

Optimal Control in Adversarial and Stochastic Environments

M. Pachter

AFIT

Wright-Patterson A.F.B., OH 45433

Collaborations: D. Casbeer, AFRL/RQ, E. Garcia AFRL/RQ, A. VonMoll, AFRL/RQ, I. Weintraub, AFRL/RQ, K. Pham AFRL/RV, AFIT Students

The views expressed on these slides are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the US Gov.

Autonomy & Adversarial Action

Three Pronged Attack

Main Thrusts

Optimization & Uncertainty

- Pattern Recognition is key to Autonomous Control

 - WTA with error prone FAC feedback

 - (With K. Kalyanam)

- Autonomous and resilient management of all-source sensors for navigation integrity – RAIM

 - (with K. Pham, AFRL/RV)

- Differential Games – optimal control in the face of adversarial action:

 - Complete Information, Complex Dynamics

 - (With D. Casbeer, E. Garcia, A. VonMoll, I. Weintraub, AFIT students)

Lady in the Lake

Martin Gardner, Scientific American, 1965

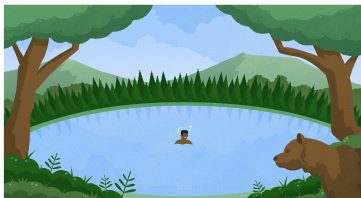
A young lady was vacationing on Circle Lake, a large artificial body of water named for its precisely circular shape. To escape from a man who was pursuing her, she got into a rowboat and rowed to the center of the lake, where a raft was anchored. The man decided to wait it out on shore. He knew she would have to come ashore eventually. Since he could run four times as fast as she could row, he assumed that it would be a simple matter to catch her as soon as her boat touched the lake's edge.

But the girl – a *mathematics* major at Radcliffe – gave some thought to her predicament. She knew that once she was on solid ground she could outrun the man: it was only necessary to devise a rowing strategy that would get her to a point on shore before he could get there. She soon hit on a simple plan, and her applied mathematics applied successfully.

What was the girl's strategy ?

Lady In The Lake

Man In The Lake¹



¹ “Mathematics can help you escape a hungry bear”, Quanta Magazine, Aug. 25 2021. [Link](#)

Lady in the Lake

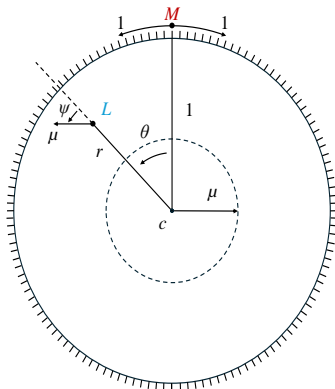


Figure: Notation

Lady In The Lake

Classical Lady in the Lake Scenario: Man (M) chases Lady (L)

State space:

$$\mathcal{R} = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$$

$$\dot{r} = \mu \cos \psi, \quad r(0) = r_0,$$

$$\dot{\theta} = \frac{\mu}{r} \sin \psi - \omega, \quad \theta(0) = \theta_0, \quad 0 \leq t \leq t_f,$$

$(r_0, \theta_0) \in \mathcal{R}$, $\psi \in [-\pi, \pi]$, and $\omega \in [-1, 1]$ and, without loss of generality, the radius of the lake is set to 1. The terminal manifold

$$\varphi(r, \theta) = r - 1 = 0$$

The Mayer cost/payoff functional is

$$J(\psi(\cdot, \cdot), \omega(\cdot, \cdot); r, \theta) = \theta_f,$$

which L wishes to maximize and M wishes to minimize.

The speed ratio parameter $\mu_{min} < \mu < 1$, μ_{min} T.B.D.

- Have DG with 2 states (r, θ) , 1 parameter μ .

Lady In The Lake

The *unique* optimal/equilibrium strategy for L is only defined when $r \geq \mu$.
 L 's optimal strategy: head away from the tangent to the circle of radius μ .

Results in a straight line in the realistic plane.

The (differentiable) Value function of the DG is

$$V(r, \theta) = \theta - \sqrt{\frac{1}{\mu^2} - 1} + \cos^{-1}(\mu) + \sqrt{\frac{r^2}{\mu^2} - 1} - \cos^{-1}\left(\frac{\mu}{r}\right).$$

$$\theta_T \equiv V(\mu, \pi)$$

$$\theta_T = \pi - \sqrt{\frac{1}{\mu^2} - 1} + \cos^{-1}(\mu)$$

L can only escape from the point $E = (\mu, \pi)$ if $\theta_T > 0$ which implies that $\mu > \mu_{\text{crit}} \approx 0.21723$. It is therefore stipulated that L 's speed

$$\mu > \mu_{\text{crit}} \approx 0.21723$$

Lady In The Lake

The special optimal/equilibrium trajectory which departs from E and exits the lake is B – in the parlance of DGs, it is a *Barrier*.

$$B(r) = \pi - \sqrt{\frac{r^2}{\mu^2} - 1} + \cos^{-1}\left(\frac{\mu}{r}\right), \quad r \in [\mu, 1].$$

If the state is such that $\theta < B(r)$ then $\theta_f < \theta_T \Rightarrow$ It is better for L to navigate to the point E and depart along B in order to achieve $\theta_f = \theta_T$. All equilibrium/optimal trajectories for the classical solution are above B . The large blank area of the state space for which no unique equilibrium trajectory exists is where L is prescribed to swim to the point E and subsequently take the B trajectory.

The curve B is a *barrier* surface – neither agent can steer the state of the system towards or across the surface B on their opponent's respective side.

- If the state (r, θ) , is below B , then L cannot force the state onto B .
- If the state is above B then M cannot force the state onto B .
- That's why she is prescribed to swim to E first and then take B .

Lady in the Lake

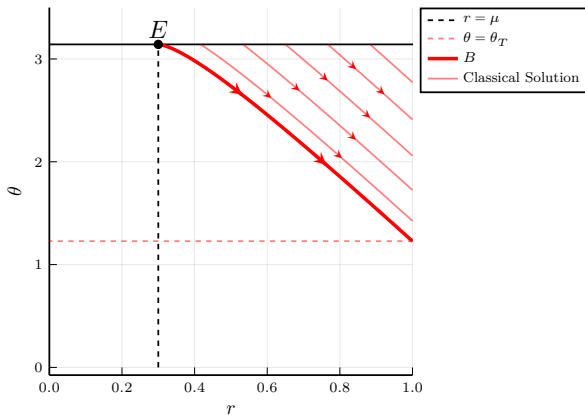


Figure: Classical Solution

Lady In The Lake

Min-Max Time to Reach the Antipodal Point E

Objective Cover the terra incognita territory: construct the optimal flow field in the region of the state space below the barrier B .

Will populate this region with trajectories which reach the point $E = (\mu, \pi)$ such that the time spent getting there is in equilibrium w.r.t. the two agents' control strategies. Singular surfaces crop up:

FL

On the FL L 's equilibrium control keeps the state on the line $\theta = \pi$ – she chooses the heading, ψ , s.t. $\dot{\theta} = 0$:

$$\sin \psi_{FL} = \frac{r}{\mu},$$

M 's equilibrium control is

$$\omega_{FL} = 1.$$

Lady In The Lake

FL Tributaries

Lemma

*Let s denotes the value of r wherein the state enters the FL.
The equilibrium heading for L along FL tributaries is given by*

$$\cos \psi^* = \pm \sqrt{1 - \frac{s^4}{\mu^2 r^2}}, \quad \sin \psi^* = \frac{s^2}{\mu r}.$$

The equilibrium control for M along FL tributaries is given by

$$\omega^* = 1.$$

*The FL tributaries enter the FL tangentially.
Equilibrium FL tributaries are straight lines in the realistic plane. \square*

Lady In The Lake

UL

Lemma

There is a Universal Line (UL) given by

$$\mathcal{U} = \{(r, \theta) \mid 0 \leq r \leq 1, \theta = 0\},$$

The strategies: L heads to the center of the lake, M doesn't move, i.e.,

$$\cos \psi_{UL} = -1, \quad \omega_{UL} = 0.$$

The equilibrium headings for L and M along UL tributaries is given by

$$\cos \psi = -1, \quad \omega = -1.$$

The UL tributaries are $\theta(r; s) = \frac{1}{\mu}r - \frac{1}{\mu}s, \quad 0 < s \leq 1.$ □

Lady In The Lake

Lemma

The straight line segment/Archimedian spiral.

$$\mathcal{P} = \left\{ (r, \theta) \mid 0 \leq r \leq 1, \theta = \frac{r}{\mu} \right\}$$

partitions the state space into two regions: one where FL tributaries exist and are optimal and one where UL tributaries exist and are optimal.

The two regions are mutually exclusive.



Lady In The Lake

Theorem The solution to the DG of max-min time to reach the antipodal point E is given by the following optimal/equilibrium control strategies and associated Value function.

$$(\cos \psi^*, \sin \psi^*) = \begin{cases} \left(\sqrt{1 - \frac{r^2}{\mu^2}}, \frac{r}{\mu} \right) & \text{if } \theta = \pi, \\ (-1, 0) & \text{if } \theta \leq \frac{r}{\mu}, \\ \left(\pm \sqrt{1 - \frac{r_f^4}{\mu^2 r^2}}, \frac{r_f^2}{\mu r} \right) & \text{otherwise.} \end{cases}$$

$$\omega^* = \begin{cases} 1 & \text{if } \theta > \frac{r}{\mu} \\ 0 & \text{if } \theta = 0 \\ \text{undef.} & \text{otherwise,} \end{cases}$$

$$t_f^* = \begin{cases} \frac{\pi}{2} - \sin^{-1} \left(\frac{r}{\mu} \right) & \text{if } \theta = \pi, \\ \frac{\pi}{2} + \frac{r}{\mu} & \text{if } \theta \leq \frac{r}{\mu}, \\ \frac{\pi}{2} - \sin^{-1} \left(\frac{s}{\mu} \right) + t_L(s), & \text{otherwise,} \end{cases}$$

Lady In The Lake

where s is the solution of the transcendental equation $\delta(s) = 0$.
The function $\delta(s) \equiv t_L(s) - t_M(s)$, where $t_L(s)$ is given by

$$t_L(s) = \frac{1}{\mu} \left(\sqrt{r^2 - \frac{s^4}{\mu^2}} + \sqrt{s^2 - \frac{s^4}{\mu^2}} \right),$$

or

$$t_L(s) = \frac{1}{\mu} \left(\sqrt{s^2 - \frac{s^4}{\mu^2}} - \sqrt{r^2 - \frac{s^4}{\mu^2}} \right),$$

and

$$t_M(s) = \theta + \cos^{-1} \left(\frac{s^2}{\mu r} \right) + \cos^{-1} \left(\frac{s}{\mu} \right) - \pi.$$

or

$$t_M(s) = \theta - \cos^{-1} \left(\frac{s^2}{\mu r} \right) + \cos^{-1} \left(\frac{s}{\mu} \right) - \pi.$$

Lady In The Lake

The function

$$\delta(s) = t_L(s) - t_M(s)$$

It is the difference of the agents' respective times of arrival. The equilibrium entry point onto the FL is thus the smallest possible root of this function, i.e.,

$$s^* = \min s \quad \text{s.t. } \delta(s) = 0, s \in (0, \mu].$$

depending on which case applies to the current state.

The corresponding case determines the sign of $\cos \psi^*$ as well.



Lady in the Lake

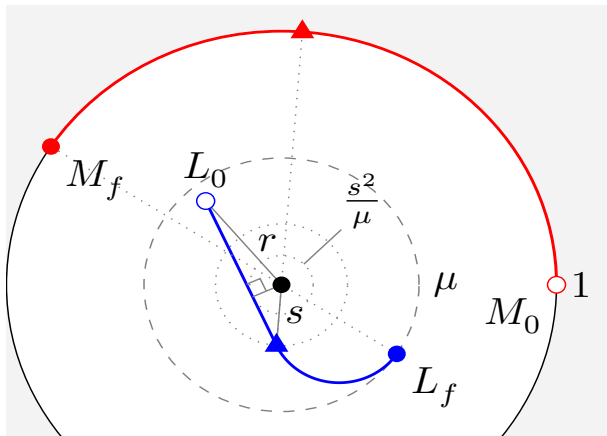


Figure: FL trajectory starting from the tributaries in the realistic plane. L initially heads toward the tangent of the circle of radius $\frac{s^2}{\mu}$.

Lady in the Lake

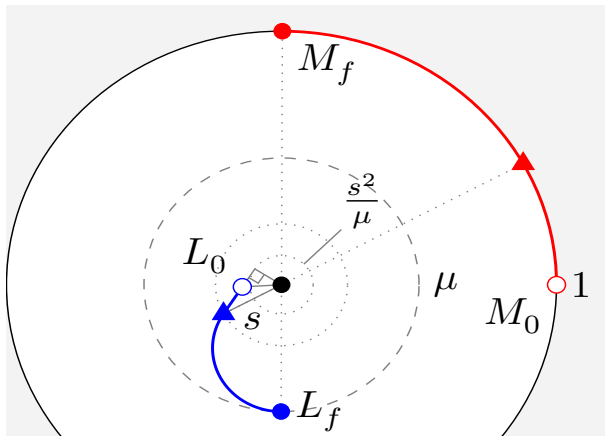


Figure: L heads away from the tangent. In both cases, open markers indicate initial positions, triangles designate positions at the moment the FL is reached and closed markers indicate terminal positions.

Lady in the Lake

Solution of the Differential Game

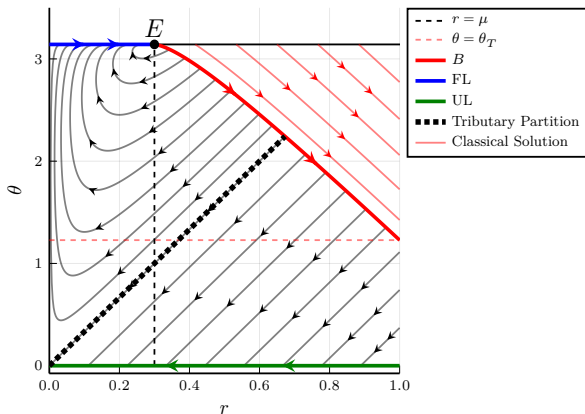


Figure: Optimal/Equilibrium Flow Field

Air Force Application: DE Weapon System

- ① A. Von Moll, M. Pachter, D. Shishika, Z. Fuchs: Guarding a Circular Target by Patrolling its Perimeter, 59th CDC on AC, Vol. 68, Issue 7, pp. 1659-1665, Jeju Island, South Korea, Dec. 11, 2020.
- ② A. Von Moll, M. Pachter, D. Shishika, Z. Fuchs: Circular Target Defense Differential Game, IEEE Trans. on AC, Vol. 68, Issue 7, pp. 4065-4078, July 2023.
- ③ A. Von Moll, Z. Fuchs, D. Shishika, D. Marty, M. Dorothy, M. Pachter,: Turret Escape Differential Game, Journal of Dynamics and Games, Editor: Vladimir Mazalov, Vol. 11, No. 2, pp. 100-114, April 2024.
- ④ A. Von Moll, A. Gerlach, C. Bakker, A. Rupe, M. Pachter: Constrained Turret Defense with Fixed Final Time, MECC 2024, Chicago, IL, 28-30 October 2024.
- ⑤ A. Von Moll and M. Pachter: Turret and Mobile Defender to Ward Off an Attacker, forthcoming.

Highlights

The complete solution of the classical Lady in the Lake DG is obtained.

- In this DG have a new feature - an *internal* Barrier/semipermeable “surface”.
- The two tributary regions are separated by a spiral of Archimedes curve.
- In this DG have a Focal Line (FL), a Universal Line (UL) + their respective tributaries.
- The UL is akin to a singular arc in optimal control, but a FL is a unique feature of DGs.
- The connection of the Lady in the Lake DG to DGs where a directed energy weapon is employed to protect a high value target is emphasized.

Non conventional pursuit-evasion DGs with a *fast* Evader introduced.



- Group Pursuit/Swarm Action/Many-on-One scenarios addressed.

Pursuit and Evasion: N-on-1



Figure: Multi Pursuers – Wolf Pack Action

2P1E

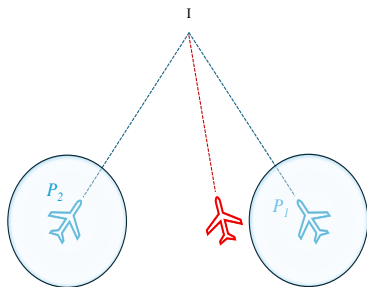
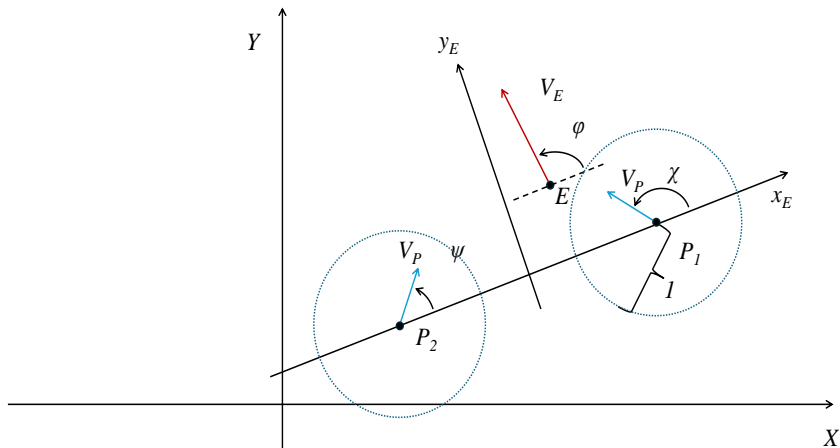


Figure: Pincer Movement Pursuit

$$2P1E, \mu \geq 1$$

Reduced state Space (x_P, x_E, y_E)



2P1E, $\mu \geq 1$

Canonical Dynamics – Strongly Nonlinear

$$\begin{aligned}\dot{x}_P &= \frac{1}{2}(\cos \xi - \cos \psi), & x_P(0) &= x_{P0} \\ \dot{x}_E &= \mu \cos \phi - \frac{1}{2}(\cos \xi + \cos \psi) + \frac{1}{2} \frac{y_E}{x_P}(\sin \xi - \sin \psi), & x_E(0) &= x_{E0} \\ \dot{y}_E &= \mu \sin \phi - \frac{1}{2}(\sin \xi + \sin \psi) - \frac{1}{2} \frac{x_E}{x_P}(\sin \xi - \sin \psi), & y_E(0) &= y_{E0}\end{aligned}$$

$\mu \geq 1 \Rightarrow$ *contact* trajectories possible.

During contact, the state space dimension is reduced to 2.

Contact \neq capture.

- Capture requires *both* P_1 and P_2 to be in contact with E.

Equal Speed: $\mu = 1$

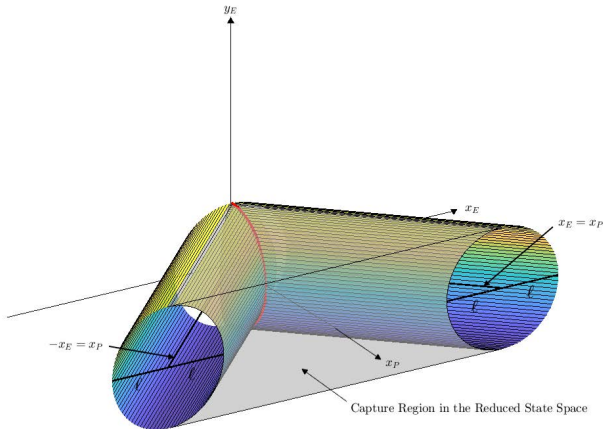


Figure: Capture Zone \mathcal{R}_c

Equal Speed: $\mu = 1$

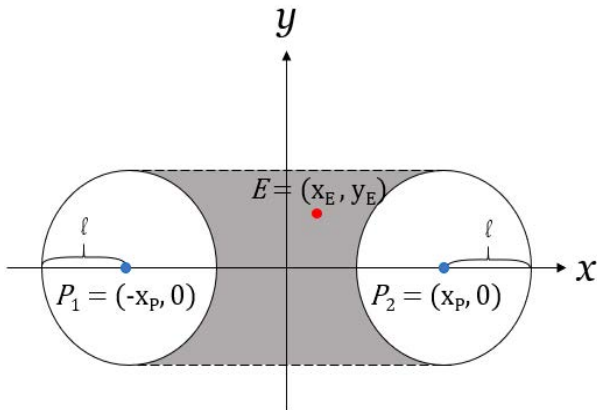


Figure: x_P -Cross Section of Region of Capturability \mathcal{R}_c

Equal Speed: $\mu = 1$

The optimal/equilibrium state feedback strategies are

$$\psi^* = \cos^{-1}\left(\frac{x_P}{t_c + 1}\right), \quad \chi^* = \pi - \cos^{-1}\left(\frac{x_P}{t_c + 1}\right), \quad \phi^* = \pi - \cos^{-1}\left(\frac{x_E}{t_c + 1}\right)$$

The Value function

$$V(x_P, x_E, y_E) = t_c(x_P, x_E, y_E)$$

where

$$t_c(x_P, x_E, y_E) = \frac{1}{2(1 - y_E^2)} [x_P^2 + y_E^2 - x_E^2 - 1 + y_E \sqrt{(x_P^2 + y_E^2 - x_E^2 - 1)^2 - 4x_E^2(1 - y_E^2)}]$$

When E is in contact with P_1 , i.e., $y_E = \sqrt{1 - (x_P - x_E)^2}$,

$$V(x_P, x_E) = \frac{x_E}{x_P - x_E}$$

Equal Speed: $\mu = 1$

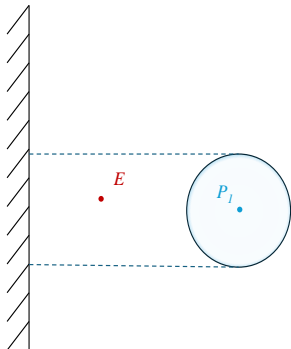


Figure: Capture Region In Deadline Game

Equal Speed: $\mu = 1$

Group Pursuit: 3P1E

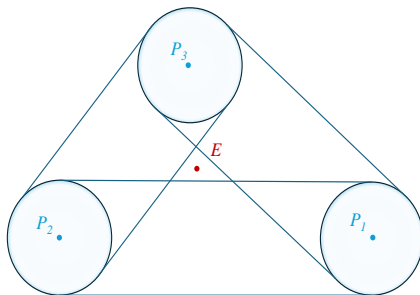


Figure: x_P -Cross Section of Region of Capturability \mathcal{R}_c

Fast Evader: $\mu > 1$

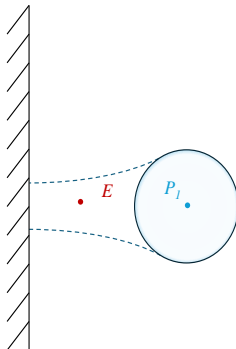


Figure: Capture Region In Deadline Game

Fast Evader: $\mu > 1$

Group Pursuit: 3P1E

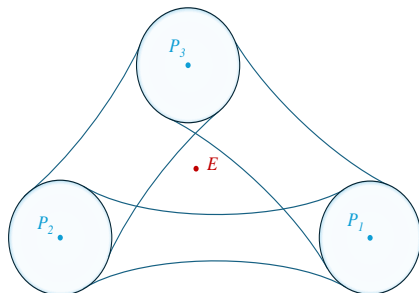


Figure: x_P -Cross Section of Region of Capturability \mathcal{R}_c

Conclusion

The complete solution of the classical Lady in the Lake DG is obtained.

- In this DG have a new feature - an *internal* Barrier/semipermeable “surface”.
- The two tributary regions are separated by a spiral of Archimedes curve.
- In this DG have a Focal Line (FL), a Universal Line (UL) + their respective tributaries.
- The UL is akin to a singular arc in optimal control, but a FL is a unique feature of DGs.
- The connection of the Lady in the Lake DG to DGs where a directed energy weapon is employed to protect a high value target is emphasized.

Non conventional pursuit-evasion DGs with a *fast* Evader introduced.



- Group Pursuit/Swarm Action/Many-on-One scenarios addressed.

Conclusion

New, off the beaten path, and currently relevant, pursuit-evasion scenarios where a *fast* Evader is engaged by a Pursuit Team are addressed.

- In these DGs a novel type of optimal/equilibrium trajectories is discerned, where the Evader is temporarily in contact with one of the Pursuers before capture is finally effected.

The closed form optimal/equilibrium strategies of the agents and the Value functions of the Lady in the Lake and the 2P1E DGs are obtained.

The optimal/equilibrium strategies are in state feedback form, are realizable in RT, and thus are actionable and useful to the war fighter.

Lady In The Lake



Lady In The Lake

IEEE Trans. on Aerospace and Electronic Systems

History Column: Hedy Lamarr

DOI: No. 10.1109/MAES.2021.3101776

Hugh Griffiths[©] and Alfonso Farina[©], University College London, London WC1E 6BT, U.K.

Hedy Lamarr (born Hedwig Eva Maria Kiesler, 1914, in Vienna, Austria) is best known as a Hollywood actress of the 1940s and 1950s. She starred in movies including *Break Town* (1940) with Clark Gable and Spencer Tracy, *The Heavenly Body* (1944), and *Samson and Delilah* (1949) opposite Victor Mature.

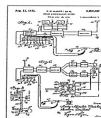


Hedy Lamarr in *The Heavenly Body*, MGM, 1944 (Public Domain).

Her early acting career was in Germany, but in 1938 she moved to Hollywood via London, and was signed by Louis B. Mayer of Metro-Goldwyn-Mayer (MGM). She was married (and divorced) six times. She was honored in 1960 with a star on the Hollywood Walk of Fame for her contribution to the motion picture industry, and she died in January 2000 at the age of 85.

Less well known is the fact that she was also a talented and innovative inventor. In 1942, she and avant-garde musician and inventor George Antheil developed and patented a communications scheme for use in a radio guidance system for torpedoes. The key novelty was that this exploited frequency-hopping both as not to be discovered by an opponent and as a counter to jamming, and this may be considered as a forerunner of spread-spectrum techniques which are now widely used in secure communications. An important aspect

of this was the means used for synchronization of the hopping codes at the transmitter and receiver, which was achieved in a similar way to that used by Antheil to synchronize two pianos whose tunes were controlled by perforated paper strips. The figure below shows a page from the patent. In the top diagram can be seen seven switched capacitors which determined the seven frequencies of the hopping pattern.



A page of drawings from Lamarr and Antheil's patent (1942).

The technique was not exploited directly, but was employed by the U.S. Navy in 1957 as part of the radio link for a sonobuoy. Lamarr and Antheil were posthumously inducted into the U.S. National Inventors Hall of Fame in 2014.

Further information may be found at:

R. Rhodes, *Hedy's Folly: The Life and Breakthrough Inventions of Hedy Lamarr*, New York, NY, USA: Doubleday, 2012.

A Tale of Two Lives, Aug. 2018, [Online]. Available: <https://physicsworld.com/a/a-tale-of-two-lives/>

A. George, "Thank this World War II-era film star for your wi-fi," *Smithsonian Magazine*, April 2019. Available: <https://www.smithsonianmag.com/smithsonian-institution/thank-world-war-ii-era-film-star-your-wi-fi-180971584/>

Bondshell, *The Hedy Lamarr Story* (documentary trailer), 2017, [Online]. Available: <https://www.youtube.com/watch?v=BKXAKITmGU>

ACKNOWLEDGMENT

The authors would like to thank Dr. E. Detoma and Dr. S. Maddio for useful exchange of additional information on the subject.

Author's current address: Hugh Griffiths and Alfonso Farina, University College London, London WC1E 6BT, U.K. (e-mail: h.griffiths@ucl.ac.uk).


Manuscript received July 10, 2021; accepted July 27, 2021, and ready for publication July 29, 2021. Review handled by Daniel O'Hagan. 0885-8958/21/\$26.00 © 2021 IEEE.

Lady In The Lake

Institute Of Navigation

vocacy item. The Alliance communicates directly with Congress, congressional committees, regulatory agencies like the Federal Communications Commission (FCC), and federal agencies that manage or use GPS.

The Alliance provides these institutions with the correct scientific details to make accurate and informed decisions. To learn more, visit <https://www.gpsalliance.org/about-us>.

As you can see, there is a lot going on beyond the scenes to ensure the future of reliable, robust, and accurate PNT. 

Frank van Graas

Follow us on Facebook and Twitter @ionavigation.



The Purpose of the ION®

Founded in 1945, the Institute of Navigation is the world's premier non-profit professional society advancing the art and science of positioning, navigation, and timing.

--- 2022 Executive Committee ---

President:
Dr. Frank van Diggelen
Executive Vice President:
Dr. Sherman Lo
Treasurer:
Dr. Frank van Graas

Eastern Region Vice President:
Dr. Jason Rife
Western Region Vice President:
Tim Murphy

Satellite Division Chair:
Patricia Doherty
Military Division Chair:
Dr. Thomas Powell
Immediate Past President:
Dr. Y. Jade Morton

--- How to Reach the ION® ---

Telephone: 703-366-2723
Facsimile: 703-366-2724

Website: ion.org
E-mail: membership@ion.org

The ION® National Office
8551 Rileys Lane, Suite 360
Manassas, VA 20108

--- The ION® National Office Staff ---

Executive Director:
Lisa Besty
Director of Information Technology:
Rick Buongiovanni
Director of Membership & Marketing:
Kenneth P. Esthus

Program/Author Liaison/Executive Assistant:
Miriam Lewis
Meeting Planner:
Megan Andrews
Assistant Editor:
Rachel Sutton

Graphic Designer:
Melanie Awerney
Newsletter Editor:
Dee Ann Davis

Opinions expressed in ION Newsletter articles and columns do not necessarily reflect an official policy of the ION or the views of any other individual ION member(s).

Last GPS III Satellite Named For Inventor, Film Star Hedy Lamarr


The last of the ten GPS III spacecraft has been named "Hedy Lamarr" to honor the inventive film star who co-developed a frequency-skipping technology that laid the foundation for WiFi, Bluetooth, and GPS.

The nickname was given to the satellite at the completion of the "core mate" production milestone, the point at which a satellite is fully assembled or "born."

Lamarr became fascinated as a child by the way machines worked, according to the National Women's History Museum website. Though her film career took off, her interest in technology continued. She dated Howard Hughes who encouraged her inventive genius and for whom she designed a new airplane wing. What she learned about weapons during her first marriage to a munitions dealer informed the spread spectrum technology she



Photo courtesy of Wikimedia

patented with musician and inventor George Antheil as a radio guidance system for Allied torpedoes. The technique aimed to make it impossible to jam guidance signals by using a piano roll to randomly change frequencies. 

Follow ION on Social Media

 **Twitter**
[@ionavigation](https://twitter.com/ionavigation)

 **LinkedIn**
[ionavigation](https://www.linkedin.com/company/ionavigation)

 **Facebook**
[ionavigation](https://www.facebook.com/ionavigation)

 **YouTube**
[instofnavigation](https://www.youtube.com/ionavigation)

 **Instagram**
[ionavigation](https://www.instagram.com/ionavigation)

Why AI in the D&C Review ?

The views expressed on this slide are those of the author and do *not* reflect the official policy or position of the US Air Force, DoD, or the US Gov.

Current state of AI

Enumeration, ML/overfitting, Heuristics/hacking, ES/no AI

-It's like alchemy

-AI is deus ex machina

So what is real AI?

- AI is decision making powered by mathematics and machine computing

An attempt at a definition in the D&C context

AI: Optimal control in the presence of uncertainty and adversarial action.

Example: Solution of a DG

The adversary is an *algorithm* and the human player is a loser

Why AI in the D&C Review ?

The views expressed on this slide are those of the author and do *not* reflect the official policy or position of the US Air Force, DoD, or the US Gov.

INTUITION

The GCD of $n^{17} + 9$ and $(n + 1)^{17} + 9 = 1$

for n up to

8424432925592889329288197322308900672459420460792432

and then for

$n = 8424432925592889329288197322308900672459420460792433$

it is

8936582237915716659950962253358945635793453256935559.

You can check your intuition at the door !

Lady in the Lake

Quanta Article, August 2021

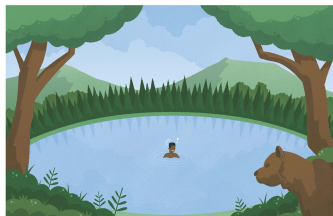
INSIGHTS PUZZLE

Math Can, in Theory, Help You Escape a Hungry Bear

By PRADDEEP MUTALIK

August 25, 2021

How readers used their geometry skills to survive a dangerous puzzle. 4 | III



James Round for Quanta Magazine

Our June Insights puzzle added a few twists to a classic puzzle made famous by Martin Gardner in his 1985 *Scientific American* column and later published in the book *The Colossal Book of Short Puzzles and Problems*. In our version, a swimmer at the center of a circular lake of radius 3.5 is attempting to escape a bear hunting him from the shore. The bear doesn't swim but can run along the circumference at 3.5 times the swimmer's speed, which is 1 unit of length per unit of time. To survive, the athlete must swim to shore before the bear reaches the same point.

Our first puzzle posed some basic questions about the swimmer's strategy. For example, what could he learn from the way squirrels spiral up a tree to escape pursuing dogs? The other puzzles explored newer questions, which led to some unexpected mathematical sleuthing.

Lady in the Lake

Solution of the Differential Game

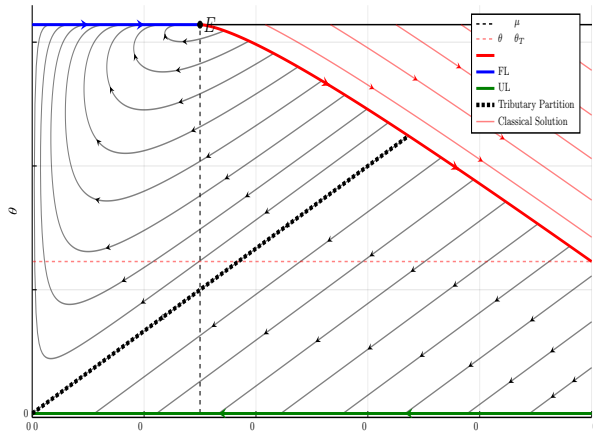


Figure: Optimal/Equilibrium Flow Field