Attitude control and No-go theorems:

differential topology + hybrid systems

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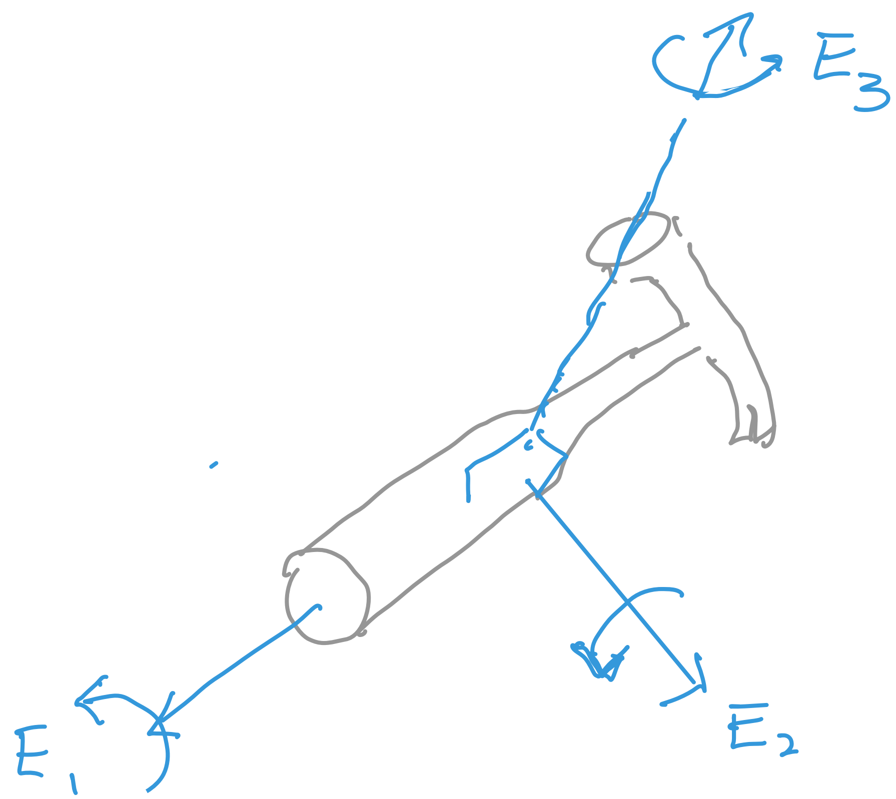
My favorite attitude control problem!

but we will work instead on:

$SO(3)$



State space for a rigid body in Euclidean 3-space.
Origin of the 3-space: the body's center of mass.
Axes: parallel to the axes of an inertial frame.

$$\dot{g} = g\omega$$

$$\dot{\omega} = \omega \times \mathbb{I}\omega + \tau$$

Kinematic control:

$$\dot{g} = gu \qquad g \in G$$

Feedback: $u = u(g)$.

Question: Can we continuosly feedback stabilize the system to $g = I =$ Identity?

Answer. NO !

This 'No' is No-go theorem 1.

Why not?
The flow of a smooth feedback stablizer , stabilizing to a desired point ("attitude") CONTRACTS the entire manifold onto that point.

$SO(3)$ is not contractible.

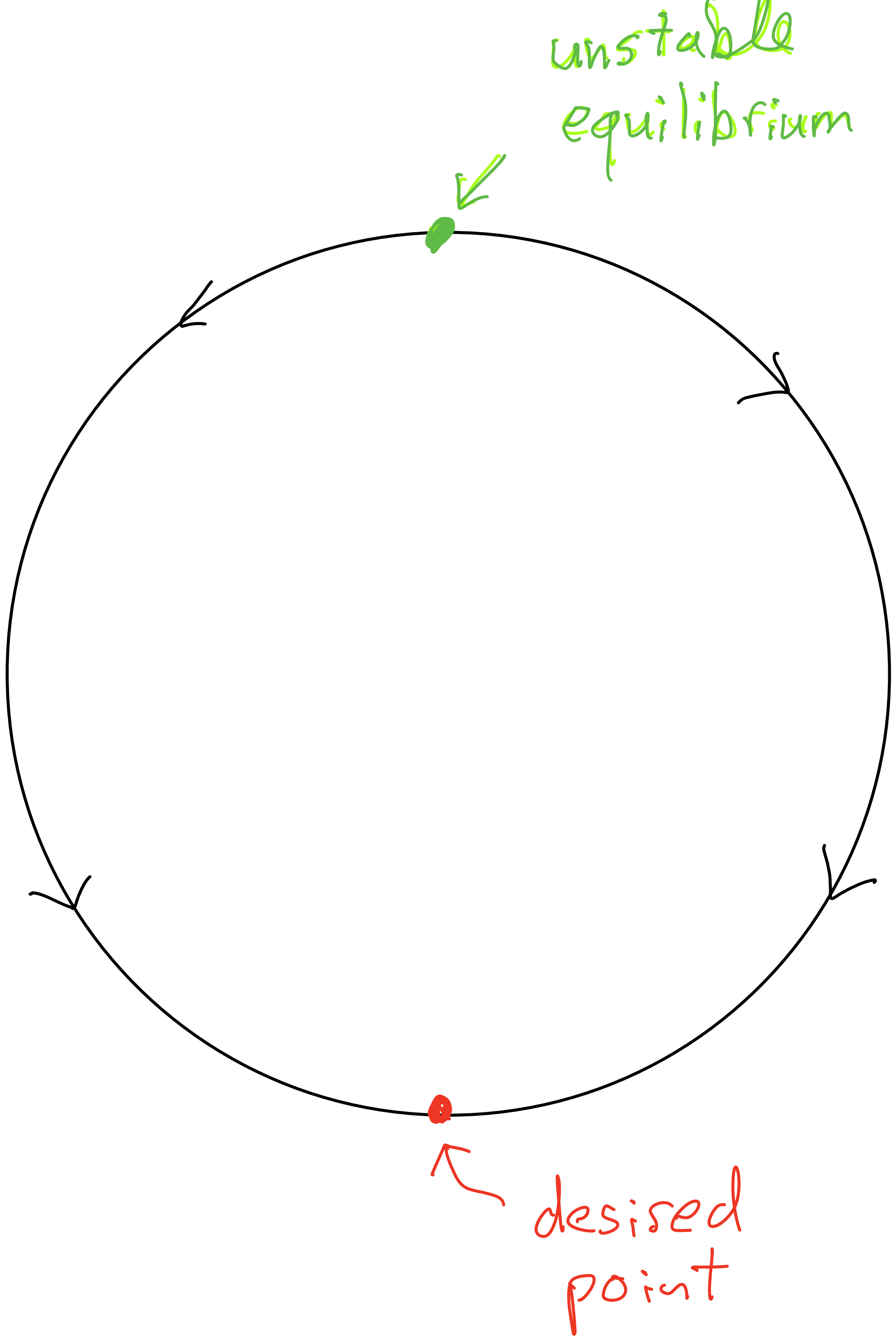
Any compact manifold without boundary is not contractible.

$\Longrightarrow$ $SO(3)$, along with all such manifolds, does not admit a continuous feedback stabilizer stabilizing to a point.

Try anyways

$$\mathbb{S}^1 = SO(2)$$

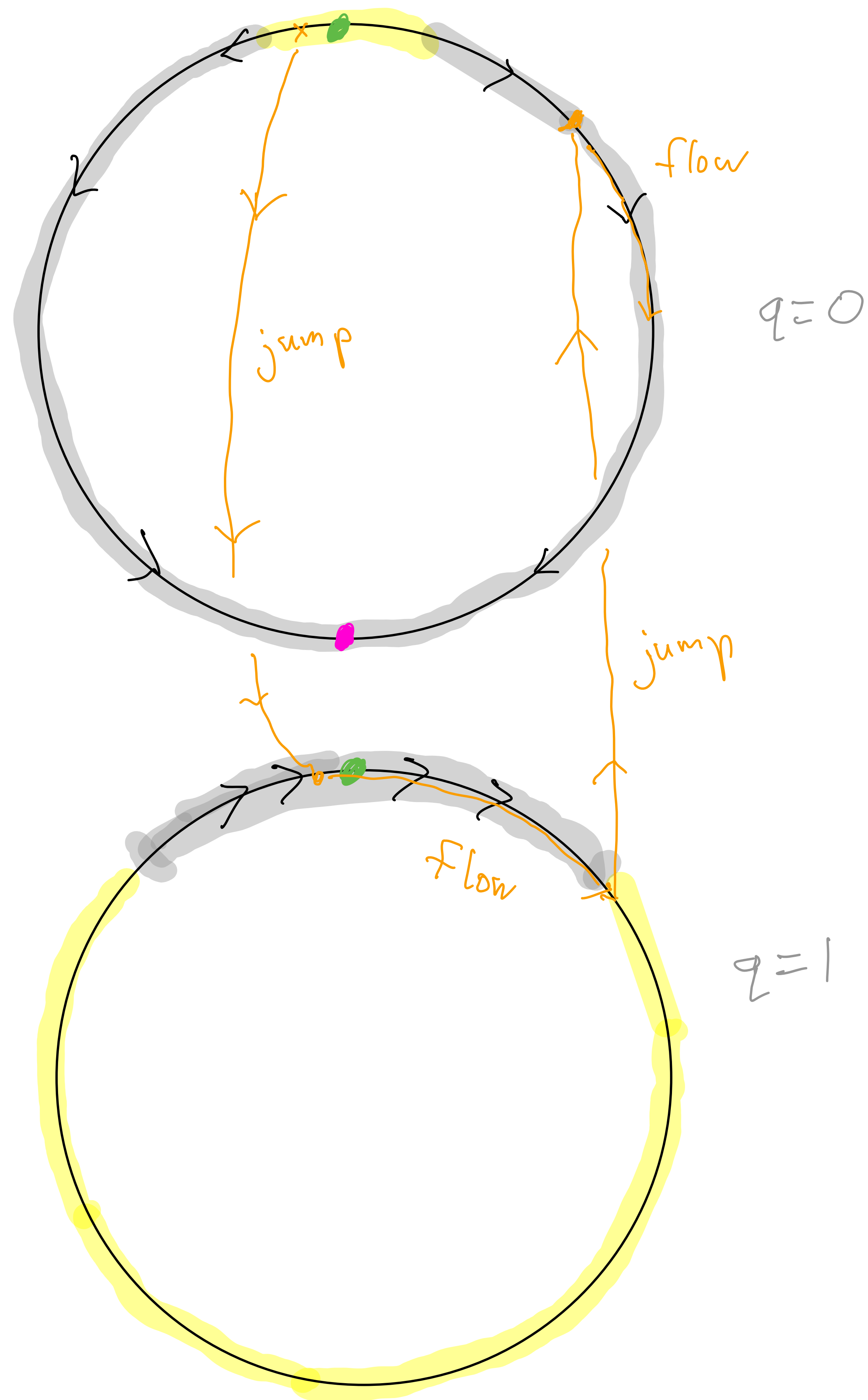
Use a gradient flow.



The entire circle minus the green point flows down to the red point

To stabilize all of $\mathbb{S}^1$ to m_0 , including the unstable equilibrium, "hybridize": introduce a "logic element" $q \in \mathbb{Z}_2 := \{0, 1\}$.

Replace $\mathbb{S}^1$ by $\mathbb{S}^1 \times \mathbb{Z}_2$.



flow
jump
jump
$q=0$
flow
$q=1$

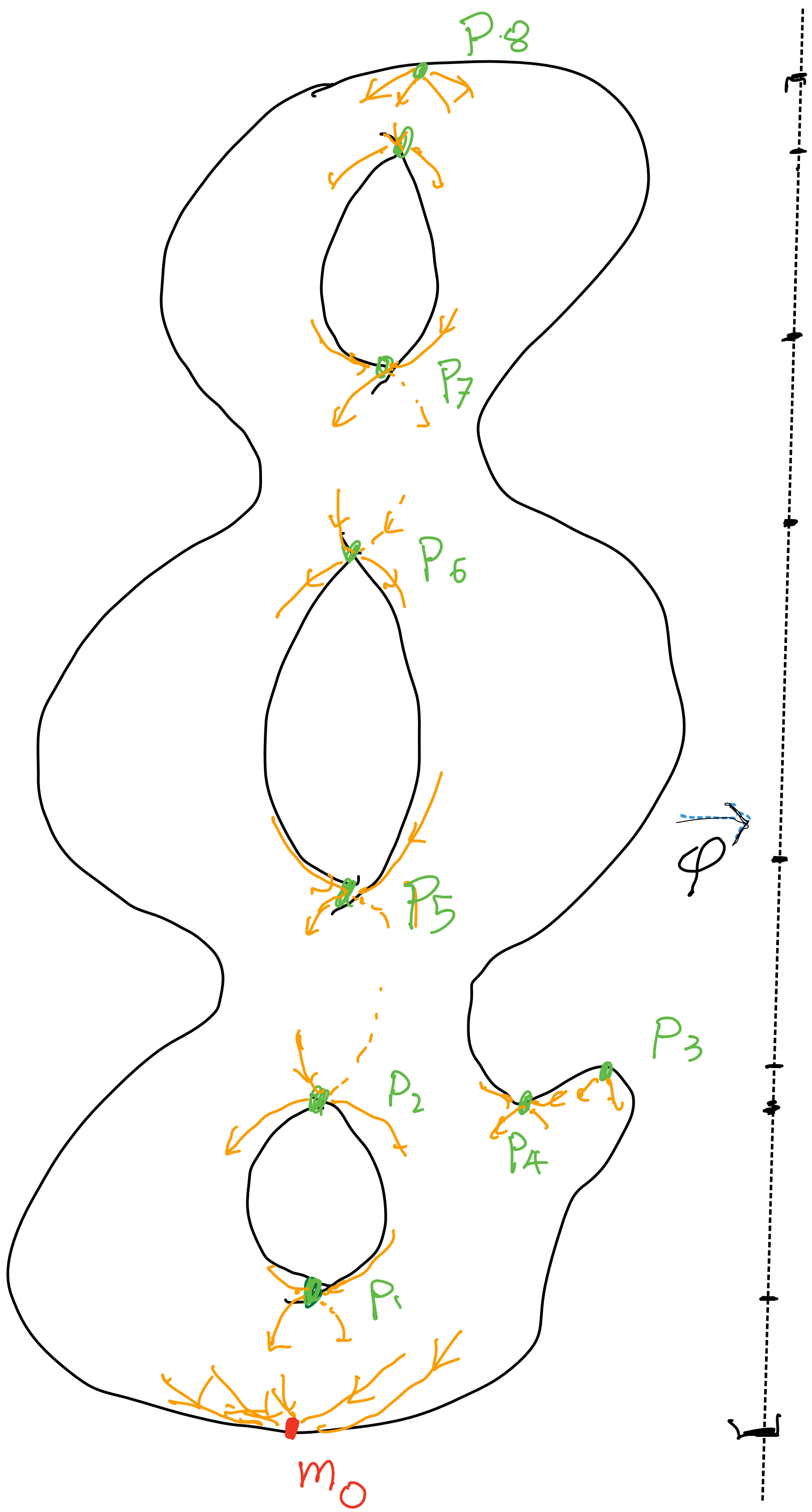
We can use MORSE THEORY to copy this example onto any manifold.

Theorem. Given a smooth connected manifold M and a goal point $m_0 \in M$ there is a hybrid system on $M \times \mathbb{Z}_2$ having $(m_0, 0)$ as global attractor.

Construction:

Starting point. MORSE THEORY yields a function ϕ on M whose unique global minimum is m_0 and whose critical point set is finite.

Result: A smooth flow on the manifold for which the desired fixed point is attractive and has basin of attraction an open dense subset of M of full measure.



p_8
p_7
p_6
p_5
p_3
p_2
p_4
p_1
m_0
φ

Instead of a single unstable equilibrium we get
k unstable equilibria where $k > 1$ unless M is a sphere.

MORSE THEORY

Definitions

A critical point p of a function ϕ is NON-DEGENERATE if the Hessian at p, is full rank.

A Morse function is a function all of whose critical points are non-degenerate.

Every smooth manifold admits a Morse function.

The critical points of a Morse function are ISOLATED hence finite in number when M is compact.

Given M connected, desired point $m_0 \in M$ choose a Morse function ϕ with m_0 as its global minimum.

Let $p_1, p_2, \ldots, p_k$ be the other critical points of ϕ

Off of the 'bad set' $B = \bigcup_{i=1}^{k} W^+(p_i)$ every point of M flows by

$$\dot{m} = -\nabla\phi(m)$$

to m_0.

The $W^+(p_i) =$ stable manifolds of the p_i are embedded submanifolds of codimension ≥ 1.
$\implies M \setminus B$ is open, dense, of full measure.

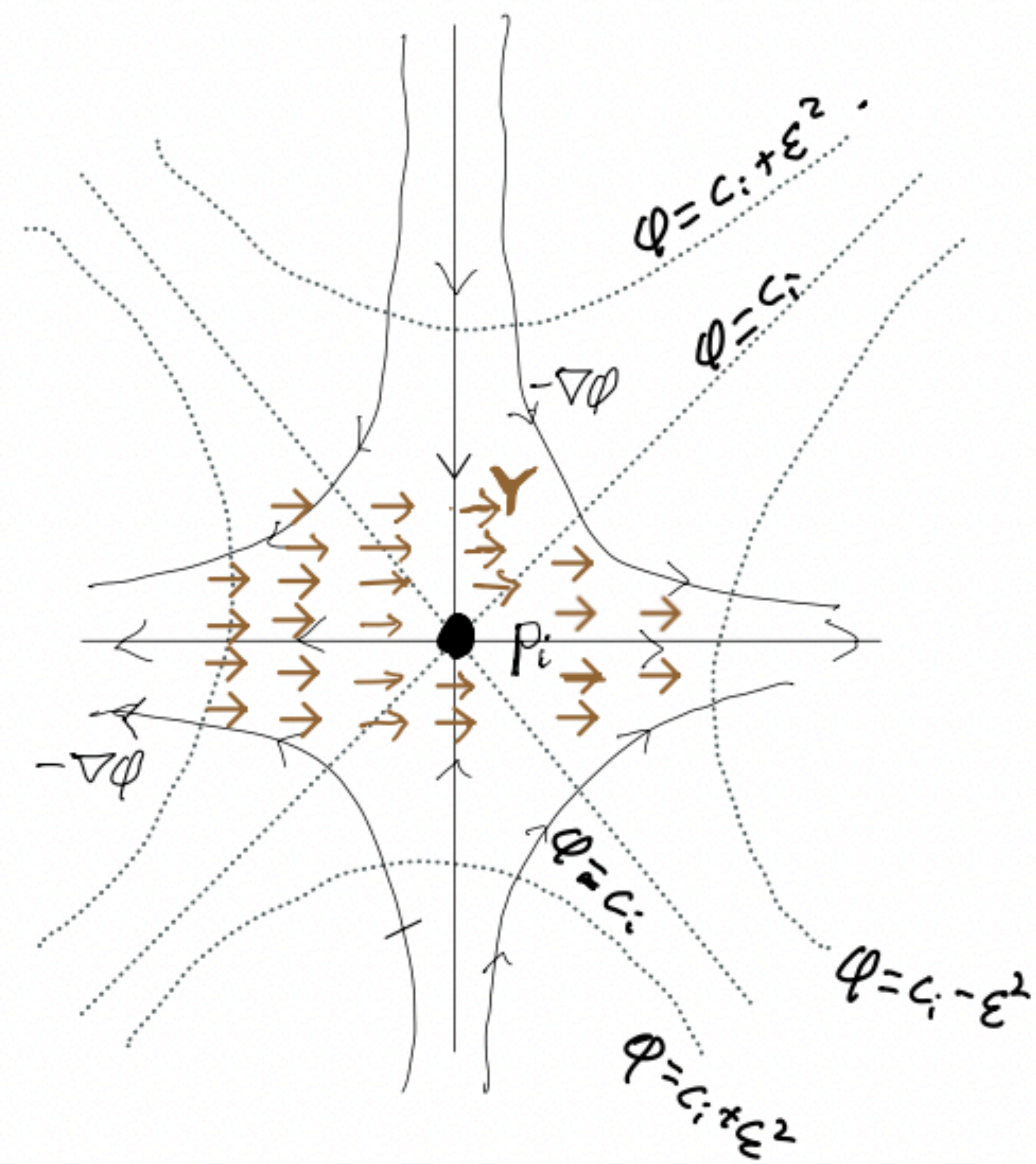
But we want ALL points to limit to m_0.

Design "gusts" blowing us past the unstable critical points p_i and their unstable manifolds.

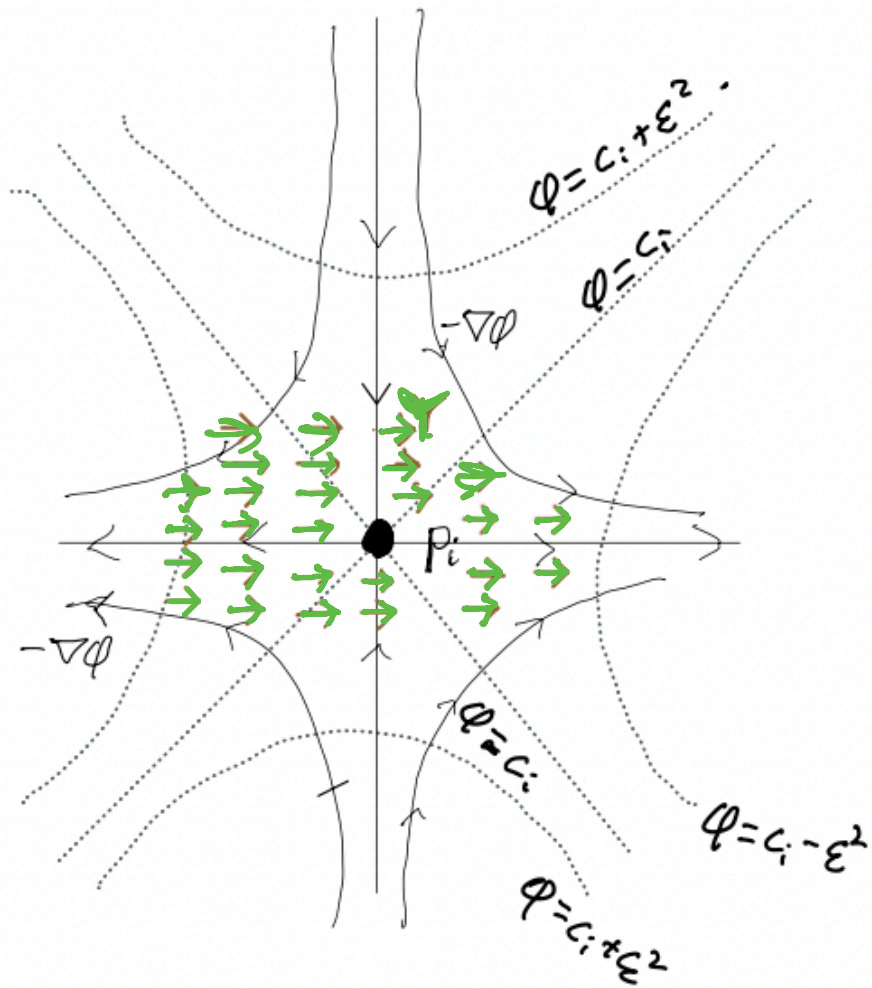
To do this use:

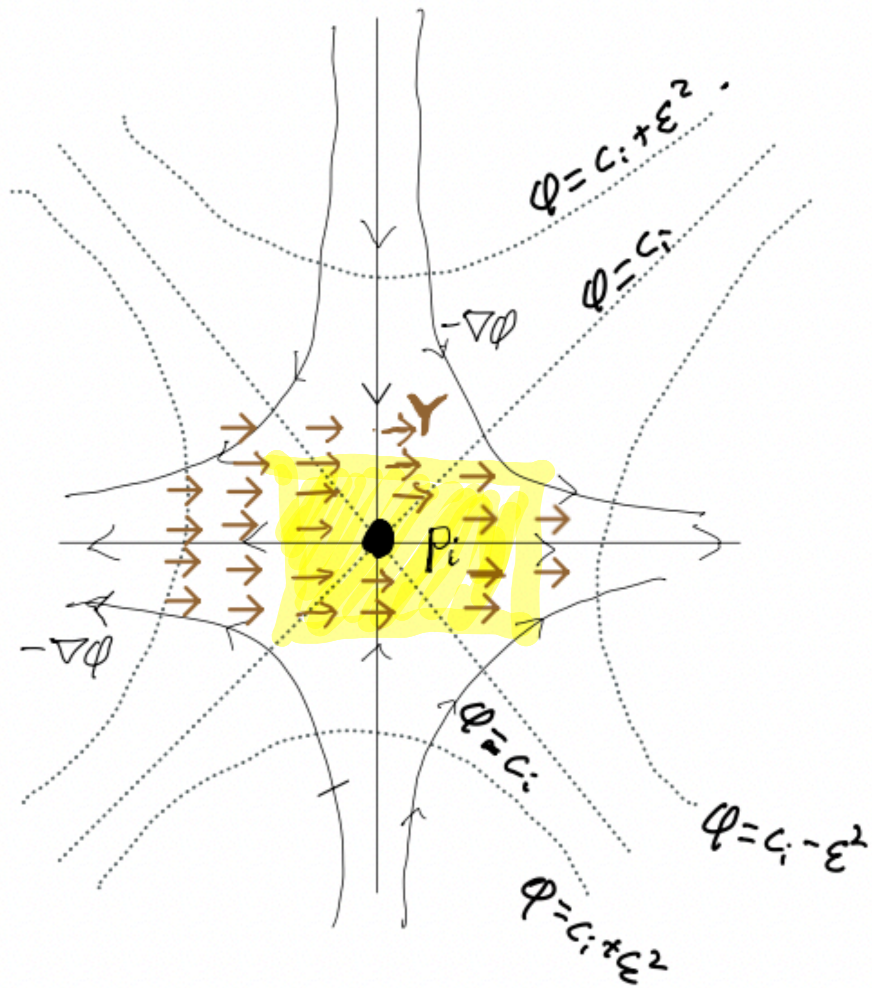
Morse Lemma. If p is a non-degenerate critical point for ϕ and $\phi(p) = c$ then $\exists$ coordinates $(x_i, y_j), 1 \leq i \leq k, 1 \leq j \leq n-k$ centered at $p \in M$ such that in these coordinates

$$\phi(x_i, y_j) = c + \sum x_i^2 - \sum y_j^2$$



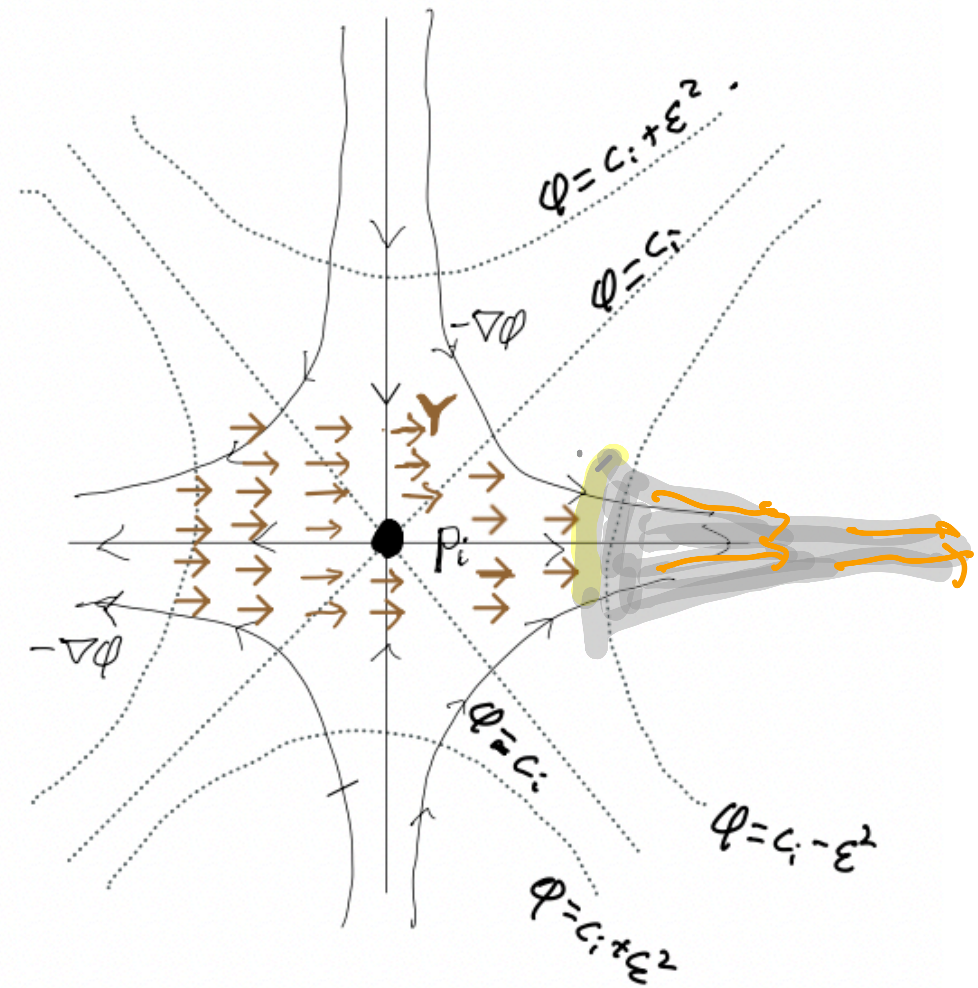
$\varphi = c_i + \varepsilon^2$
$\varphi = c_i$
$-\nabla\varphi$
p_i
$-\nabla\varphi$
$\varphi = c_i$
$\varphi = c_i - \varepsilon^2$
$\varphi = c_i + \varepsilon^2$


φ=c_i+ε^2
φ=c_i
-∇φ
p_i
-∇φ
φ=c_i
φ=c_i-ε^2
φ=c_i+ε^2



φ=c_i+ε²
φ=c_i
-∇φ
p_i
-∇φ
φ=c_i
φ=c_i-ε²
φ=c_i+ε²

Then flow w $-\nabla\varphi$



$$\dot{m} = \begin{cases} -\nabla\phi(m), \text{ if } q = 0 \text{ and } m \in Flow(0) \\ Y(m), \text{ if } q = 1 \text{ and } m \in Flow(1) \end{cases}$$

$$Jump(0) := M \setminus Flow(0)$$

$$Jump(1) := M \setminus Flow(1)$$

If $q = 0$ or $q = 1$ and if $m \in Jump(q)$ then JUMP:

$$\text{apply } (m, q) \mapsto (m, \bar{q})$$

$$\bar{1} := 0; \bar{0} = 1$$

$$Jump(0) \subset\subset Flow(1)$$

$$Flow(1) = \bigcup_{i=1}^{k} U(p_i) \text{ where } U(p_i) \text{ is a Morse nbhd of } p_i$$

$$Flow(1) \setminus Jump(0) = Flow(1) \cap Flow(0)$$

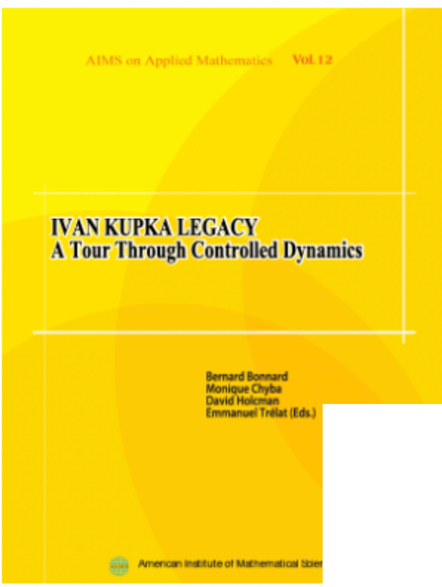
Theorem. The hybrid system on $M \times \mathbb{Z}_2$ just constructed has a single global attractor, the point $(m_0, 0)$, which is Lyapunov stable.
Moreover, this stability property is robust with respect to measurement and system errors.

Estimates regarding tolerances to measurement and system errors specified in terms of

$diam(Flow(1) \cap Flow(0))$, ϕ and $\|\nabla\phi\|$.

REFERENCES/ Advertising station break:

1. **google search: AIMS on Applied Math volume 12**



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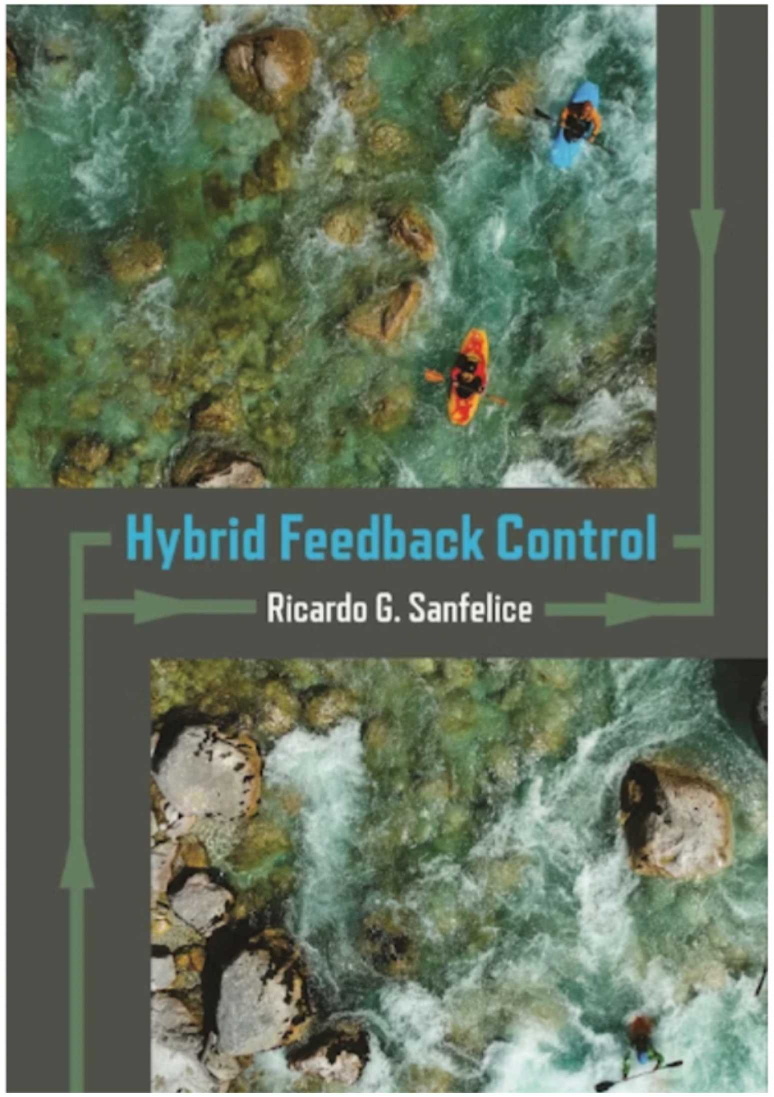
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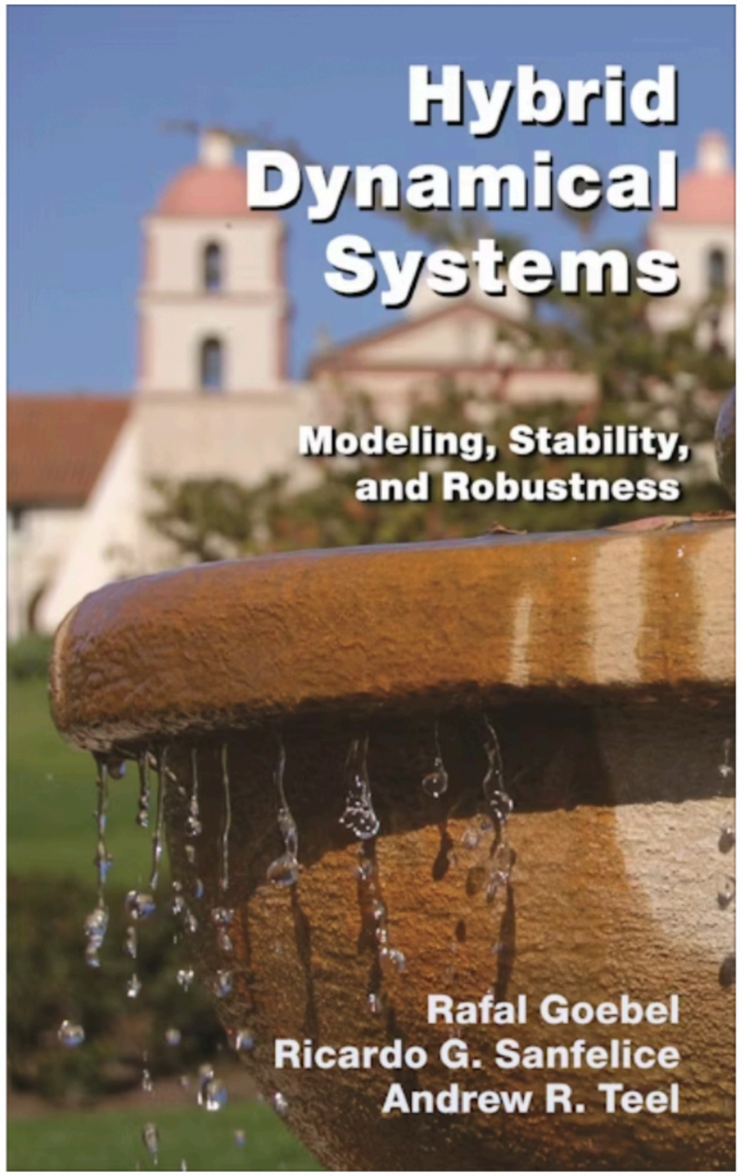
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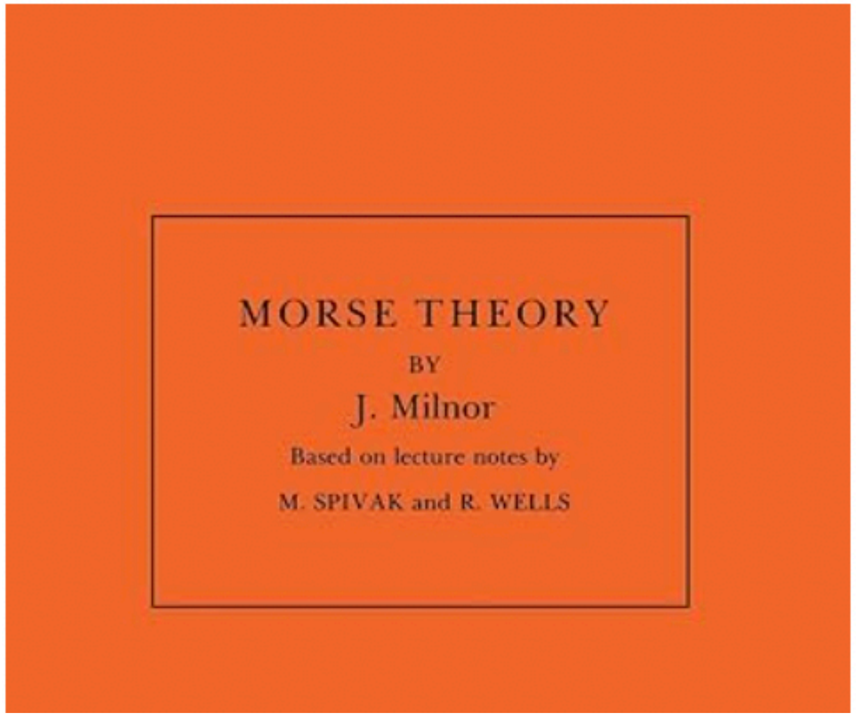

MORSE THEORY
BY
J. Milnor
Based on lecture notes by
M. SPIVAK and R. WELLS

QUESTIONS??

“Instead of considering this hybrid system, why not consider the equivalent situation [I THINK] of the discontinuous vector field obtained by using the gust vector field in the Morse neighborhoods of the unstable equilibria and the gradient vector field outside them?

“Wouldn’t the discontinuous v-field yield the same robustness and global stability properties?
If so, what is the advantage here of the hybrid viewpoint over simple discontinuous vector fields ? ”
-anonymous program attendee MK

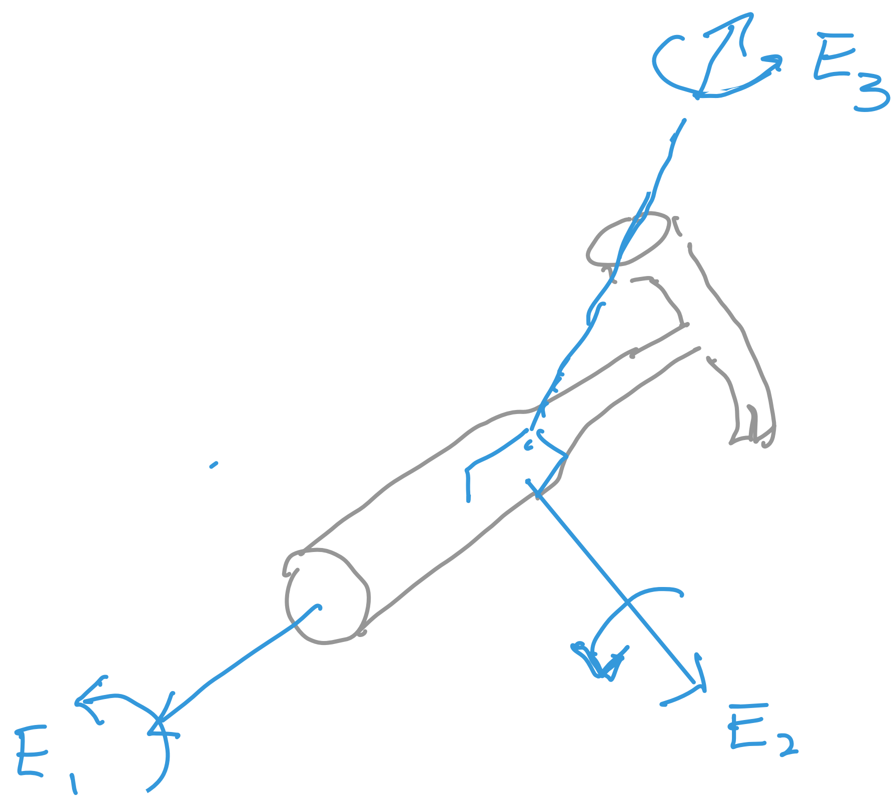
"Wouldn't the discontinuous v-field yield the same ~~robustness and~~ global stability properties?

YES!

Hybridizing allows the domains of the gust and gradient flows to overlap. The overlap is what yields robustness. Without overlap, we can construct chattering-type local attractors via `designer' noise, equilibria which kill the global stability of m_0.

Apply to $M = SO(3)$

$m_0 = Id \in SO(3)$



$exp(\theta E_i) =$ Rotation about axis of E_i counterclockwise by θ radians

WAY 1. Follow the above procedure.
Construct a Morse function on $SO(3)$

$$\phi(R) = tr(PR) \qquad ; P \text{ a fixed weight matrix}$$

If P is symmetric with distinct eigenvalues then ϕ is Morse.

Example: $P = diag(0, -1, -2)$
Compute: $d\phi(R) = 0 \iff R$ is diagonal.

$$\begin{aligned} SO(3) \cap \{diagonals\} &= \text{Klein 4 -group.} \\ &= \{diag(1,1,1), diag(-1,-1,1), \\ &\qquad diag(1,-1,-1), diag(-1,-1,1)\} \\ &= \{m_0, p_1, p_2, p_3\} \end{aligned}$$

Critical values of ϕ: $\quad -(\pm 0 \pm 1 \pm 2)$

with an even number of minus signs

$$3$$

$$1$$

$$-1$$

$$-3 \quad \leftarrow Id$$

Four is the minimum number of critical points a Morse function can have on $SO(3)$: $b_i(SO(3)) = 1, i = 0, 1, 2, 3$

Then implement Morse lemma near p_1, p_2, p_3 to construct gust field Y to blow away from nbhd(p_i)

Way 2. $\phi(R) = -tr(R)$

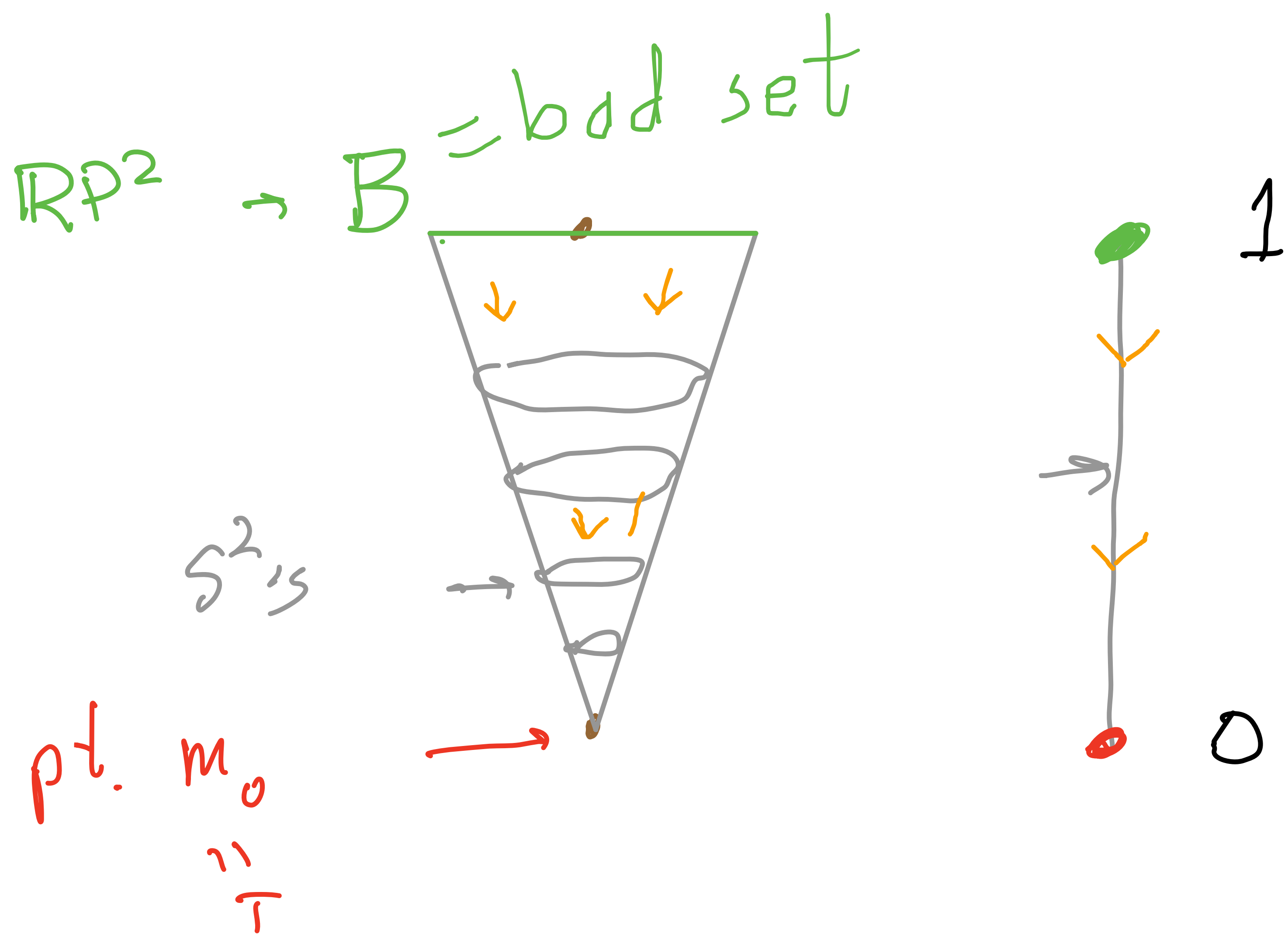
Critical values: points

 1 ; global max; a surface

-3 : global min $= Id =$ goal

Rescale

$$\phi(R) = \frac{1}{4}\left(3 - \operatorname{tr} R\right)$$

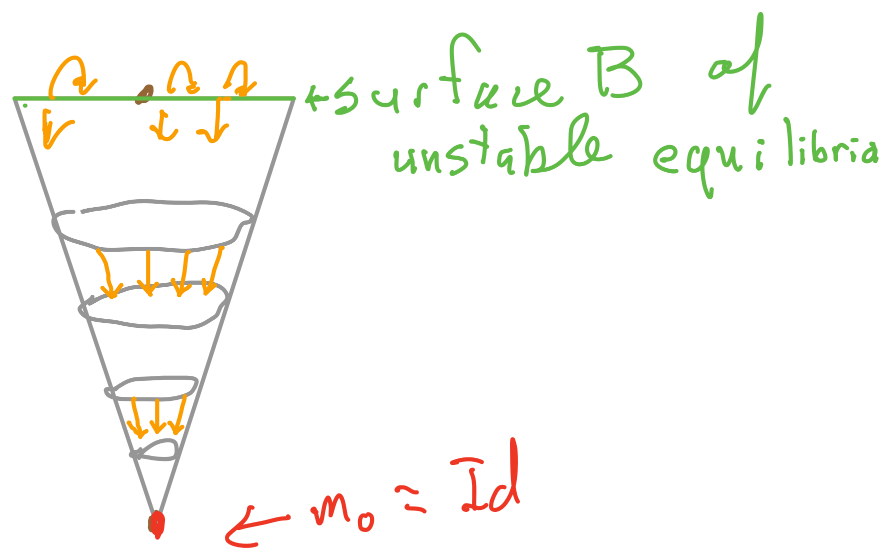


Cannot be Morse since crit pt set is not discrete.

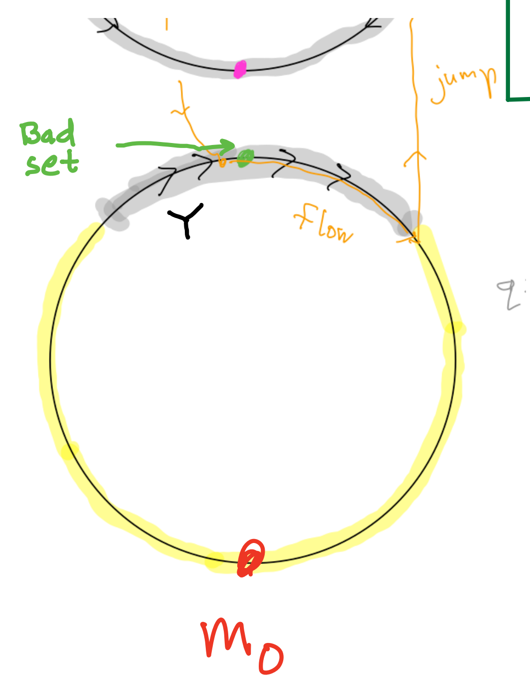
Rather φ is "Morse-Bott":

for $b \in B$, $d^2\varphi_b\big|_{(T_bB)^\perp}$ is nondeg. (<0)

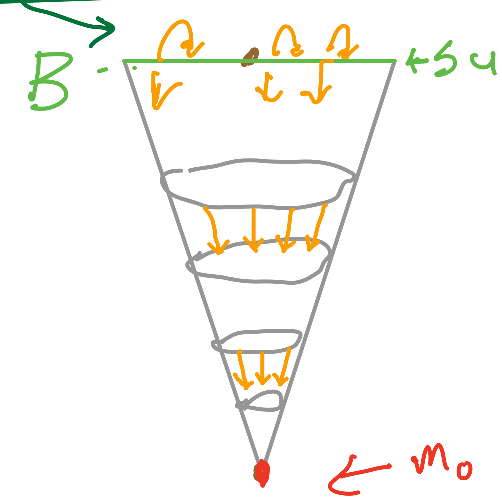
& only other crit. pt is I which is non degenerate (global min)



Try to copy S^1 game



¿Can we blow B away with a 'gust' $\psi_t = \exp(tY)$ of wind Y?



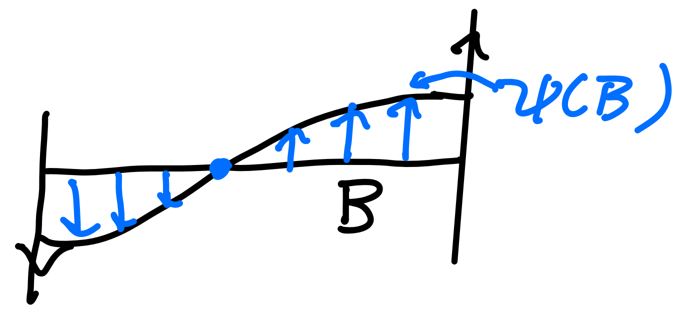
i.e. ¿∃ isotopy $\Psi = \psi_t$ such that
$\Psi(B) \cap B = \emptyset$?

No-go Theorem 2. This problem has no solution. Indeed if $\psi : SO(3) \to SO(3)$ is any homeomorphism then $\psi(B) \cap B \neq \emptyset$.

Picture.

$(B, SO(3)) \simeq (\mathbb{RP}^2, \mathbb{RP}^3)$ & the normal bundle of $\mathbb{RP}^2$ in $\mathbb{RP}^3$ is like the Möbius strip in $\mathbb{R}^3$: unorientable; $\nexists$ vanishing normal vector fields.

No!



No isotopies displacing B from itself.

Indeed no Homeomorphisms displace B from itself.

Proof. $(B, SO(3)) \cong (\mathbb{RP}^2, \mathbb{RP}^3)$.

For $j = 0, 1, 2, 3$: $H_j := H_j(\mathbb{RP}^3, \mathbb{Z}_2) = \mathbb{Z}_2$
with non-zero element $[\mathbb{RP}^j] \in H_j$
So $[\psi_* B] = [B] \neq 0$.

Intersection product $H_2 \times H_2 \to H_1$;
$[\mathbb{RP}^2] \cap [\mathbb{RP}^2] = [\mathbb{RP}^1]$
But if $\psi(B) \cap B = \emptyset$ then $[\mathbb{RP}^2] \cap [\mathbb{RP}^2] = 0$,
Contradiction.

Remark:

This homological "non-displacement theorem" is the non-linear version of the basic (linear) incident axioms of Projective geometry.

$\mathbb{RP}^3$ = real projective 3-space.
Any two distinct planes intersect in a unique line.

Planes: $\mathbb{RP}^2$ and its images under projective transformations (linear transformation of $\mathbb{R}^4$
Lines: $\mathbb{RP}^1$ and its and its images under projective transformations

Piyush P. Jirwankar went ahead anyways…

and got it to work! How could that happen?

I got the quantifiers wrong!

$$\exists t > 0 \, \forall b \in B \qquad \Psi_t(b) \notin B$$

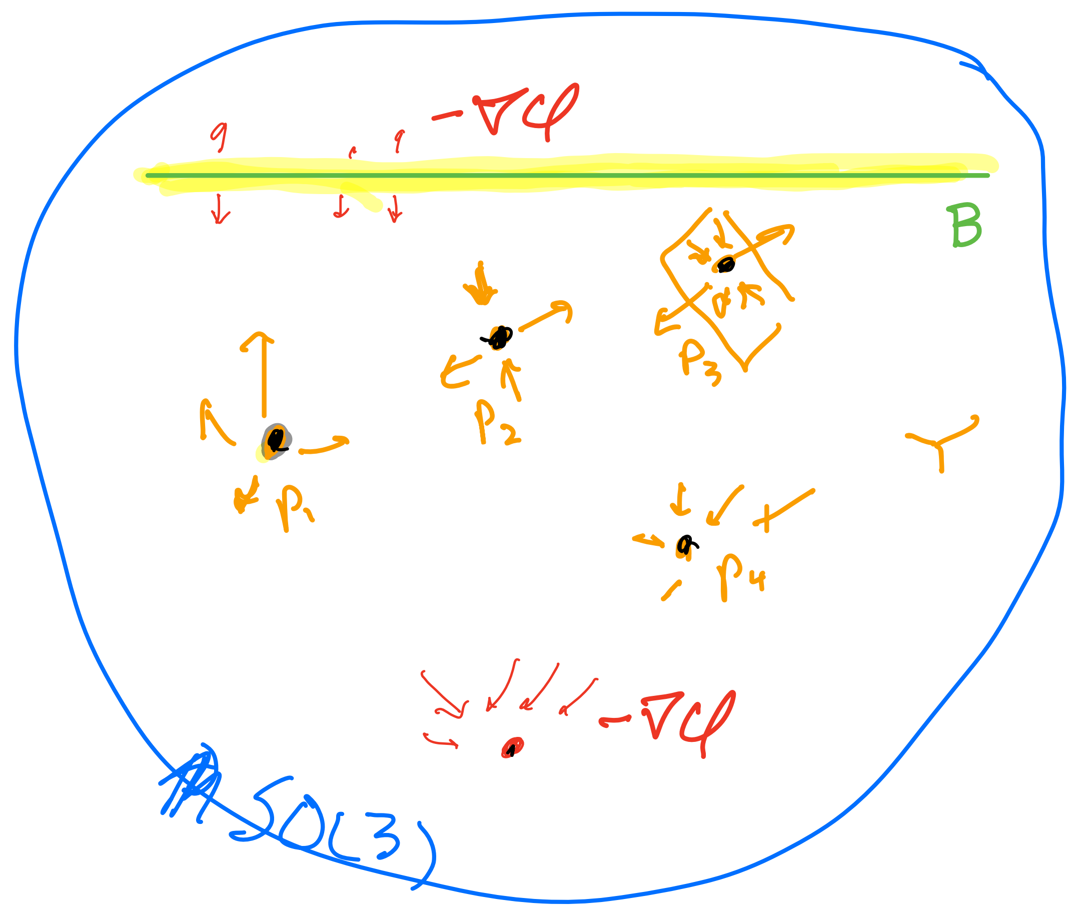
Impossible, by `No-go theorem 2'

But what we need:

$$\forall b \in B \; \exists t = t(b) > 0 \qquad \Psi_t(b) \notin B$$

Choose v-fld Y s.t.
the ω-limit set of ANY
point is disjoint from B.

Here $Y = -\nabla f$, $Crit(f) \cap B = \emptyset$



Key

M ∖ Flow(0)

cc Flow(1)

To construct Y, take any Morse fn whose critical pts are <u>disjoint</u> from B.

For example, 'rotate' our previous Morse fn

$$f(R) = tr(PR)$$

which had crit. pts

$$p_i = \exp(\pi E_i) \in B,$$

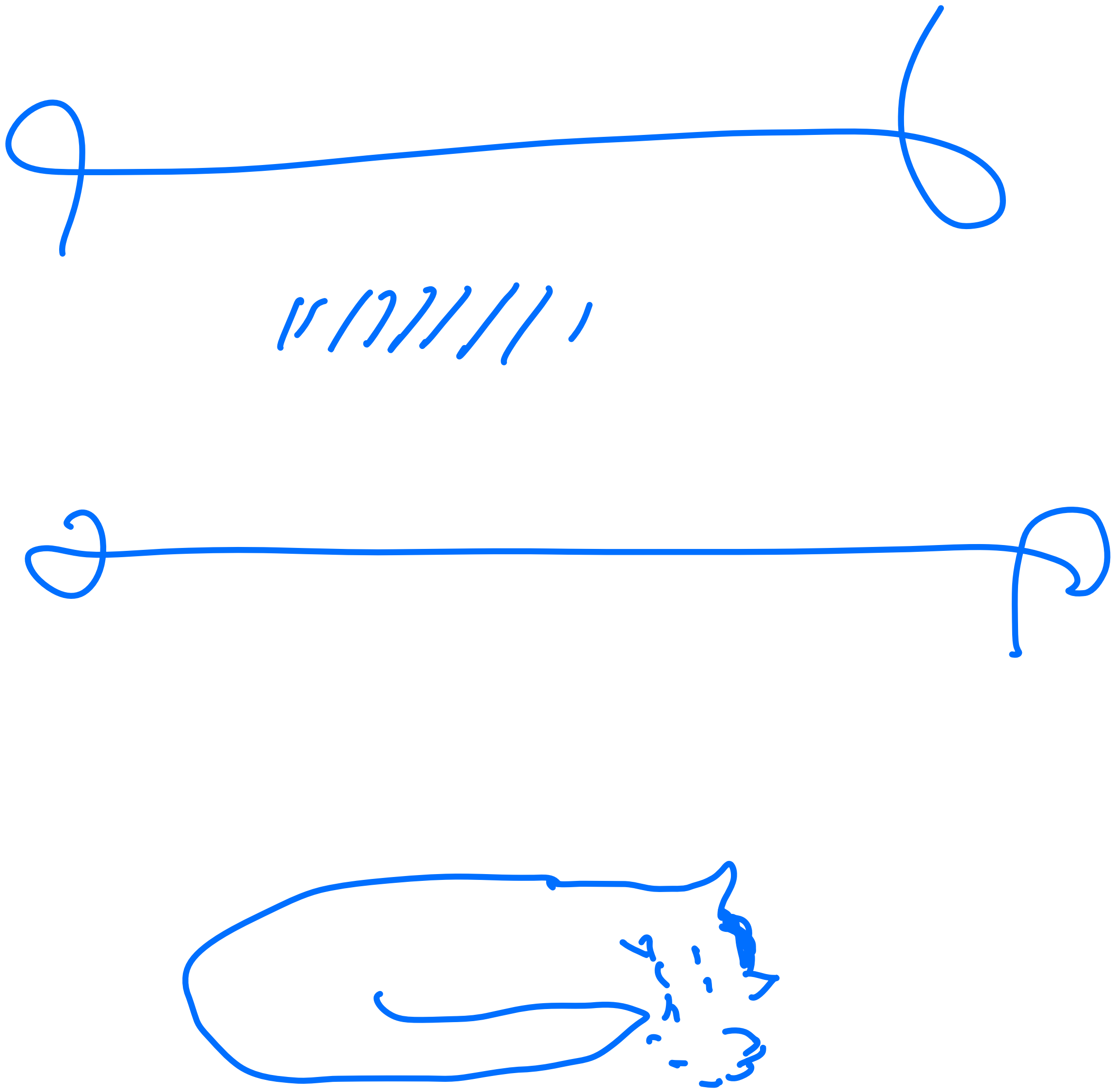
by setting

$$\tilde{f}(R) = tr(g_0 PR)$$

w/ $g_0 p_i \notin B$, $i = 1, 2, 3$.

& $g_0 I \notin B$.

a.e. g_0 works!



OVERFLOW

Alternatively

$$\mathbb{S}^3 = \text{unit quaternions}$$

$$SO(3) = \mathbb{S}^3 / \pm I = \mathbb{RP}^3$$

However the critical points are in the wrong place:
They are $Id, exp(\pi E_1), exp(\pi E_2), exp(\pi E_3)$
and $exp(\pi E_i) \in B$.

To knock them off of B just translate by almost any element $g_0 \in SO(3)$.

$$f(R) = tr(g_0 P R)$$

Any g_0 such that

$$g_0, g_0 exp(\pi E_1), g_0 exp(\pi E_2) g_0 exp(\pi E_3) \notin B$$

$$exp(0\vec{N}) = Id.$$
$$exp(\pi\vec{N}) = exp(\pi(-\vec{N}))$$
$$Tr(exp(\theta\vec{N})) = 1 + 2\cos(\theta).$$

$exp(B^3) = SO(3)$
where B^3 is the closed ball of radius π.
Moreoever exp is a diffeo on the interior of this ball.

View $(\theta, \vec{N}) \in [0, \infty) \times \mathbb{S}^2$ as spherical coordinates for $\mathbb{R}^3$

$$(\pi, \vec{N}) \sim (\pi, -\vec{N})$$

$$\implies SO(3) \cong (B^3/\sim) \cong \mathbb{RP}^3$$