisfies these homogeneous boundary conditions for $i, j = 1, \ldots, 4$

$$
\begin{aligned}
P_{2,j}(0, y_2) &= 0, \qquad & P_{i,2}(y_1, 0) &= 0 \\
\frac{\partial^2 P_{2,j}}{\partial y_1^2}(0, y_2) &= 0, \qquad & \frac{\partial^2 P_{i,2}}{\partial y_2^2}(y_1, 0) &= 0 \\
\frac{\partial^2 P_{2,j}}{\partial y_1^2}(L, y_2) &= 0, \qquad & \frac{\partial^2 P_{i,2}}{\partial y_2^2}(y_1, L) &= 0 \\
\frac{\partial^3 P_{2,j}}{\partial y_1^3}(L, y_2) &= 0, \qquad & \frac{\partial^3 P_{i,2}}{\partial y_2^3}(y_1, L) &= 0 \\
\frac{\partial P_{4,j}}{\partial y_1}(0, y_2) &= 0, \qquad & \frac{\partial P_{i,4}}{\partial y_2}(y_1, 0) &= 0 \\
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$$

These are just the homogeneous version of the boundary conditions that $z(y, t)$ satisfies.

Linear Quadratic Regulator

If there is a $u(t)$ such that $z(y,t) \to 0$ as $t \to \infty$ then by the Fundamental Theorem of Calculus

$$
\begin{aligned}
0 \;=\; & \iint_{\mathcal{S}} z^T(y_1,0)M^{-1}P(y_1,y_2)M^{-1}z(y_2,0)\ dA \\
& + \int_0^\infty \frac{\partial}{\partial t} \iint_{\mathcal{S}} z^T(y_1,t)M^{-1}P(y_1,y_2)M^{-1}z(y_2,t)\ dA\ dt
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\end{aligned}
$$

We bring the time differentiation inside the spatial integrals and obtain

$$
\begin{aligned}
0 \; = \; & \iint_{\mathcal{S}} z^T(y_1,0)M^{-1}P(y_1,y_2)M^{-1}z(y_2,0)\; dA \\
& + \int_0^\infty \iint_{\mathcal{S}} z'(y_1,t)D_1^T P(y_1,y_2)M^{-1}z(y,t)\; dA \\
& + \int_0^\infty \iint_{\mathcal{S}} z'(y,t)M^{-1}P(y_1,y_2)D_2 z(y_2,t)\; dA
\end{aligned}
$$

where D_i is the matrix differential operator

$$
D_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -EI\frac{\partial^4}{\partial y_i^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & GJ\frac{\partial^2}{\partial y_i^2} & 0 \end{bmatrix}
$$

Integration by Parts

We integrate this by parts several times taking into account the boundary conditions on $z(y,t)$ and $P(y_1,y_2)$.

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Let $P_{:,j}(y_1,y_2)$ denote the j^{th} column and $P_{i,:}(y_1,y_2)$ denote the i^{th} row of $P(y_1,y_2)$ then

$$
\begin{aligned}
0 &= \iint_{\mathcal{S}} z^T(y_1,0)M^{-1}P(y_1,y_2)M^{-1}z(y_2,0)\,dA \\
&+ \int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)M^{-1}\begin{bmatrix} -EI\frac{\partial^4 P_{2,:}}{\partial y_1^4}(y_1,y_2) \\ P_{1,:}(y_1,y_2) \\ GJ\frac{\partial^2 P_{4,:}}{\partial y_1^2}(y_1,y_2) \\ P_{3,:}(y_1,y_2) \end{bmatrix} z(y_2,t)\,dA\,dt \\
&+ \int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)M^{-1} \\
&\times \begin{bmatrix} -EI\frac{\partial^4 P_{:,2}}{\partial y_2^4}(y_1,y_2) & P_{:,1}(y_1,y_2) & GJ\frac{\partial^2 P_{:,4}}{\partial y_2^2}(y_1,y_2) & P_{:,3}(y_1,y_2) \end{bmatrix} z(y_2,t)\,dA\,dt \\
&+ \int_0^\infty \int_0^L u^T(t)B\begin{bmatrix} EI\frac{\partial P_{2,:}}{\partial y_2}(y_1,0) \\ GJP_{4,:}(y_1,0) \end{bmatrix} M^{-1}z(y_2,t)\,dy_2\,dt \\
&+ \int_0^\infty \int_0^L z^T(y_1,t)M^{-1}\begin{bmatrix} EI\frac{\partial P_{:,2}}{\partial y_2}(y_1,0 & GJP_{:,4}(y_1,0) \end{bmatrix} Bu(t)\,dy_1\,dt
\end{aligned}
$$

New Criterion

We add the right side of this last equation to the criterion to be minimized to obtain an equivalent criterion to be minimized

$$
\begin{aligned}
0 &= \iint_{\mathcal{S}} z^T(y_1,0)M^{-1}P(y_1,y_2)M^{-1}z(y_2,0)\, dA \\
&+ \int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)Q(y_1,y_2)z(y_2,t)\, dA + u^T(t)Ru(t)\, dt \\
&+ \int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)M^{-1} \begin{bmatrix} -EI\frac{\partial^4 P_{2,:}}{\partial y_1^4}(y_1,y_2) \\ P_{1,:}(y_1,y_2) \\ GJ\frac{\partial^2 P_{4,:}}{\partial y_1^2}(y_1,y_2) \\ P_{3,:}(y_1,y_2) \end{bmatrix} z(y_2,t)\, dA\, dt \\
&+ \int_0^\infty \iint_{\mathcal{S}} z^T(y_1,t)M^{-1} \\
&\times \begin{bmatrix} -EI\frac{\partial^4 P_{:,2}}{\partial y_2^4}(y_1,y_2) & P_{:,1}(y_1,y_2) & GJ\frac{\partial^2 P_{:,4}}{\partial y_2^2}(y_1,y_2) & P_{:,3}(y_1,y_2) \end{bmatrix} z(y_2,t)\, dA\, dt \\
&+ \int_0^\infty \int_0^L u^T(t)B \begin{bmatrix} EI\frac{\partial P_{2,:}}{\partial y_2}(y_1,0) \\ GJP_{4,:}(y_1,0) \end{bmatrix} M^{-1}z(y_2,t)\, dy_2\, dt \\
&+ \int_0^\infty \int_0^L z^T(y_1,t)M^{-1} \begin{bmatrix} EI\frac{\partial P_{:,2}}{\partial y_2}(y_1,0 & GJP_{:,4}(y_1,0) \end{bmatrix} Bu(t)\, dy_1\, dt
\end{aligned}
$$

Completing the Square

We wish to find a 2×4 matrix valued function $K(y)$ such that the time integrand of the equivalent criterion is equal to a perfect square of the form

$$\iint_{\mathcal{S}} \left(u(t) - K(y_1)z(y_1,t)\right)^T R \left(u(t) - K(y_2)z(y_2,t)\right) \; dA$$

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The terms quadratic in $u(t)$ match so we equate terms bilinear in $u^T(t)$ and $z(y_2,t)$.

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The terms quadratic in $u(t)$ match so we equate terms bilinear in $u^T(t)$ and $z(y_2,t)$.

This yields

$$-RK(y_2) \;\; = \;\; B \begin{bmatrix} EI\frac{\partial P_{2,:}}{\partial y_1}(0,y_2) \\ GJP_{4,:}(0,y_2) \end{bmatrix} M^{-1}$$

so we assume that

$$K(y_2) \;\; = \;\; -R^{-1}B \begin{bmatrix} EI\frac{\partial P_{2,:}}{\partial y_1}(0,y_2) \\ GJP_{4,:}(0,y_2) \end{bmatrix} M^{-1}$$

Riccati PDE

Then by equating terms bilinear in $z^T(y_1,t)$ and $z(y_2,t)$ we obtain the Riccati PDE for quadratic Fredholm kernel $P(y_1,y_2)$ of the optimal cost,

$$
\begin{aligned}
& M^{-1}\begin{bmatrix} -EI\frac{\partial^4 P_{:,2}}{\partial y_2^4}(y_1,y_2) & P_{:,1}(y_1,y_2) & GJ\frac{\partial^2 P_{:,4}}{\partial y_2^2}(y_1,y_2) & P_{:,3}(y_1,y_2) \end{bmatrix} \\
& + \begin{bmatrix} -EI\frac{\partial^4 P_{2,:}}{\partial y_1^4}(y_1,y_2) \\ P_{1,:}(y_1,y_2) \\ GJ\frac{\partial^2 P_{4,:}}{\partial y_1^2}(y_1,y_2) \\ P_{3,:}(y_1,y_2) \end{bmatrix} M^{-1} + Q(y_1,y_2) \\
& = M^{-1}\begin{bmatrix} EI\frac{\partial P_{:,2}}{\partial y_2}(y_1,0) & GJP_{:,4}(y_1,0) \end{bmatrix} BR^{-1}B \begin{bmatrix} EI\frac{\partial P_{2,:}}{\partial y_1}(0,y_2) \\ GJP_{4,:}(0,y_2) \end{bmatrix} M^{-1}
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& + \left[\begin{array}{c} -EI\frac{\partial^4 P_{2,:}}{\partial y_1^4}(y_1,y_2) \\ P_{1,:}(y_1,y_2) \\ GJ\frac{\partial^2 P_{4,:}}{\partial y_1^2}(y_1,y_2) \\ P_{3,:}(y_1,y_2) \end{array}\right] M^{-1} + Q(y_1,y_2) \\
& = M^{-1}\left[\begin{array}{cc} EI\frac{\partial P_{:,2}}{\partial y_2}(y_1,0) & GJP_{:,4}(y_1,0) \end{array}\right] BR^{-1}B \left[\begin{array}{c} EI\frac{\partial P_{2,:}}{\partial y_1}(0,y_2) \\ GJP_{4,:}(0,y_2) \end{array}\right] M^{-1}
\end{aligned}
$$

This is an elliptic PDE with a quadratic nonlinearity.

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\begin{aligned}
& M^{-1}\left[\begin{array}{cccc} -EI\frac{\partial^4 P_{:,2}}{\partial y_2^4}(y_1,y_2) & P_{:,1}(y_1,y_2) & GJ\frac{\partial^2 P_{:,4}}{\partial y_2^2}(y_1,y_2) & P_{:,3}(y_1,y_2) \end{array}\right] \\
& +\left[\begin{array}{c} -EI\frac{\partial^4 P_{2,:}}{\partial y_1^4}(y_1,y_2) \\ P_{1,:}(y_1,y_2) \\ GJ\frac{\partial^2 P_{4,:}}{\partial y_1^2}(y_1,y_2) \\ P_{3,:}(y_1,y_2) \end{array}\right] M^{-1} + Q(y_1,y_2) \\
& = M^{-1}\left[\begin{array}{cc} EI\frac{\partial P_{:,2}}{\partial y_2}(y_1,0) & GJP_{:,4}(y_1,0) \end{array}\right] BR^{-1}B \left[\begin{array}{c} EI\frac{\partial P_{2,:}}{\partial y_1}(0,y_2) \\ GJP_{4,:}(0,y_2) \end{array}\right] M^{-1}
\end{aligned}
$$

This is an elliptic PDE with a quadratic nonlinearity.

Since we only assumed that $P(y_1,y_2)$ is continuous this PDE and its homogeneous boundary conditions are to be interpreted in the weak sense.

Fourier Analysis

Fourier methods are well-suited to solving weak PDEs.

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$$-\frac{\partial^4}{\partial y^4}, \qquad \frac{\partial^2}{\partial y^2}$$

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subject to the appropriate boundary conditions.

All of the eigenvalues of these operators are nonpositive. Since the temporal partial differential operator $\frac{\partial^2}{\partial t^2}$ is second order this implies that all of the eigenvalues of the inertially coupled beam are imaginary.

Fourth Order PDO

The partial differential operator $-\frac{\partial^4}{\partial y^4}$ is self-adjoint when subject to the appropriate boundary conditions

$$\phi(0) = 0, \frac{\partial^2 \phi}{\partial y^2}(0) = 0, \frac{\partial^2 \phi}{\partial y^2}(L) = 0, \frac{\partial^3 \phi}{\partial y^3}(L) = 0$$

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There is exactly one root $\beta_m L \in [m\pi, (m + 1/2)\pi)$. As $m \to \infty$ the m^{th} root is quickly converging to $(m\pi + \frac{\pi}{4})$.

Fourth Order PDO

The corresponding orthogonal but not orthonormal eigenfunctions are quickly converging to

$$\Phi_m(y) \;\approx\; \sin\beta_m y + d\sinh\beta_m y$$

where

$$d \;\approx\; \frac{(-1)^{m+1}\frac{\sqrt{2}}{2}}{\sinh(m\pi + \frac{\pi}{4})}$$

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Note we are using the symbol m in two different senses. Previously we used m for mass per unit span but now we are also using it as an integer index.

The correct interpretation will be clear from context.

Second Order PDO

The appropriate boundary conditions for the second order operator

$$\frac{\partial^2}{\partial y^2}$$

are

$$\frac{\partial \theta}{\partial y}(0,t) = 0, \qquad \frac{\partial \theta}{\partial y}(L,t) = 0$$

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The corresponding orthogonal but not orthonormal eigenfunctions are

$$\Theta_n(y) \;\; = \;\; \cos \frac{n\pi y}{L}$$

Series Solution of the Riccati PDE

Suppose $Q(y_1, y_2)$ is diagonal, $Q_{i,j}(y_1, y_2) = Q_{j,i}(y_1, y_2) = 0$ if $i \neq j$, and it has an expansion of the form

$$
\begin{aligned}
Q(y_1, y_2) &= \sum_{m=1}^{\infty} \begin{bmatrix} Q_{1,1}^{m,m} & 0 & 0 & 0 \\ 0 & Q_{2,2}^{m,m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Phi_m(y_1)\Phi_m(y_2) \\
&\quad + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{3,3}^{n,n} & 0 \\ 0 & 0 & 0 & Q_{4,4}^{n,n} \end{bmatrix} \Theta_n(y_1)\Theta_n(y_2)
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$$

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We could consider more general $Q(y_1, y_2)$ but to keep the exposition relatively simple we do not.

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&\quad + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{3,3}^{n,n} & 0 \\ 0 & 0 & 0 & Q_{4,4}^{n,n} \end{bmatrix} \Theta_n(y_1)\Theta_n(y_2)
\end{aligned}
$$

We could consider more general $Q(y_1, y_2)$ but to keep the exposition relatively simple we do not.

Notice that the ranges of the indices m and n are different.

Series Solution of the Riccati PDE

We also assume that the solution $P(y_1, y_2)$ of the Riccati PDE has a similar but more complicated expansion. When $i, j = 1, 2$

$$P_{i,j}(y_1, y_2) = \sum_{m_1, m_2 = 1}^{\infty} P_{i,j}^{m_1, m_2} \Phi_{m_1}(y_1) \Phi_{m_2}(y_2)$$

When $i = 1, 2$ and $j = 3, 4$

$$P_{i,j}(y_1, y_2) = \sum_{m_1 = 1, n_2 = 0}^{\infty} P_{i,j}^{m_1, n_2} \Phi_{m_1}(y_1) \Theta_{n_2}(y_2)$$

When $i = 3, 4$ and $j = 1, 2$

$$P_{i,j}(y_1, y_2) = \sum_{n_1 = 0, m_2 = 1}^{\infty} P_{i,j}^{n_1, m_2} \Theta_{n_1}(y_1) \Phi_{m_2}(y_2)$$

When $i, j = 3, 4$

$$P_{i,j}(y_1, y_2) = \sum_{n_1 = 0, n_2 = 0}^{\infty} P_{i,j}^{n_1, n_2} \Theta_{n_1}(y_1) \Theta_{n_2}(y_2)$$

Series Solution of the Riccati PDE

We plug these expansions into Riccati PDE and collect similar terms to obtain an infinite dimensional algebraic Riccati equation which has four coupled components.

Series Solution of the Riccati PDE

We plug these expansions into Riccati PDE and collect similar terms to obtain an infinite dimensional algebraic Riccati equation which has four coupled components.

The $\Phi_{m_1}(y_1)\Phi_{m_2}(y_2)$ component is

$$
\begin{bmatrix}
\nu_{m_2} EIP^{m_1,m_2}_{1,2} & P^{m_1,m_2}_{1,1} & 0 & 0 \\
\nu_{m_2} I_y EIP^{m_1,m_2}_{2,2} & I_y P^{m_1,m_2}_{2,1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\nu_{m_2} S_y EIP^{m_1,m_2}_{2,2} & S_y P^{m_1,m_2}_{2,1} & 0 & 0
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
\nu_{m_1} EIP^{m_1,m_2}_{2,1} & \nu_{m_1} I_y EIP^{m_1,m_2}_{2,2} & 0 & \nu_{m_1} S_y EIP^{m_1,m_2}_{2,2} \\
P^{m_1,m_2}_{1,1} & I_y P^{m_1,m_2}_{1,2} & 0 & S_y P^{m_1,m_2}_{1,2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
Q^{m_1,m_2}_{1,1} & 0 & 0 & 0 \\
0 & Q^{m_1,m_2}_{2,2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
EIP^{m_1,m_4}_{1,2} \Phi'_{m_4}(0) & GJP^{m_1,n_4}_{1,4} \Theta_{n_4}(0) \\
I_y EIP^{m_1,m_4}_{2,2} \Phi'_{m_4}(0) & I_y GJP^{m_1,n_4}_{2,4} \Theta_{n_4}(0) \\
0 & 0 \\
S_y EIP^{m_1,m_4}_{2,2} \Phi'_{m_4}(0) & S_y GJP^{m_1,n_4}_{2,4} \Theta_{n_4}(0)
\end{bmatrix} \Gamma
$$

$$
\times \begin{bmatrix}
EIP^{m_3,m_2}_{2,1} \Phi'_{m_3}(0) & I_y EIP^{m_3,m_2}_{2,2} \Phi'_{m_3}(0) & 0 & S_y EIP^{m_3,m_2}_{2,2} \Phi'_{m_3}(0) \\
GJP^{n_3,m_2}_{4,1} \Theta_{n_3}(0) & I_y GJP^{n_3,m_2}_{4,2} \Theta_{n_3}(0) & 0 & S_y GJP^{n_3,m_2}_{4,2} \Theta_{n_3}(0)
\end{bmatrix}
$$

Algebraic Riccati Equation

where

$$\boldsymbol{\Gamma} = \frac{1}{mI_y - S_y^2}\left[\begin{array}{cc} B_1^2 R_1^{-1} & 0 \\ 0 & B_2^2 R_2^{-1} \end{array}\right]$$

Algebraic Riccati Equation

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$$\Gamma = \frac{1}{mI_y - S_y^2}\begin{bmatrix} B_1^2 R_1^{-1} & 0 \\ 0 & B_2^2 R_2^{-1} \end{bmatrix}$$

The $\Phi_{m_1}(y_1)\Theta_{n_2}(y_2)$ component is

$$\begin{bmatrix} 0 & 0 & \eta_{n_2} GJP_{1,4}^{m_1,n_2} & P_{1,3}^{m_1,n_2} \\ 0 & 0 & \eta_{n_2} GJP_{2,4}^{m_1,n_2} & P_{2,3}^{m_1,n_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_y \eta_{n_2} GJP_{2,4}^{m_1,n_2} & S_y P_{2,3}^{m_1,n_2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \nu_{m_1} S_y EIP_{2,4}^{m_1,n_2} & \nu_{m_1} EIP_{2,3}^{m_1,n_2} & \nu_{m_1} EIP_{2,4}^{m_1,n_2} \\ 0 & S_y P_{1,4}^{m_1,n_2} & P_{1,3}^{m_1,m_2} & P_{1,4}^{m_1,m_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} EIP_{1,2}^{m_1,m_4} \Phi'_{m_4}(0) & GJP_{1,4}^{m_1,n_4} \Theta_{n_4}(0) \\ I_y EIP_{2,2}^{m_1,m_4} \Phi'_{m_4}(0) & I_y GJP_{2,4}^{m_1,n_4} \Theta_{n_4}(0) \\ 0 & 0 \\ S_y EIP_{2,2}^{m_1,m_4} \Phi'_{m_4}(0) & S_y GJP_{2,4}^{m_1,n_4} \Theta_{n_4}(0) \end{bmatrix} \Gamma$$

$$\begin{bmatrix} 0 & S_y EIP_{2,4}^{m_3,n_2} \Phi'_{m_3}(0) & EIP_{2,3}^{m_3,n_2} \Phi'_{m_3}(0) & mEIP_{2,4}^{m_3,n_2} \Phi'_{m_3}(0) \\ 0 & GJP_{4,4}^{n_3,n_2} \Theta_{n_3}(0) & GJP_{4,3}^{n_3,n_2} \Theta_{n_3}(0) & mGJP_{4,4}^{n_3,n_2} \Theta_{n_3}(0) \end{bmatrix}$$

Algebraic Riccati Equation

The $\Theta_{n_1}(y_1)\Phi_{m_2}(y_2)$ component is

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\nu_{m_2} S_y EIP_{4,2}^{n_1,m_2} & S_y P_{4,1}^{n_1,m_2} & 0 & 0 \\
\nu_{m_2} EIP_{3,2}^{n_1,m_2} & P_{3,1}^{n_1,m_2} & 0 & 0 \\
\nu_{m_2} EIP_{4,2}^{n_1,m_2} & P_{4,1}^{n_1,m_2} & 0 & 0
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\eta_{n_1} GJP_{4,1}^{n_1,m_2} & \eta_{n_1} I_y GJP_{4,2}^{n_1,m_2} & 0 & \eta_{n_1} S_y GJP_{4,2}^{n_1,m_2} \\
P_{3,1}^{n_1,m_2} & I_y P_{3,2}^{n_1,m_2} & 0 & S_y P_{3,2}^{n_1,m_2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 \\
S_y EIP_{4,2}^{n_1,m_4} \Phi'_{m_4}(0) & S_y GJP_{4,4}^{n_1,n_4} \Theta_{n_4}(0) \\
EIP_{3,2}^{n_1,m_4} \Phi'_{m_4}(0) & GJP_{3,4}^{n_1,n_4} \Theta_{n_4}(0) \\
mEIP_{4,2}^{n_1,m_4} \Phi'_{m_4}(0) & mGJP_{4,4}^{n_1,n_4} \Theta_{n_4}(0)
\end{bmatrix} \Gamma
$$

$$
\times \begin{bmatrix}
EIP_{2,1}^{m_3,m_2} \Phi'_{m_3}(0) & I_y EIP_{2,2}^{m_3,m_2} \Phi'_{m_1}(0) & 0 & S_y EIP_{2,2}^{m_3,m_2} \Phi'_{m_3}(0) \\
GJP_{4,1}^{n_3,m_2} \Theta_{n_3}(0) & I_y GJP_{4,2}^{n_3,m_2} \Theta_{n_3}(0) & 0 & S_y GJP_{4,2}^{n_3,m_2} \Theta_{n_3}(0)
\end{bmatrix}
$$

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$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & S_y\eta_{n_2}GJP^{n_1,n_2}_{4,4} & S_yP^{n_1,n_2}_{4,3} \\
0 & 0 & \eta_{n_2}GJP^{n_1,n_2}_{3,4} & P^{n_1,n_2}_{3,3} \\
0 & 0 & \eta_{n_2}GJP^{n_1,n_2}_{4,4} & P^{n_1,n_2}_{4,3}
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \eta_{n_1}S_yGJP^{n_1,n_2}_{4,4} & \eta_{n_1}GJP^{n_1,n_2}_{4,3} & \eta_{n_1}mGJP^{n_1,n_2}_{4,4} \\
0 & S_yP^{n_1,n_2}_{3,4} & P^{n_1,n_2}_{3,3} & mP^{n_1,n_2}_{3,4}
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & Q^{n_1,n_2}_{3,3} & \\
0 & 0 & 0 & Q^{n_1,n_2}_{4,4}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
0 & 0 \\
S_yEIP^{n_1,m_4}_{4,2}\Phi'_{m_4}(0) & S_yGJP^{n_1,n_4}_{4,4}\Theta_{n_4}(0) \\
EIP^{n_1,m_4}_{3,2}\Phi'_{m_4}(0) & GJP^{n_1,n_4}_{3,4}\Theta_{n_4}(0) \\
mEIP^{n_1,m_4}_{4,2}\Phi'_{m_4}(0) & mGJP^{n_1,n_4}_{4,4}\Theta_{n_4}(0)
\end{bmatrix}\Gamma
$$

$$
\times\begin{bmatrix}
0 & S_yEIP^{m_3,n_2}_{2,4}\Phi'_{m_3}(0) & EIP^{m_3,n_2}_{2,3}\Phi'_{m_3}(0) & mEIP^{m_3,n_2}_{2,4}\Phi'_{m_3}(0) \\
0 & S_yGJP^{n_3,n_2}_{4,4}\Theta_{n_3}(0) & GJP^{n_3,n_2}_{4,3}\Theta_{n_3}(0) & mGJP^{n_3,n_2}_{4,4}\Theta_{n_3}(0)
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Policy Iteration

We can approximately solve this algebraic Riccati equation by policy iteration. This method could also be seen as value iteration.

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To find an initial estimate $\left(P_{i,j}^{m_1,m_2}\right)^{(0)}$, $\left(P_{i,j}^{n_1,n_2}\right)^{(0)}$ of the Fourier coeficients of the kernel $P(y_1, y_2)$ of the optimal cost we specify that

$$\left(P_{i,j}^{m_1,m_2}\right)^{(0)} = 0 \qquad \text{unless } m_1 = m_2 \text{ and } i = j$$

$$\left(P_{i,j}^{n_1,n_2}\right)^{(0)} = 0 \qquad \text{unless } n_1 = n_2 \text{ and } i = j$$

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$$\left(P_{i,j}^{m_1,n_2}\right)^{(0)} = \qquad \left(P_{i,j}^{n_1,m_2}\right)^{(0)} = 0$$

and solve the thus simplified algebraic Riccati equations for $\left(P_{i,i}^{m,m}\right)^{(0)}$ and $\left(P_{i,i}^{n,n}\right)^{(0)}$.

Policy Iteration

Succesive iterates are found by plugging $\left(P_{i,j}^{m_1,m_2}\right)^{(k)}$ and $\left(P_{i,j}^{n_1,n_2}\right)^{(k)}$ into the right side of the algebraic Riccati equations and plugging $\left(P_{i,i}^{m_1,m_2}\right)^{(k+1)}$ and $\left(P_{i,i}^{n_1,n_2}\right)^{(k+1)}$ into the left side.

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It is crucial that the estimates

$$
\begin{aligned}
P^{(k)}(y_1.y_2) \quad &= \quad \sum_{m_1,m_2=1}^{\infty} \left(P^{m_1,m_2}\right)^{(k)} \Phi_{m_1}(y_1)\Phi_{m_2}(y_2) \\
&\quad + \sum_{m_1=1}^{\infty}\sum_{n_2=0}^{\infty} \left(P^{m_1,n_2}\right)^{(k)} \Phi_{m_1}(y_1)\Theta_{n_2}(y_2) \\
&\quad + \sum_{n_1=0}^{\infty}\sum_{m_2=0}^{\infty} \left(P^{n_1,m_2}\right)^{(k)} \Theta_{n_1}(y_1)\Phi_{m_2}(y_2) \\
&\quad + \sum_{n_1,n_2=1}^{\infty} \left(P^{n_1,n_2}\right)^{(k)} \Theta_{n_1}(y_1)\Theta_{n_2}(y_2)
\end{aligned}
$$

of the kernel of the optimal cost are continuous.

Policy Iteration

If they are continuous then

$$\iint_{\mathcal{S}} z^T(y_1,0)P^{(k)}(y_1,y_2)z(y_2,0)\ dA$$

is bounded for any continuous initial condition $z(y,0)$ which implies that the feedback with kernel

$$K^{(k)}(y_2) \quad = \quad -R^{-1}B\begin{bmatrix} EI\frac{\partial P^{(k)}_{2,:}}{\partial y_1}(0,y_2) \\ GJP^{(k)}_{4,:}(0,y_2) \end{bmatrix} M^{-1}$$

has moved all the closed eigenvalues into open left half plane.

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has moved all the closed eigenvalues into open left half plane.

This implies asymptotic stability but not exponential stability.

Two Theorems

Theorem: The series for $P^{(0)}_{i,i}(y_1, y_2)$ with $i = 1, 2$ converges to a continuous function if there exist positive numbers q and $r > 8$ such that

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We believe that similar theorems are true for $P^{(k)}_{i,j}(y_1, y_2)$ but we really don't need them to prove asymptotic stability becaues value iteration implies that

$$\iint_S z^T(y_1, 0) P^{(k)}(y_1, y_2) z(y_2, 0)\, dA \leq \iint_S z^T(y_1, 0) P^{(0)}(y_1, y_2) z(y_2, 0)\, dA$$

Approximating Finite Dimensional LQR

We construct a finite dimensional LQR whose algebraic Riccati equation is a truncation of the above infinite dimensional algebraic Riccati equation.

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We choose an $N > 0$ and construct a linear system with state $\zeta = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T$ where $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is each of dimension N. So the finite dimensional state ζ is of dimension $4N$.

Approximating Finite Dimensional LQR

The dynamics is

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
m\dot{\zeta}_2 - S_y\dot{\zeta}_4 &= F_1\zeta_1 + G_1u_1 \\
\dot{\zeta}_3 &= \zeta_4 \\
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where

$$F_1 = EI\begin{bmatrix} \nu_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \nu_N \end{bmatrix}, \qquad G_1 = \begin{bmatrix} \Phi_1'(0) \\ \vdots \\ \Phi_N'(0) \end{bmatrix}$$

$$F_2 = GJ\begin{bmatrix} \eta_0 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \eta_{N-1} \end{bmatrix}, \qquad G_2 = \begin{bmatrix} \Theta_0(0) \\ \vdots \\ \Theta_{N-1}(0) \end{bmatrix}$$

Approximating Finite Dimensional LQR

This finite dimensional system approximates the infinite dimensional system in the following manner

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\begin{aligned}
z_1(y,t) &\approx \begin{bmatrix} \Phi_1(y) & \dots & \Phi_N(y) \end{bmatrix} \zeta_1(t) \\
z_2(y,t) &\approx \begin{bmatrix} \Phi_1(y) & \dots & \Phi_N(y) \end{bmatrix} \zeta_2(t) \\
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Recall

$$
\begin{bmatrix}
z_1(y,t) & \text{vertical displacement} & h(y,t) \\
z_2(y,t) & \text{vertical velocity} & \dot{h}(y,t) \\
z_3(y,t) & \text{angle of attack} & \alpha(y,t) \\
z_4(y,t) & \text{angular velocity of attack} & \dot{\alpha}(y,t)
\end{bmatrix}
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Example

We consider a $N = 4$ approximation which leads to a 16 dimensional system. We take Q to be a 16×16 identity matrix, R a 2×2 identity matrix and all constants to equal 1 except $S_y = 1/2$ so that M is invertible.

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The 16 open and closed loop poles are

$$\left[\begin{array}{cc} \text{Open Loop Poles} & \text{Closed Loop Poles} \\ \pm 3.61i & -1.14 \pm 3.50i \\ \pm 20.67i & -1.65 \pm 20.51i \\ \pm 7.24i & -1.85 \pm 7.24i \\ \pm 10.87i & -2.78 \pm 10.87i \\ \pm 14.50i & -2.96 \pm 13.25i \\ \pm 66.75i & -4.29 \pm 66.72i \\ \pm 139.14i & -6.52 \pm 139.11i \\ \pm 237.84 & -8.74 \pm 237.00i \end{array}\right]$$

Finite Dimensional State $\zeta(t)$

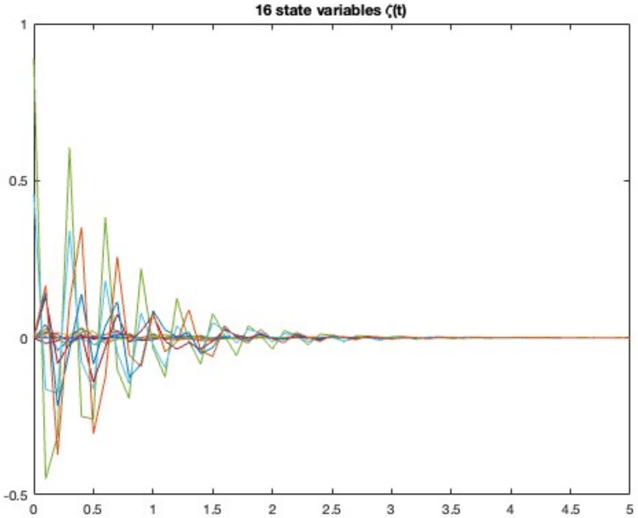
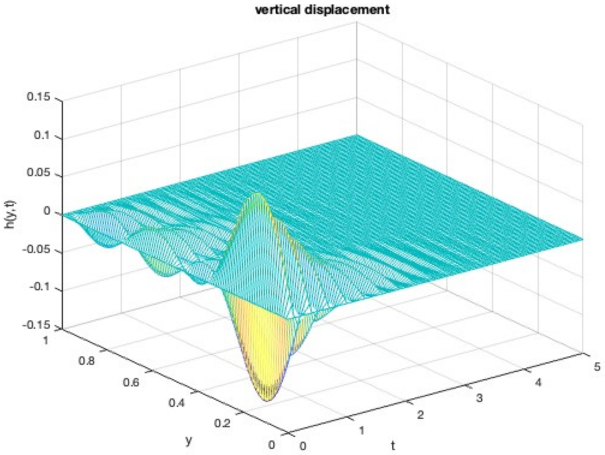


Figure: **Finite Dimensional State** $\zeta(t)$

Vertical Displacement $h(y,t)$



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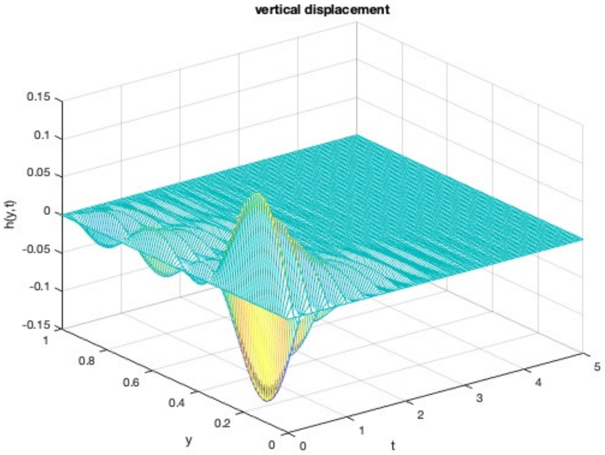
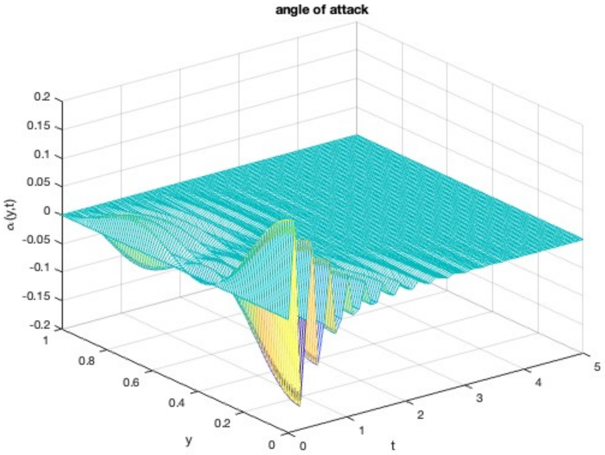


The control input at the root of the beam uses the ripples to stabilize the vertical displacement of the tip.

Angle of Attack $\boldsymbol{\alpha(y,t)}$



Again the control input at the root of the beam uses the ripples to stabilize the torsion at the tip.

Kalman Filtering

In the above discussion we made the unreasonable assumption that it was possible to measure the full state $z(y,t)$ at every location $y \in [0,L]$ and every time $t \geq 0$ so we could use full state feedback.

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The case when the measurements are only available at discrete times is mathematically simpler and the methods that we present can be extended to such discrete time measurents.

Given the state estimate $\hat{z}(y,t)$ we use it in place of the true state $z(y,t)$ in our feedback law.

Model

We assume that we can approximately measure the vertical velocity $z_2(L,t)$ and the angular velocity $z_4(L,t)$ at the tip of the beam.

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We express the inertially coupled bending and twisting dynamics in a more standard form. We also add a driving noise input $v(t) = [v_1(t), v_2(t)]^T$ (wind gusts) and obtain

$$\frac{\partial z}{\partial t}(y,t) \;\; = \;\; \mathcal{A}z(y,t) + \mathcal{B}(y)v(t)$$

where the partial differential operator is $\mathcal{A} = M^{-1}D$ and

$$\mathcal{B}(y) \;\; = \;\; \begin{bmatrix} 0 & 0 \\ \mathcal{B}_{11}(y) & \mathcal{B}_{12}(y) \\ 0 & 0 \\ \mathcal{B}_{21}(y) & \mathcal{B}_{22}(y) \end{bmatrix}$$

Boundary Conditions

The appropriate boundary conditions are homgeneous

$$\begin{aligned} z_1(0,t) &= 0, & \frac{\partial^2 z_1}{\partial y^2}(0,t) &= 0 \\ \frac{\partial^2 z_1}{\partial y^2}(L,t) &= 0, & \frac{\partial^3 z_1}{\partial y^3}(L,t) &= 0 \\ \frac{\partial z_3}{\partial y}(0,t) &= 0, & \frac{\partial z_3}{\partial y}(L,t) &= 0 \end{aligned}$$

Optimal Estimate

Because of the linear, Gaussian assumptions we expect that the optimal least squares estimate $\hat{z}(y,t)$ of $z(y,t)$ is a linear functional of the past observations,

$$\begin{aligned}\hat{z}(y,t) &= \int_0^\infty \mathcal{L}(y,s)\psi(t-s)\,ds \\ &= \int_0^\infty \mathcal{L}(y,s)\left(\begin{bmatrix} z_2(L,t) \\ z_4(L,t)\end{bmatrix} + \mathcal{D}\begin{bmatrix} w_1(t-s) \\ w_2(t-s)\end{bmatrix}\right)\,ds\end{aligned}$$

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\end{aligned}$$

for some 4×2 matrix valued function $\mathcal{L}(y,s)$.

Given such a $\mathcal{L}(y,s)$ we define a 4×4 matrix valued function $\mathcal{H}(y,y_1,s)$ by the partial differential equation

$$\frac{\partial \mathcal{H}}{\partial s}(y,y_1,s) = \mathcal{H}(y,y_1,s)\mathcal{A}_1 - \mathcal{L}(y,s) \begin{bmatrix} 0 & 0 \\ \delta(y_1 - L) & 0 \\ 0 & 0 \\ 0 & \delta(y_1 - L) \end{bmatrix}$$

Boundary and Initial Conditions

This PDE is subject to the homogeneous boundary conditions

$$\mathcal{H}_{i,2}(y,0,t) = 0, \qquad \frac{\partial^2 \mathcal{H}_{i,2}}{\partial y_1^2}(y,0,t) = 0$$

$$\frac{\partial^2 \mathcal{H}_{i,2}}{\partial y_1^2}(y,L,t) = 0, \qquad \frac{\partial^3 \mathcal{H}_{i,2}}{\partial y_1^3}(y,L,t) = 0$$

$$\frac{\partial \mathcal{H}_{i,4}}{\partial y_1}(y,0,t) = 0, \qquad \frac{\partial \mathcal{H}_{i,4}}{\partial y_1}(y,L,t) = 0$$

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and the initial condition

$$\mathcal{H}(y,y_1,0) = \delta(y-y_1) I^{4\times 4}$$

Integration by Parts

After integration by parts with respect to s and y_1 we obtain a formula for the estimation error $\tilde{z}(y,t) = z(y,t) - \hat{z}(y,t)$,

$$\begin{aligned}\tilde{z}(y,t) &= -\int_0^\infty \int_0^L \mathcal{H}(y,y_1,s)\mathcal{B}(y_1)v(t-s)\ dy_1\ ds \\ &\quad -\int_0^\infty \mathcal{L}(y,s)\mathcal{D}\begin{bmatrix} w_1(t-s) \\ w_2(t-s)\end{bmatrix}\ ds\end{aligned}$$

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\end{aligned}
$$

Because $v(t)$ and $w(t)$ are standard white Gaussian noises, the error covariance which we seek to minimize is

$$
\begin{aligned}
&\int_0^\infty \iint_{\mathcal{S}} \mathcal{H}(y,y_1,s)\mathcal{B}(y_1)\mathcal{B}^T(y_2)\mathcal{H}^T(y,y_2,s)\ dA\ ds \\
&+\int_0^\infty \mathcal{L}(y,s)\mathcal{D}\mathcal{D}^T\mathcal{L}^T(y,s)\ ds
\end{aligned}
$$

Family of Adjoint LQRs

For each $y \in [0, L]$ and for each pair of corresponding rows of $\mathcal{H}(y, y_1, s)$ and $\mathcal{L}(y, s)$ we have an LQR in adjoint form with state the i^{th} row of $\mathcal{H}(y, y_1, s)$ and with control the i^{th} row of $\mathcal{L}(y, s)$, linear dynamics and quadratic criterion.

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But we can leave this adjoint LQR in matrix form as the optimal feedback gain $\mathcal{K}(y, y_1)$ is the same for all rows,

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But if the coefficient of the driving noise does not vary with y, $\mathcal{B}(y) = \mathcal{B}$ then $\mathcal{H}(y, y_1, s) = \mathcal{H}(y_1, s)$ and $\mathcal{K}(y, y_1) = \mathcal{K}(y_1)$.

Kalman Filter

After some manipulations we can show that the optimal filter takes the standard form as a copy of the original dynamics driven by the innovations $\tilde{\psi}(t)$

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This is dynamic compensation.

Error Dynamics

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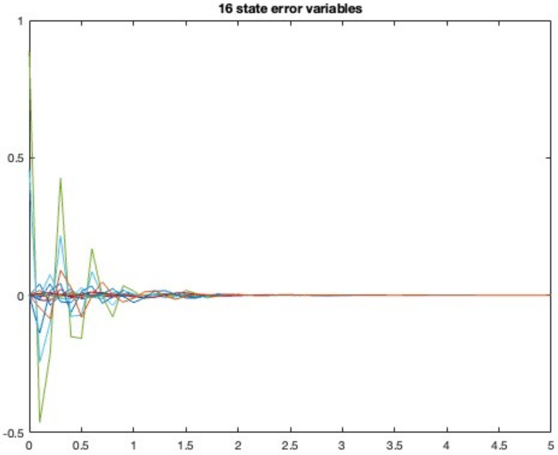
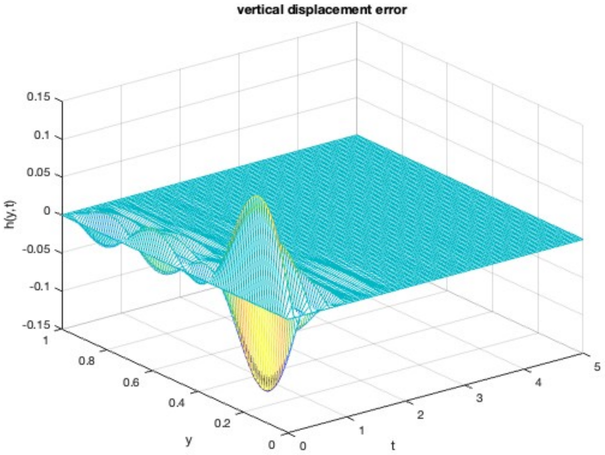
If the full state feedback aymptotically stabilizes the system and if the error dynamics of the Kalman filter is asymptotically stable then the dynamic compensator aymptotically stabilizes the system.

Error



Vertical Displacement Estimation Error



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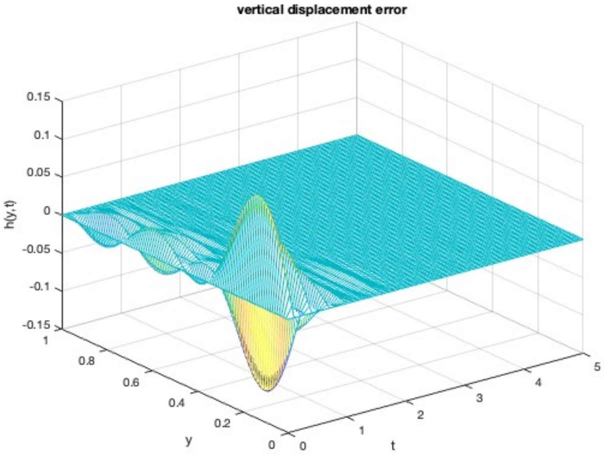
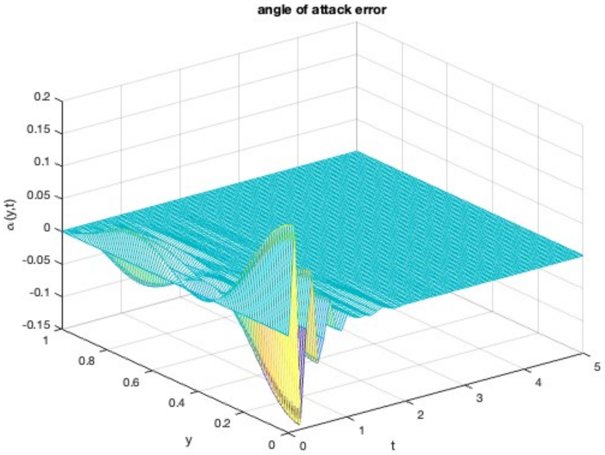


Notice that the estimation error is smaller near the sensor at the tip, $y = 1$.

Angle of Attack Estimation Error



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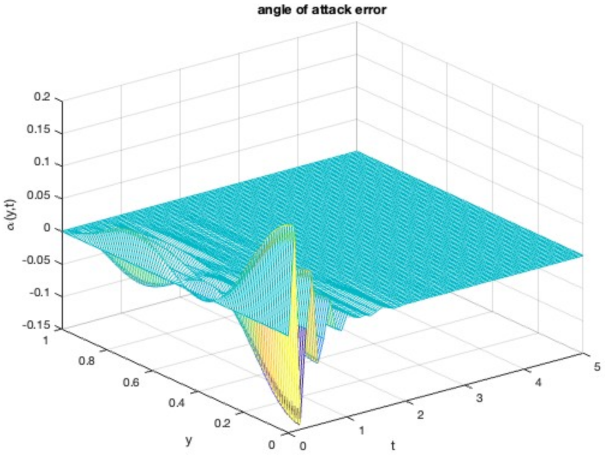

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We will treat this with nonLinear nonQuadratic Regulation (nLnQR).

Wagner's Model in State Space Form

The lift per unit span is

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\begin{aligned}
L &= \rho_\infty \pi b \left(-\ddot{h} + U_\infty \dot{\alpha} - ba\ddot{\alpha}\right) \\
&\quad + \pi \rho_\infty U_\infty \left(-\dot{h} + U_\infty \alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right) \\
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where ψ is the aerodynamic output.

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The aerodynamics state is $\boldsymbol{\xi} = [\boldsymbol{\xi_1}, \boldsymbol{\xi_2}]^T$.

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Model from Hossein Modaress-Aval et al, 2019.
Constants from Brunton and Rowley, 2012.

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This will raise the state dimension from four to six. We redfine the state to be

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But the aerodynamics introduces no new partial diffential operators so we will expand in the same eigenfunctions $\Phi_m(y), \Theta_n(y)$.

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Think Mathematically

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Act Computationally

Thank You

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Questions

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