

# Obstructions to feedback stabilization<sup>1</sup>

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## Other feedback stabilization work

- ▶ **discontinuous** (Clarke, Ledyaev, Sontag, Subbotin 1997),
- ▶ **exponential** (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- ▶ **time-varying** (Coron 1992),
- ▶ **global** (Byrnes 2008, Baryshnikov 2023)
- ▶ **practical** (Hamzi, Krener 2004),
- ▶ **infinite-dimensional** (Krstic, Bekiaris-Liberis 2013),
- ▶ **hybrid** (Sanfelice, Teel, Goebel 2008),
- ▶

versions of the stabilization problem are not considered here.

### Some excellent introductions / surveys:

- ▶ Sontag (1990). Feedback stabilization of nonlinear systems.
- ▶ Krener (1994). Nonlinear stabilizability and detectability (§2).
- ▶ Sontag (2009). Stability and feedback stabilization.
- ▶ Jongeneel, Moulay (2023). Topological obstructions to stability and stabilization.



# Obstructions to feedback stabilization

Brockett's obstruction and beyond

Coron's & Mansouri's obstructions and beyond  
descendents of the homotopy theorem

relationships within the fiber bundle picture of control

the curious case of periodic orbits



## Limitations of the classical results

The results of Brockett, Coron, Mansouri rely on parallelizability of  $\mathbb{R}^n$  to view vector fields and control systems as  $\mathbb{R}^n$ -valued.

Moreover, they apply only to the case of stabilizing a point or more generally submanifolds of  $\mathbb{R}^n$  w/ nonzero Euler characteristic.

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

How to test for stabilizability in such general settings?<sup>2</sup>

- ▶ **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- ▶ **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).
- ▶ **Relationships between these** (MDK arXiv:2312.16752).

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<sup>2</sup>An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.



# Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \quad (1)$$

where  $X \ni x$  is a smooth manifold and  $f$  is smooth.

1. **Controllability<sup>3</sup> problem:** Given  $a, b \in X$ , find  $u(t)$  s.t.  $x(T) = b$  if  $x(0) = a$  for some  $T > 0$ .

$$a \rightsquigarrow b$$

2. **Stabilizability problem:** Given a compact subset  $A \subset X$ , find smooth  $u(x)$  s.t.  $A$  is **asymptotically stable<sup>4</sup>** for the **closed-loop vector field**  $F(x) = f(x, u(x))$ . [▶ Link](#)

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<sup>3</sup>Hermann and Krener (1977): definitive nonlinear controllability results.

<sup>4</sup> $A$  is invariant, and for every open  $W \supset A$  there is an open  $V \supset A$  s.t. all forward  $F$ -trajectories initialized in  $V$  are contained in  $W$  and converge to  $A$ .



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# The stabilization conjecture and Brockett's obstruction

Often  $A = \{x_*\}$  is a point,  $X = \mathbb{R}^n$  in the stabilization problem.

**Stabilization conjecture (pre-1983):** a reasonably strong form of controllability implies smooth stabilizability of a point.

**Example:** the “Heisenberg system” or “nonholonomic integrator”

$$\left. \begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?



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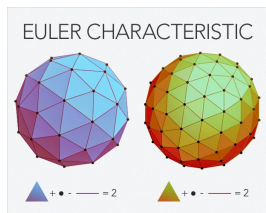
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**Theorem (Brockett).** If a point is stabilizable, then  $\text{image}(f)$  is a neighborhood of 0. (In the example,  $(0, 0, \varepsilon) \notin \text{image}(f)$ .)

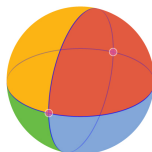


# A primer on the Euler characteristic<sup>5</sup>

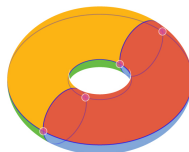
Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



Euler Characteristic ( $\chi$ ) = Faces + Corners - Edges



$$\chi = 4 + 2 - 4 = 2$$



$$\chi = 4 + 4 - 8 = 0$$

**Notation:**  $\chi(S) :=$  Euler characteristic of  $S$ .

**Examples:**  $\chi(\bullet) = 1$ ,  $\chi(\mathbb{S}^1) = 0$ ,  $\chi(\mathbb{S}^2) = 2$ ,  $\chi(\text{figure 8}) = -1$

**Theorem (Poincaré, Hopf):** if  $S$  is a compact smooth manifold with boundary  $\partial S$ , then  $\chi(S) = 0 \iff$  there exists a nowhere-zero smooth vector field on  $S$  pointing inward at  $\partial S$ .

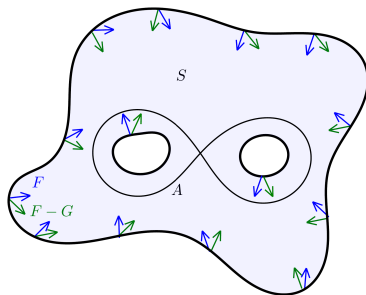
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<sup>5</sup>Figures from Quanta Magazine.



# Proof of (an extension of) Brockett's obstruction

**Adversary thm (MDK & Koditschek 2022):** Let cpct  $A \subset X$  be stabilizable. Then  $\chi(A)$  is well-defined. If  $\chi(A) \neq 0$ , then for any small enough vector field  $F$ ,  $F(x_0) = f(x_0, u_0)$  for some  $x_0, u_0$ .



**Proof:** Assume  $\exists$  stabilizing  $u(x)$ . Define  $F(x) := f(x, u(x))$ .  
Lyapunov function theory  $\implies \exists$  compact smooth domain  $S \supset A$   
s.t.  $F$  points inward at  $\partial S$  and  $\chi(S) = \chi(A) \neq 0$ . Continuity  $\implies$   
 $F - G$  points inward at  $\partial S$  if  $G$  is small  $\implies F - G$  has a zero by  
Poincaré-Hopf  $\implies \exists x_0$  s.t.  $G(x_0) = F(x_0) = f(x_0, u(x_0))$ .



## Examples

### Heisenberg system

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\quad (2)$$

### Kinematic differential drive robot

$$\begin{aligned}\dot{x} &= u \cos \theta \\ \dot{y} &= u \sin \theta \\ \dot{\theta} &= v\end{aligned}\quad (3)$$

The right side of (2)  $\neq F_\varepsilon := (0, 0, \varepsilon)$  for any  $\varepsilon > 0$ .

The right side of (3)  $\neq F_\varepsilon := (\varepsilon \sin \theta, -\varepsilon \cos \theta, 0)$  for any  $\varepsilon > 0$ .

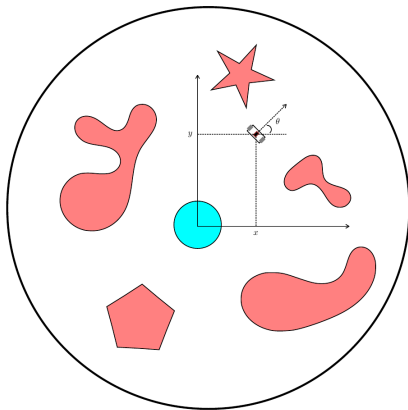
so adversary thm  $\implies A$  not stabilizable if  $\chi(A) \neq 0$ . E.g., if  $A$  is a **stabilizable connected submanifold,  $A$  is a circle or torus (!)**

**Other applications:** any stabilizable compact set has zero Euler characteristic for satellite orientation with  $\leq 2$  thrusters, for nonholonomic dynamics with  $\geq 1$  global constraint 1-form,...



## Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some “bad” set immediately enter some “good” set.



E.g., impossible for this differential drive robot to aim within  $\pm 179$  degrees of the origin while “strictly” avoiding obstacles via  $u(x)$ .



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## Coron's & Mansouri's obstructions...

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : f(x, u) \neq 0\}.$$

**Theorem (Coron 1990).** If  $n > 1$  and a point is stabilizable,

$$(\mathbb{Z} \cong) \quad H_{n-1}(\mathbb{R}^n \setminus \{0\}) = f_*(H_{n-1}(\Sigma)).$$

↑

**Theorem (Mansouri 2010).** If a closed codimension  $> 1$  submanifold  $A \subset \mathbb{R}^n$  is stabilizable,

$$(\chi(A) \cdot \mathbb{Z} \cong) \quad \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \subset f_*(H_{n-1}(\Sigma)).$$



...and beyond

$$\Sigma := \{(x, u) : f(x, u) \neq 0\}$$

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↑

Let compact  $A \subset X$  be attractor for *some* smooth vector field  $G$ .

**Homology theorem (MDK 2023).** If  $A$  is stabilizable, then for all small enough neighborhoods  $N \subset X$  of  $A$ ,

$$G_* H_\bullet(N \setminus A) \subset f_* H_\bullet(\Sigma_{N \setminus A}) \text{ in } H_\bullet(T_{N \setminus A} X \setminus 0_{TX}).$$



## Descendents of the homotopy theorem

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↑

**Lemma (MDK 2023).** If  $A$  is asymptotically stable for smooth vector fields  $F, G$ , then for all small enough nbhds  $N \subset X$  of  $A$ ,

$$G_* H_\bullet(N \setminus A) = F_* H_\bullet(N \setminus A) \text{ in } H_\bullet(T_{N \setminus A} X \setminus 0_{TX}).$$

↑

the homotopy theorem...



**Homotopy theorem (MDK 2023).** Let compact  $A \subset X$  be asymptotically stable for smooth vector fields  $F, G$ . There exists an open  $W \supset A$  s.t.  $F|_{W \setminus A}, G|_{W \setminus A}$  are homotopic through nowhere-zero vector fields.



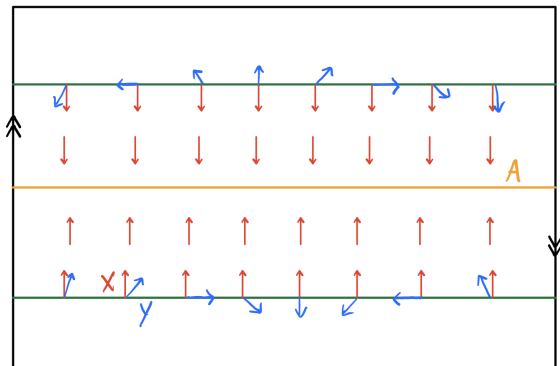
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**Remarks.**

- ▶ Also true with “smooth” replaced by “locally Lipschitz” or more generally “uniquely integrable continuous”
- ▶ For the following example the MDK 2023 results rule out stabilizability while the other results in this talk cannot.



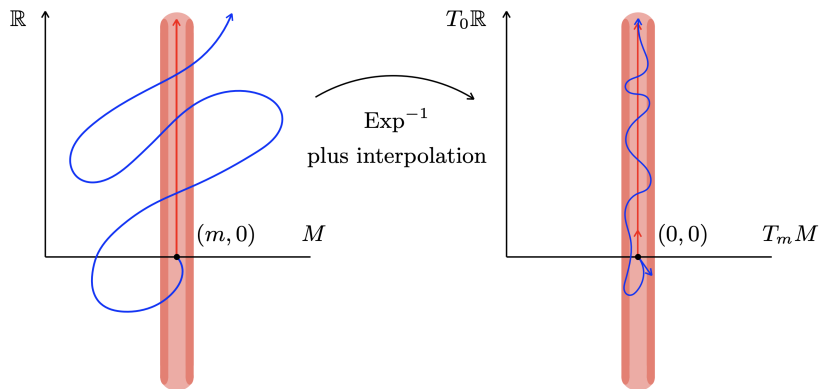
# Möbius strip example



$G \neq F$  since  $F \curvearrowright$  twice  
 around  $\bigcirc$  w.r.t.  $G$  while  $G \curvearrowright$   
 zero times w.r.t.  $G \Rightarrow$   $A$  is not  
asymptotically stable for  $F$  by the  
 homotopy theorem.



# Proof of the homotopy theorem





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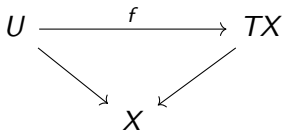
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## Relationships between the obstructions<sup>6</sup>

All MDK results actually apply in the **fiber bundle picture of control** (and much more generally...)



- ▶ Coron proved that his obstruction is stronger than Brockett's.
- ▶ **Q:** Is the homology theorem stronger than the adversary theorem?

**A:**

- ▶ Surprise: **NO** in the *fiber* bundle picture of control—they are independent (even for  $A = \{x_*\}$ ).
- ▶ **YES** in the *vector* bundle picture of control if  $\chi(A) \neq 0$  and  $X$  orientable.

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<sup>6</sup>MDK (2023). Relationships between necessary conditions for feedback stabilizability. arXiv:2312.16752



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## Can the new obstructions rule out stabilizability of periodic orbits?

If  $A$  = image of a periodic orbit with the same orientation for  $F$  and  $G$ , the straight-line homotopy over a sufficiently small open  $W \supset A$  satisfies the homotopy theorem's conclusion regardless of whether  $A$  is attracting, repelling, or neither for  $F$  or  $G$ .

**$\implies$  homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, **all other results in this talk also give no information.**



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...Could it be that periodic orbits might be “easy” to stabilize?



# Periodic orbits are sometimes easier to stabilize

Indeed, at least *sometimes*:

**Theorem (Anthony M. Bloch & MDK, in preparation).**

For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!



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**THANK YOU**