

Hybrid Lie-Bracket Averaging for Model-Free Optimization and Control

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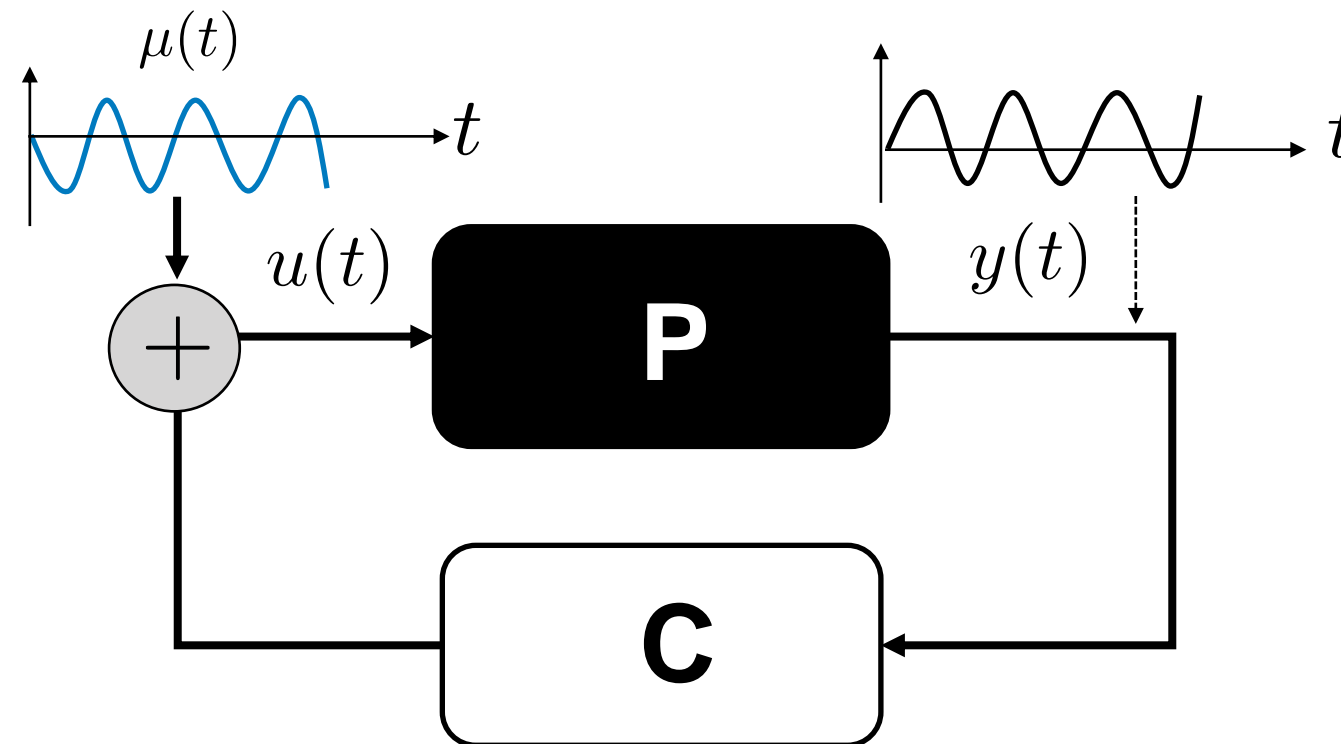
Averaging Tools: Exploration vs Exploitation

❑ We are interested in **controllers** that incorporate **real-time “exploration”** for the purpose of **learning** and **adaptation**

- Avoiding **persistence of excitation (PE)** assumptions on solutions...

❑ **Averaging theory:**

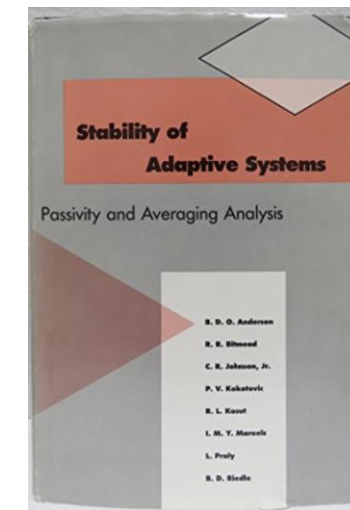
- Extremum-Seeking Control
- Vibrational Control
- Indirect Adaptive Control
- Zeroth-order Optimization



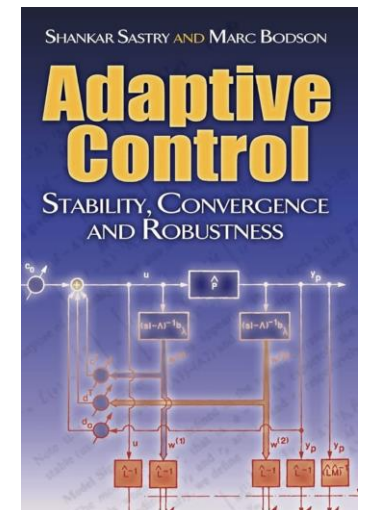
$$\dot{x} = f(\mu(t), x)$$

❑ Traditionally studied via **first-order (“standard”) averaging**:

$$\dot{\bar{x}} = f_{\text{ave}}(\bar{x}) = \frac{1}{T} \int_0^T f(\mu(t), \bar{x}) dt$$



(1986)

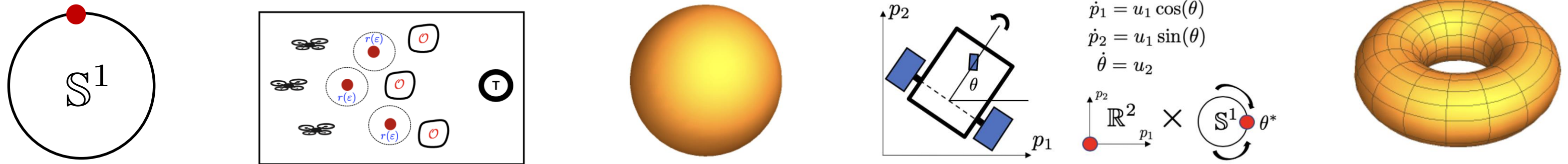


(1990)

Averaging Tools: Exploration vs Exploitation

❑ Some limitations of standard averaging:

- Directly adding “exploration policies” to the **states** or **inputs** of the plant might not be possible (or it might be difficult) under topological constraints



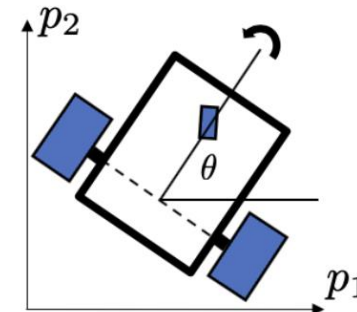
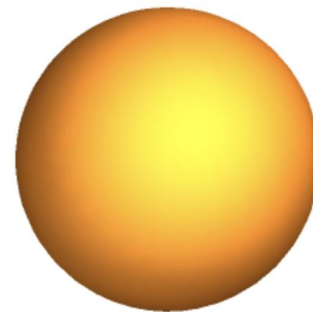
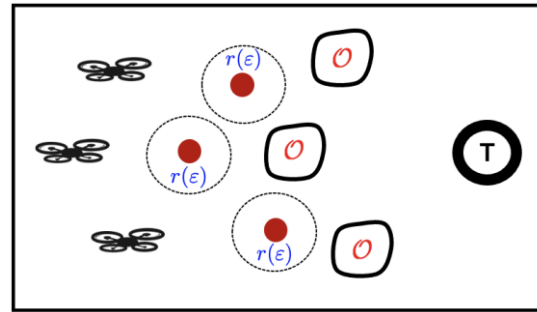
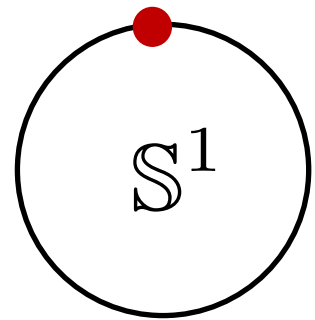
1. Ochoa, Poveda, “*Robust Global Optimization on Smooth Compact Manifolds via Hybrid Gradient-Free Dynamics*”, Automatica, 2024
2. Ochoa, Poveda, “*Momentum-Based Nash Set-Seeking over Networks via Multi-Time Scale Hybrid Inclusions*”, IEEE TAC, 2024
3. Chen, Poveda, Li, “*Continuous-Time Zeroth-Order Dynamics with Projection Maps*”, IEEE TAC (provisionally accepted), 2024
4. Abdelgalil, Ochoa, Poveda, “*Multi-Time Scale Control and Optimization via Singular Perturbations and Averaging Theory: From ODEs to Hybrid Dynamical Systems*”, Annual Reviews in Control, 2023.

- For some systems of interest **classic averaging** is **uninformative**: $\frac{1}{T} \int_0^T f(\mu(t), t) dt = 0$

Averaging Tools: Exploration vs Exploitation

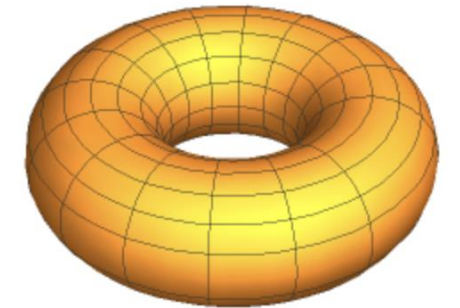
❑ Some limitations of standard averaging:

- Directly adding “exploration policies” to the **states** or **inputs** of the plant might not be possible (or it might be difficult) under topological constraints



$$\begin{aligned}\dot{p}_1 &= u_1 \cos(\theta) \\ \dot{p}_2 &= u_1 \sin(\theta) \\ \dot{\theta} &= u_2\end{aligned}$$

$\mathbb{R}^2 \times S^1$



❑ An alternative approach: (second-order) “Lie-bracket” averaging:

LIE BRACKET EXTENSIONS AND AVERAGING: THE SINGLE-BRACKET CASE

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Department of Mathematics
Rutgers University

- Bogoliubov , Mitropolskii, 1961; Sanders, Verhulst, Murdock, 2007
- Dürr, Stankovic, Ebenbauer, Johansson, 2013, 2014, 2015
- Scheinker, Krstic, 2014; Suttner, 2017; Grushkovskaya, 2018

but most existing tools were restricted to ODEs...

Today:

1. Introducing Lie-bracket Averaging for Hybrid Systems

“On Lie-Bracket Averaging for a Class of Hybrid Dynamical Systems with Applications to Model-Free Control and Optimization”, Arxiv 2023

2. Introducing new hybrid algorithms for model-free optimization and regulation

“Hybrid Minimum-Seeking in Synergistic Lyapunov Functions: Robust Global Stabilization under Unknown Control Directions”, Arxiv 2024

3. Introducing novel global (practical) stability properties for Lie-bracket averaging systems and algorithms

“Initialization-free Lie-bracket Extremum Seeking”, Systems and Control Letters, Vol. 191, pp. 105881

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**Hybrid
Systems**

2. Introducing new hybrid algorithms for model-free optimization and regulation

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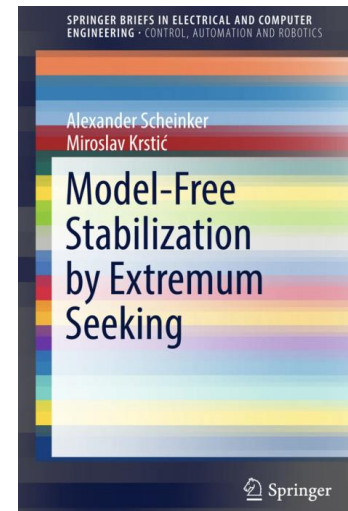
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Hybrid Systems



Hybrid Systems

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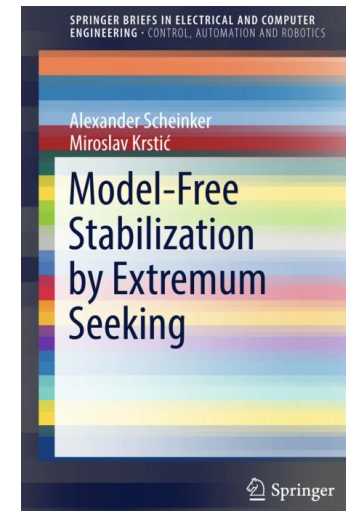
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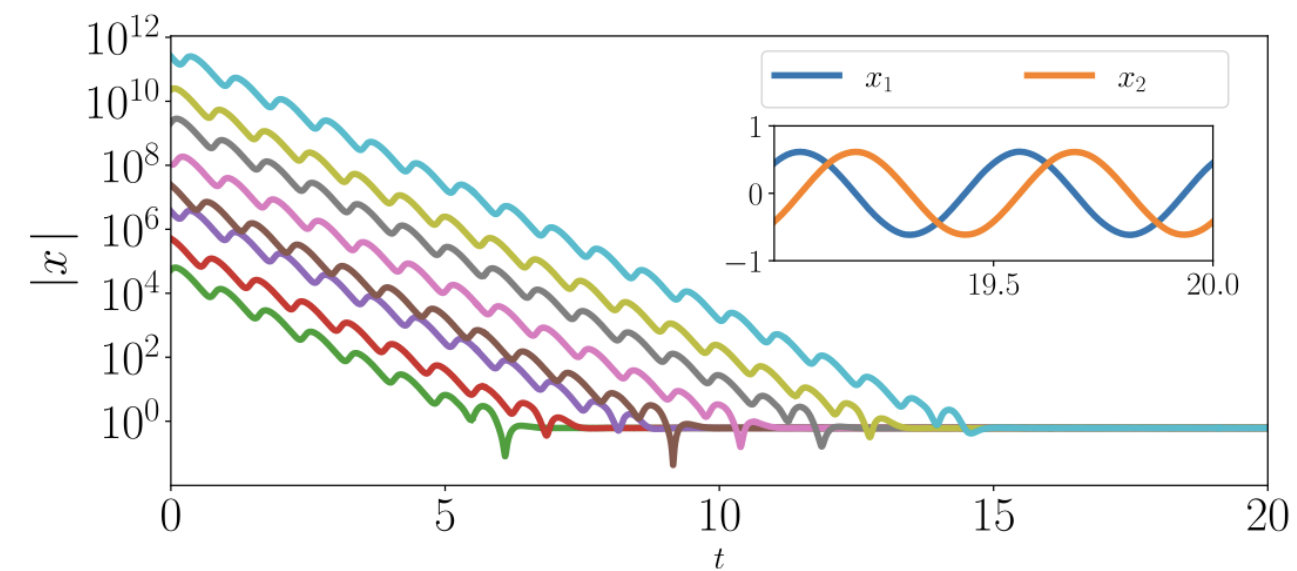
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Hybrid Systems



Hybrid Systems



Joint work with:



Mahmoud Abdelgalil

Postdoc (UCSD)

Today:

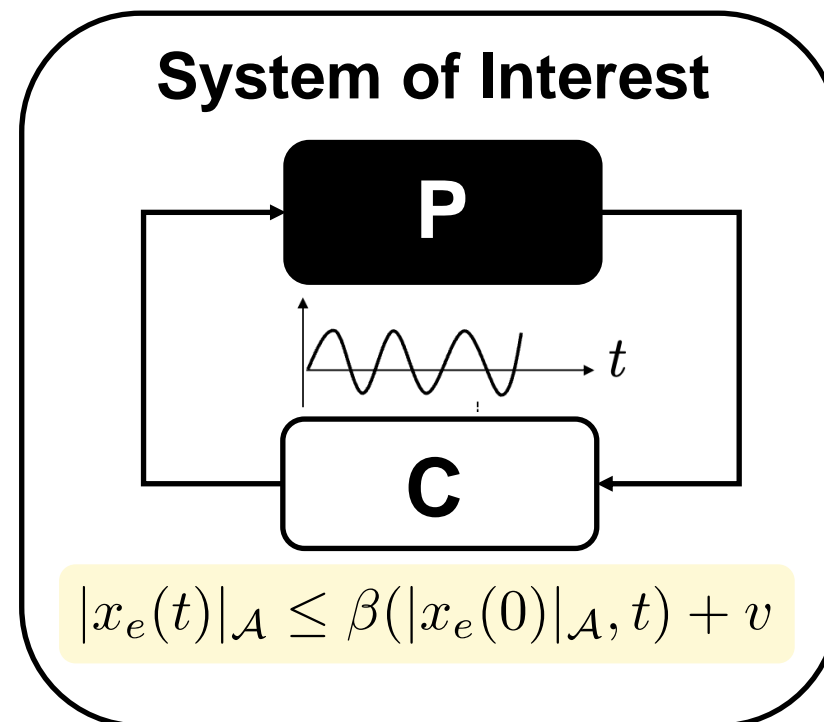
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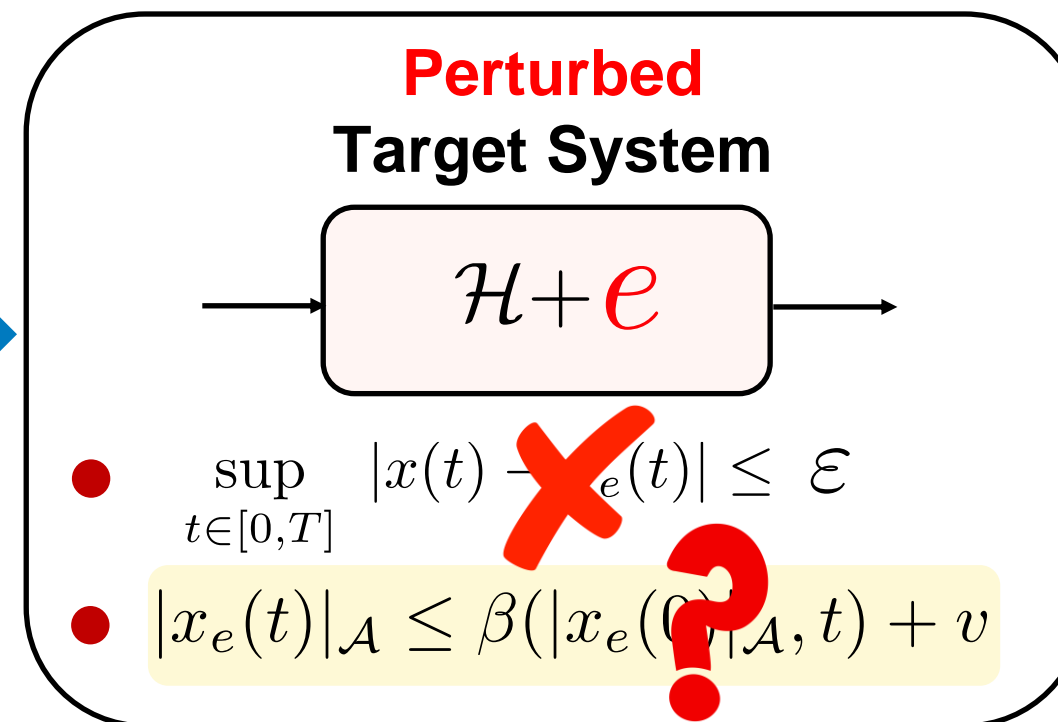
Lie-Bracket Averaging for Hybrid Control?

□ **Key question:** How to design **adaptive/learning-based hybrid controllers** using averaging theory?

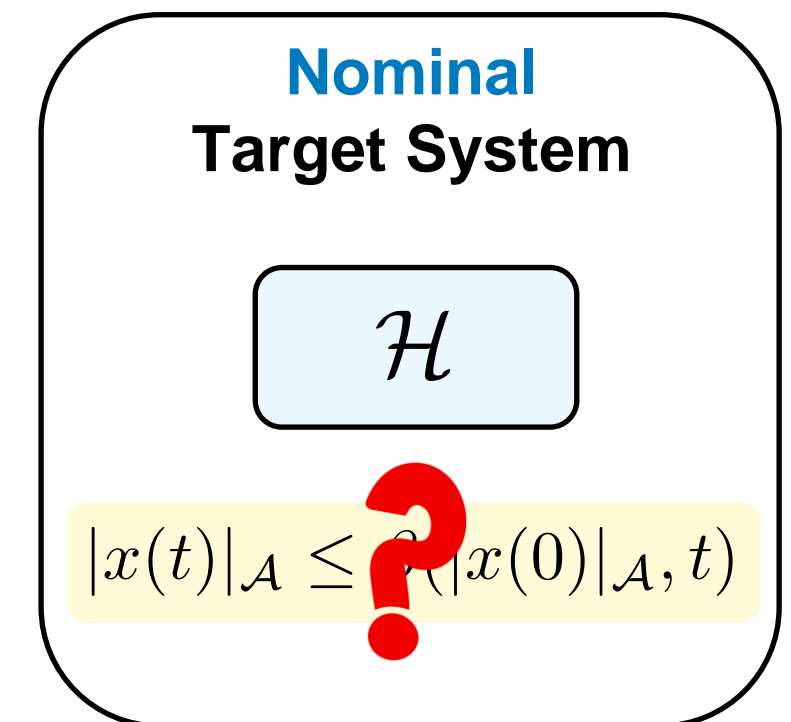
Given:



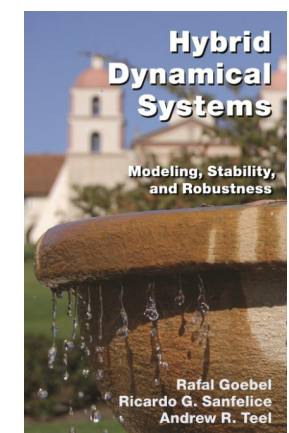
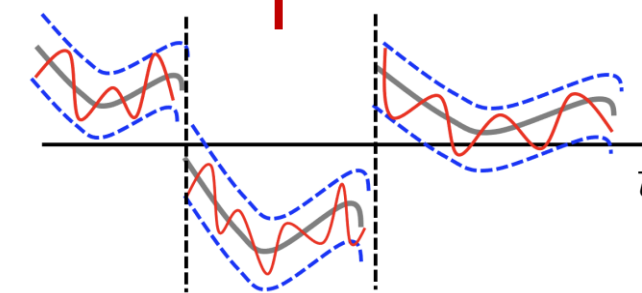
Exploit Robustness Properties:



Identify or Design:



*Second-order (Lie-bracket)
averaging tools for hybrid systems*



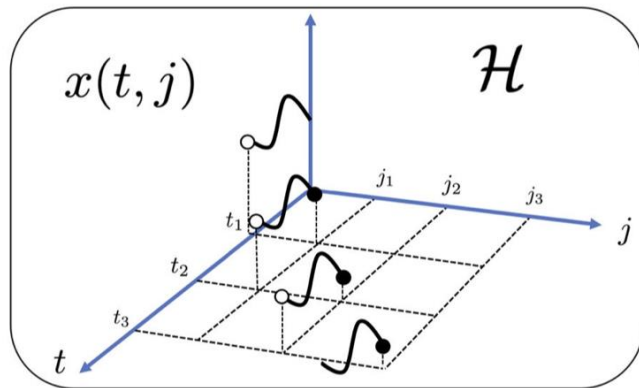
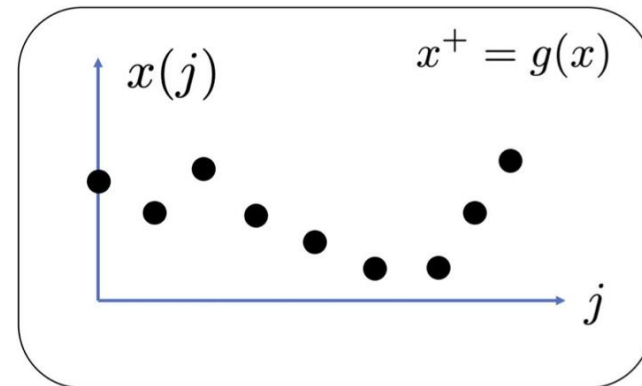
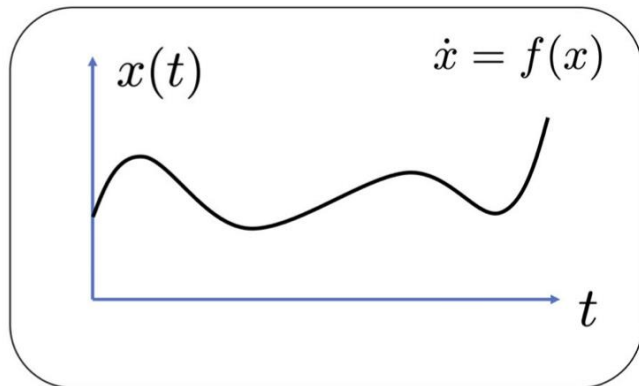
Hybrid Dynamical Systems

□ Hybrid dynamical systems combine **continuous-time** dynamics and **discrete-time** dynamics:

$$x \in C \leftarrow \text{Flow Set} \quad \dot{x} \in F(x) \leftarrow \text{Flow Map}$$

$$x \in D \leftarrow \text{Jump Set} \quad x^+ \in G(x) \leftarrow \text{Jump Map}$$

□ *Solutions* are defined on **hybrid-time domains (HTDs)**:



■ *Stability bounds* are also expressed via **HTDs**:

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j)$$

■ We study stability of **compact sets**:

$$|x(t, j)|_{\mathcal{A}} = \min_{s \in \mathcal{A}} |x(t, j) - s|$$

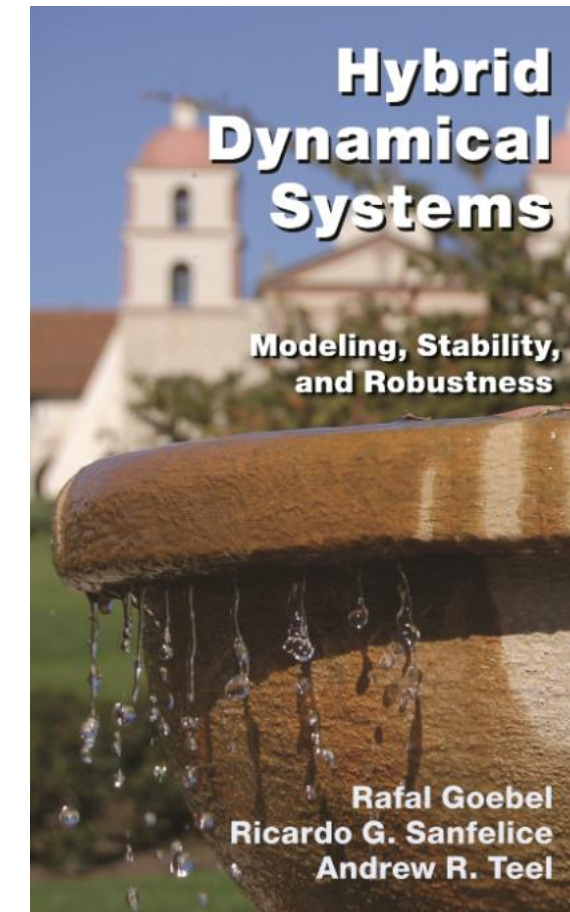
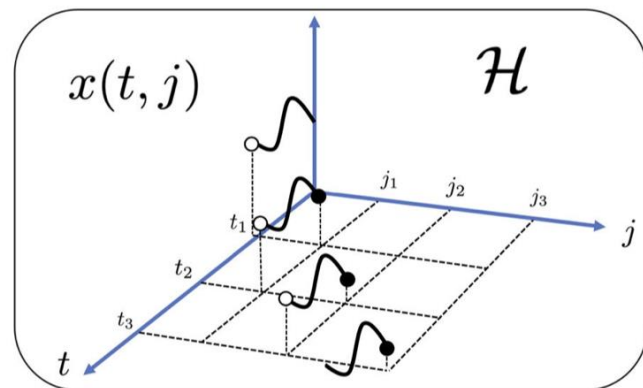
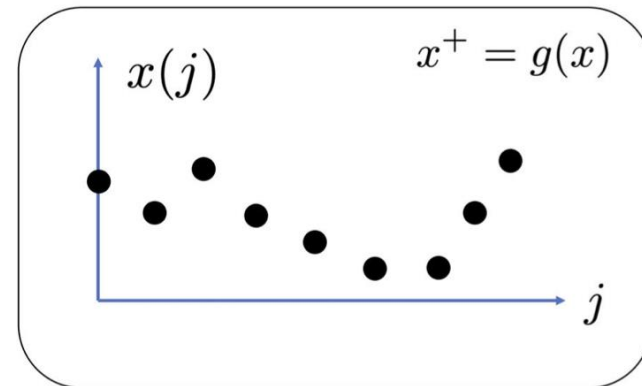
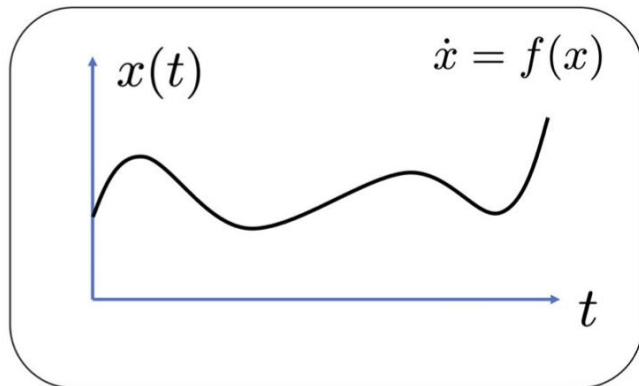
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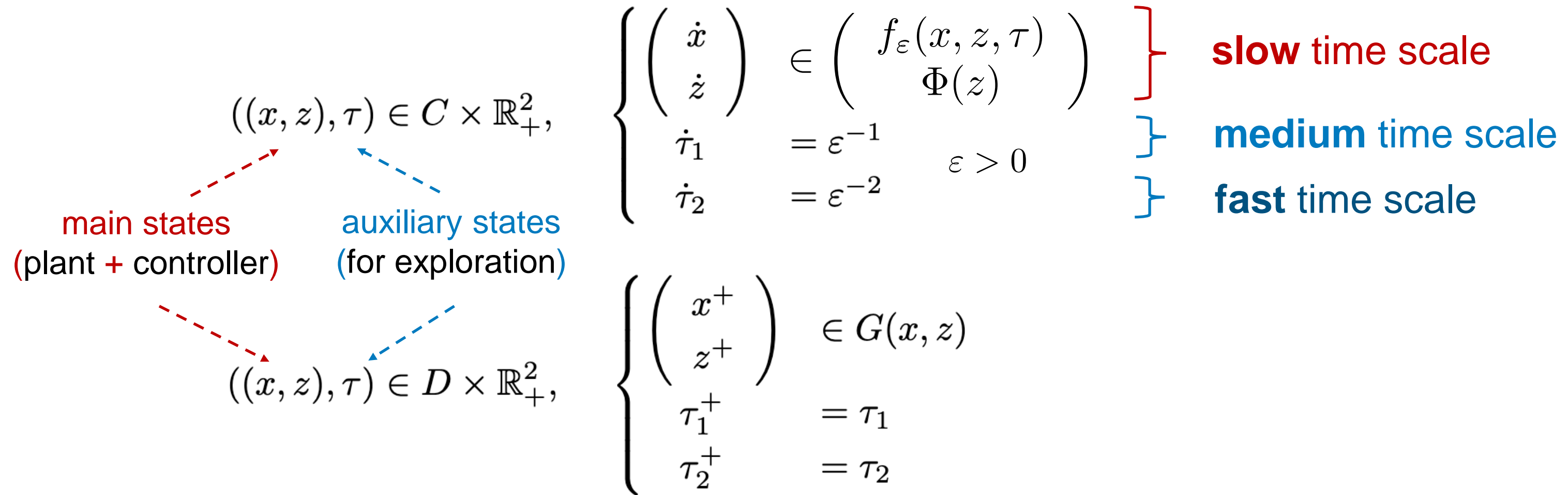
$$x \in D \leftarrow \text{Jump Set} \quad x^+ \in G(x) \leftarrow \text{Jump Map}$$

□ *Solutions* are defined on **hybrid-time domains (HTDs)**:



Lie-Bracket Averaging on Hybrid Dynamical Systems

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Lie-Bracket Averaging on Hybrid Dynamical Systems

$$\begin{aligned} ((x, z), \tau) \in C \times \mathbb{R}_+^2, & \quad \left\{ \begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \in \begin{pmatrix} f_\varepsilon(x, z, \tau) \\ \Phi(z) \end{pmatrix} \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{array} \right. \\ ((x, z), \tau) \in D \times \mathbb{R}_+^2, & \quad \left\{ \begin{array}{l} \begin{pmatrix} x^+ \\ z^+ \end{pmatrix} \in G(x, z) \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{array} \right. \end{aligned}$$

Φ, G, D, C are **application-dependent**
encode “if-then” rules and **discrete-time** dynamics

Lie-Bracket Averaging on Hybrid Dynamical Systems

$$((x, z), \tau) \in C \times \mathbb{R}_+^2, \quad \left\{ \begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \in \begin{pmatrix} f_\varepsilon(x, z, \tau) \\ \Phi(z) \end{pmatrix} \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{array} \right. \quad \left. \vphantom{\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix}} \right\} \text{ induces oscillatory behaviors} \\ \text{("exploration")}$$

$$((x, z), \tau) \in D \times \mathbb{R}_+^2, \quad \left\{ \begin{array}{l} \begin{pmatrix} x^+ \\ z^+ \end{pmatrix} \in G(x, z) \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{array} \right.$$

Exploration function:

$$f_\varepsilon(x, z, \tau) = \varepsilon^{-1} \underbrace{\phi_1(x, z, \tau_1, \tau_2)}_{\text{Periodic in } \tau_1, \tau_2} + \underbrace{\phi_2(x, z, \tau_1, \tau_2)}$$

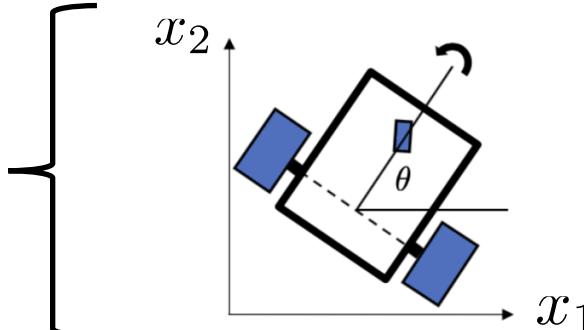
Non-Biased Exploration Condition

ϕ_1 *also satisfies:*

$$\int_0^{T_2} \phi_1(x, z, \tau_1, s_2) ds_2 = 0,$$

Lie-Bracket Averaging on Hybrid Dynamical Systems

□ Example:

PLANT: 

$$\begin{aligned} \dot{x}_1 &= u x_3, & \dot{x}_2 &= u x_4, & \dot{x}_3 &= \Omega x_4, & \dot{x}_4 &= -\Omega x_3, \\ x_p &:= (x_1, x_2) \in \mathbb{R}^2 & (x_3, x_4) &\in \mathbb{S}^1 \\ \text{Position} & & \text{Orientation} \end{aligned}$$

Consider the following **control law**, which uses a potential field J :

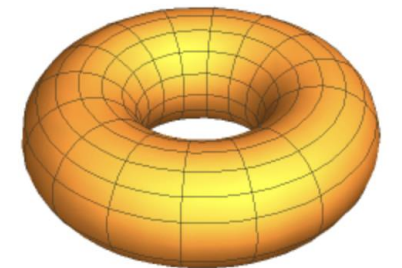
$$u(x_p, \tau_2) = \varepsilon^{-1} \cos(\tau_2 + J(x_p)) \quad \phi_2 = 0$$

Closed-loop:

$$\dot{x} = f_\varepsilon(x, z, \tau) = \varepsilon^{-1} \phi_1(x, z, \tau_1, \tau_2) + \phi_2(x, z, \tau_1, \tau_2)$$

□ More general **plants**: $\dot{x} = \sum_{i=1}^r b_i(x, \tau_1) u_i(x, \tau_2) \quad u_i(x, \tau_2) = \varepsilon^{-1} \sqrt{2\omega_i} \cos(\omega_i \tau_2 + J(x)) \quad \phi_2 = 0$

□ Function ϕ_2 can encode **non-controllable dynamics**: $\begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^r \end{pmatrix} = \begin{pmatrix} (1 + \alpha_1 u_1) S x^1 \\ \vdots \\ (1 + \alpha_r u_r) S x^r \end{pmatrix}$



□ More complex **plant dynamics** can be considered using multi-layered architectures

Lie-Bracket Averaging on Hybrid Dynamical Systems

$$\left. \begin{aligned} ((x, z), \tau) \in C \times \mathbb{R}_+^2, & \quad \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \in \begin{pmatrix} f_\varepsilon(x, z, \tau) \\ \Phi(z) \end{pmatrix} \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{cases} \\ ((x, z), \tau) \in D \times \mathbb{R}_+^2, & \quad \begin{cases} \begin{pmatrix} x^+ \\ z^+ \end{pmatrix} \in G(x, z) \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{cases} \end{aligned} \right\} : \mathcal{H}_S$$

performs real-time
“**exploration**”
with plant in the
loop


We would like to study \mathcal{H}_S using a simpler
non-oscillating **hybrid** system
(“**exploitation**”)

Lie-Bracket Averaging on Hybrid Dynamical Systems

□ To study \mathcal{H}_S , we consider the following *Hybrid Lie-Bracket* system:

$$\mathcal{H}_{\text{LB}} : \quad (\bar{x}, \bar{z}) \in C, \quad \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} \in \begin{pmatrix} \bar{f}(\bar{x}, \bar{z}) \\ \Phi(\bar{z}) \end{pmatrix} \quad (\bar{x}, \bar{z}) \in D, \quad \begin{pmatrix} \bar{x}^+ \\ \bar{z}^+ \end{pmatrix} \in G(\bar{x}, \bar{z})$$

Same elements from \mathcal{H}_S



Lie-Bracket Averaging on Hybrid Dynamical Systems

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Definition: Using $\theta = (\bar{x}, \bar{z})$, the **average map** is defined as:

$$\bar{f}(\theta) := \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \left(\phi_2(\theta, \tau) + \frac{1}{2} [u_1, \phi_1]_x(\theta, \tau) \right) d\tau_2 d\tau_1$$

Lie-Bracket (LB):

$$[u_1, \phi_1]_x(\cdot) =$$

$$u_1 := \int_0^{\tau_2} \phi_1(\bar{x}, \bar{z}, \tau_1, s_2) ds_2$$

- **LB** encodes **gradients** (new info available for decision-making and stabilization)
- With proper design, **LB** can also encode **geometric/safety constraints**

Examples: Model-Free Optimization of $J(x)$

$$\bar{f}(\theta) := \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \left(\phi_2(\theta, \tau) + \frac{1}{2} [u_1, \phi_1]_x(\theta, \tau) \right) d\tau_2 d\tau_1 \quad \theta = (\bar{x}, \bar{z})$$

a) From cost **measurements** to **gradients**:

average map

$$\phi_1(x, \tau_1, \tau_2) = \sum_{i=1}^n \sqrt{\frac{2}{T_i}} \cos\left(\frac{\tau_2}{T_i} + J(x)\right) e_i \quad \phi_2(x, \tau_1, \tau_2) = 0 \quad \rightarrow \quad \dot{\bar{x}} = \bar{f}(\bar{x}) = -\nabla J(\bar{x})$$

$\bar{x} \in \mathbb{R}^n$

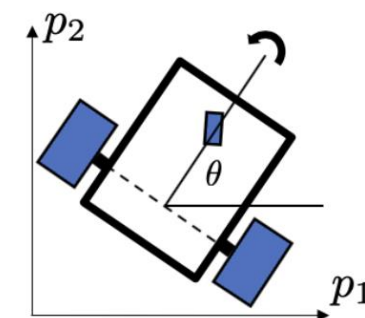
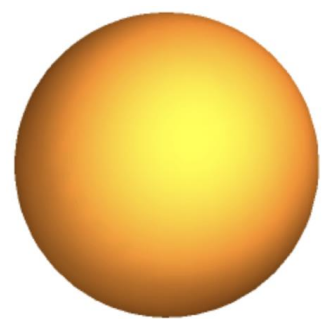
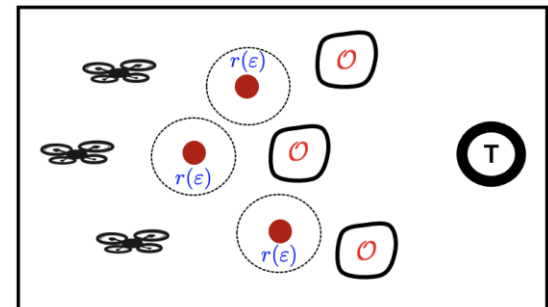
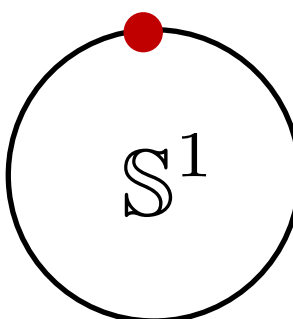
b) From cost **measurements** to **gradients** with **geometric** constraints:

average map

$$\phi_1(x, \tau_1, \tau_2) = \begin{pmatrix} \sqrt{w_1} \cos(w_1 \tau_2 + \kappa J_1(x)) \\ \sqrt{w_2} \cos(w_2 \tau_2 + \kappa J_2(x)) \\ \vdots \\ \sqrt{w_r} \cos(w_r \tau_2 + \kappa J_r(x)) \end{pmatrix} Sx \quad \phi_2(x, \tau_1, \tau_2) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} Sx \quad \rightarrow \quad \dot{\bar{x}} = \bar{f}(\bar{x}) \in T_{\bar{x}}(\text{torus})$$

$\bar{x} \in \text{torus}$

c) We can incorporate different types of **geometric** constraints:



$$\begin{aligned} \dot{p}_1 &= u_1 \cos(\theta) \\ \dot{p}_2 &= u_1 \sin(\theta) \\ \dot{\theta} &= u_2 \end{aligned}$$

$\mathbb{R}^2 \times S^1$

Back to the Hybrid Lie-Bracket System (“Exploitation”)

- Using the **average map**, we can now design the **hybrid Lie-bracket system** to incorporate “**if-then**” rules to solve the **main task of interest**:

$$\mathcal{H}_{\text{LB}}: (\bar{x}, \bar{z}) \in C \quad \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} \in \begin{pmatrix} \bar{f}(\bar{x}, \bar{z}) \\ \Phi(\bar{z}) \end{pmatrix} \quad (\bar{x}, \bar{z}) \in D \quad \begin{pmatrix} \bar{x}^+ \\ \bar{z}^+ \end{pmatrix} \in G(\bar{x}, \bar{z})$$

- Let a compact set \mathcal{A} encode the **main task of interest** (e.g., set of minimizers of J)
- Design C, D, Φ, G and the **average map** \bar{f} to **stabilize** \mathcal{A}

Key Stability Assumption: The set \mathcal{A} is UGAS for the hybrid system \mathcal{H}_{LB} :

$$\theta = (\bar{x}, \bar{z}) \quad |\theta(t, j)|_{\mathcal{A}} \leq \beta(|\theta(0, 0)|_{\mathcal{A}}, t + j) \quad \beta \in \mathcal{KL}$$

Key Regularity Assumption:

C, D are closed sets

G, Φ are OSC and LB

Φ is convex

Hybrid Seeking Systems:

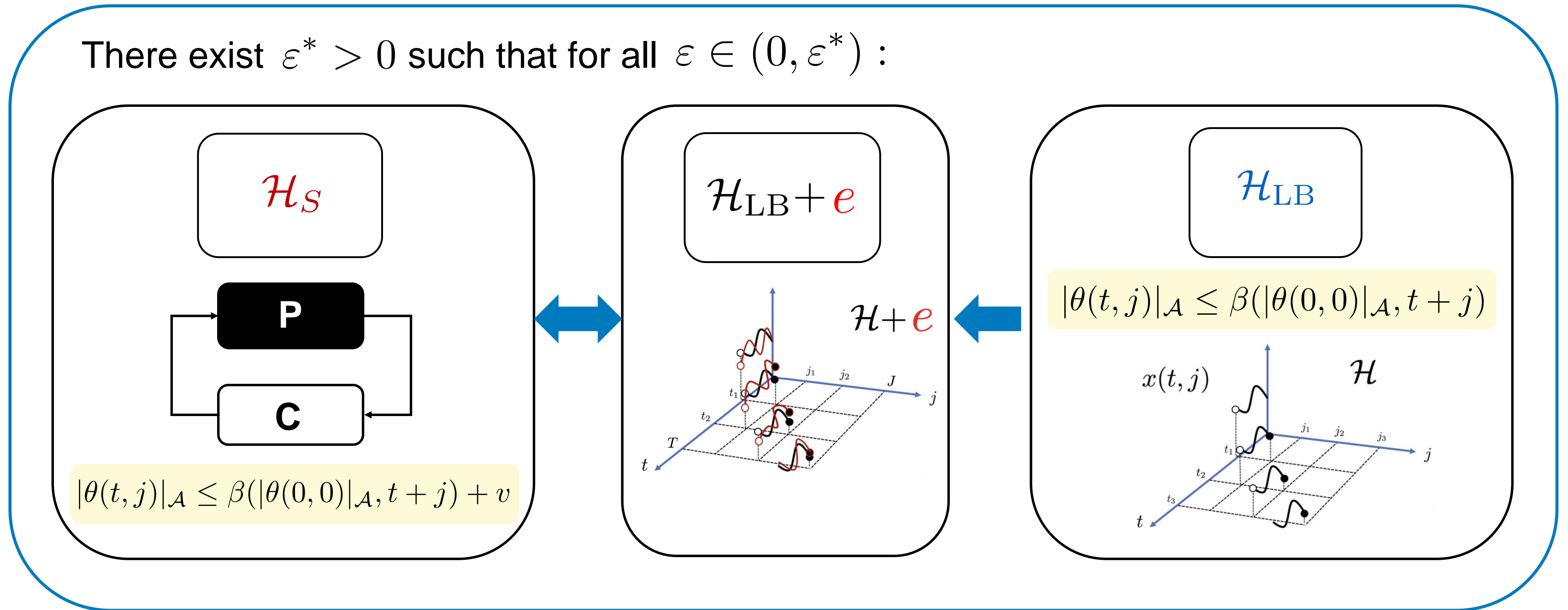
$$\begin{array}{l}
 ((x, z), \tau) \in \overset{\checkmark}{C} \times \mathbb{R}_+^2, \quad \left\{ \begin{array}{l} \left(\begin{array}{l} \dot{x} \\ \dot{z} \end{array} \right) \in \left(\begin{array}{l} f_\varepsilon(x, z, \tau) \\ \Phi(z) \end{array} \right) \checkmark \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{array} \right. \\
 \\
 ((x, z), \tau) \in \overset{\checkmark}{D} \times \mathbb{R}_+^2, \quad \left\{ \begin{array}{l} \left(\begin{array}{l} x^+ \\ z^+ \end{array} \right) \in G(x, z) \checkmark \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{array} \right.
 \end{array}
 \left. \vphantom{\begin{array}{l} ((x, z), \tau) \in \overset{\checkmark}{C} \times \mathbb{R}_+^2, \\ ((x, z), \tau) \in \overset{\checkmark}{D} \times \mathbb{R}_+^2, \end{array}} \right\} : \mathcal{H}_S \quad \left(\begin{array}{l} \text{performs real-time} \\ \text{"exploration"} \end{array} \right)$$

$$f_\varepsilon(x, z, \tau) = \varepsilon^{-1} \phi_1(x, z, \tau_1, \tau_2) + \phi_2(x, z, \tau_1, \tau_2) \quad \longrightarrow \quad \bar{f} \quad \checkmark$$

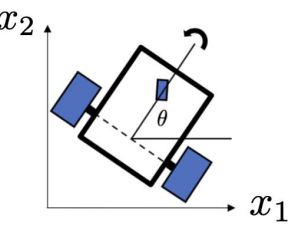
□ When can we predict the behavior of \mathcal{H}_S based on the behavior of \mathcal{H}_{LB} ?

Main Result 1: Lie-bracket Averaging for HDS

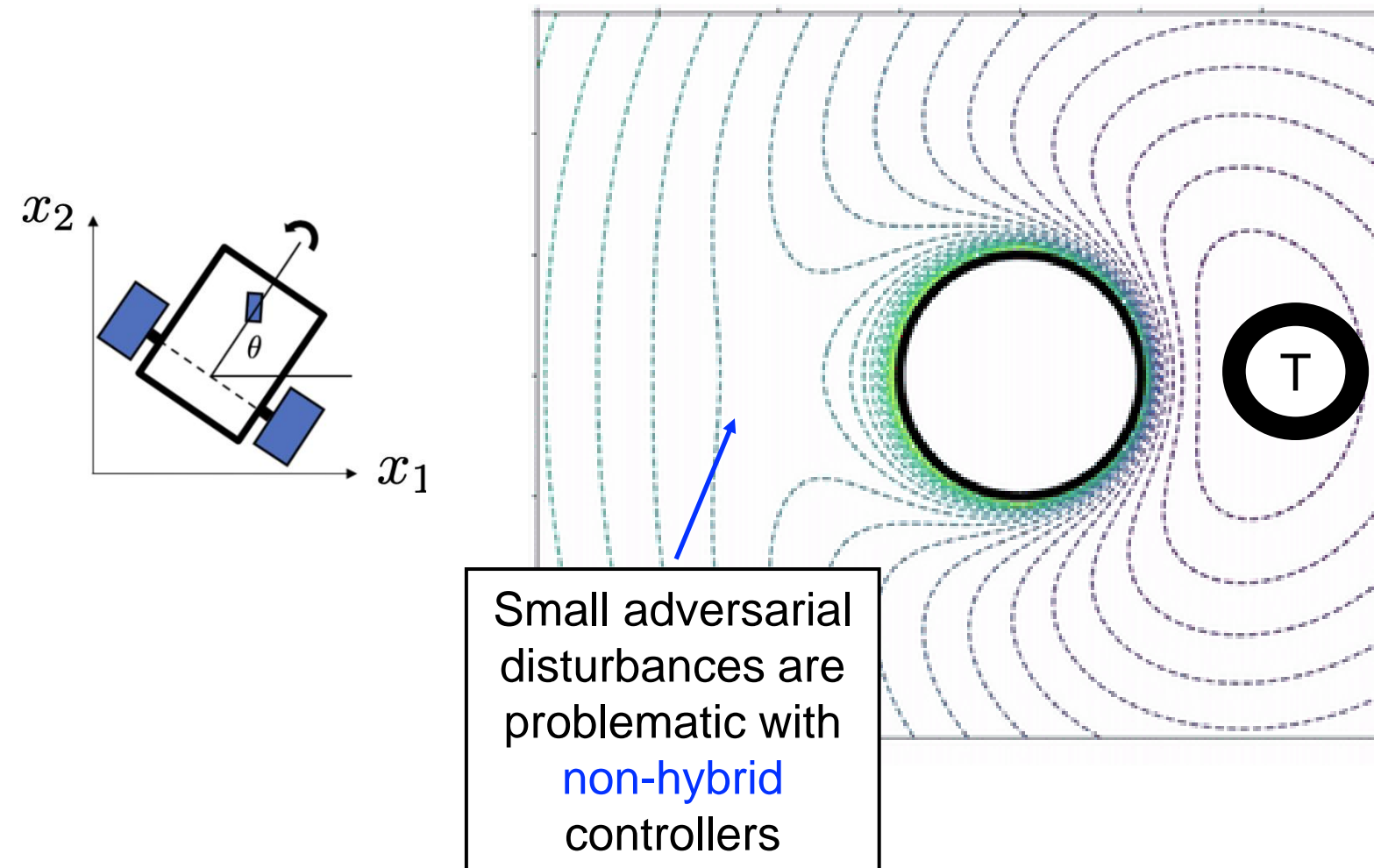
There exist $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$:



Example: Localizing a Gas Leak While Avoiding an Obstacle



$$\dot{x}_1 = u x_3, \quad \dot{x}_2 = u x_4, \quad \dot{x}_3 = \Omega x_4, \quad \dot{x}_4 = -\Omega x_3,$$

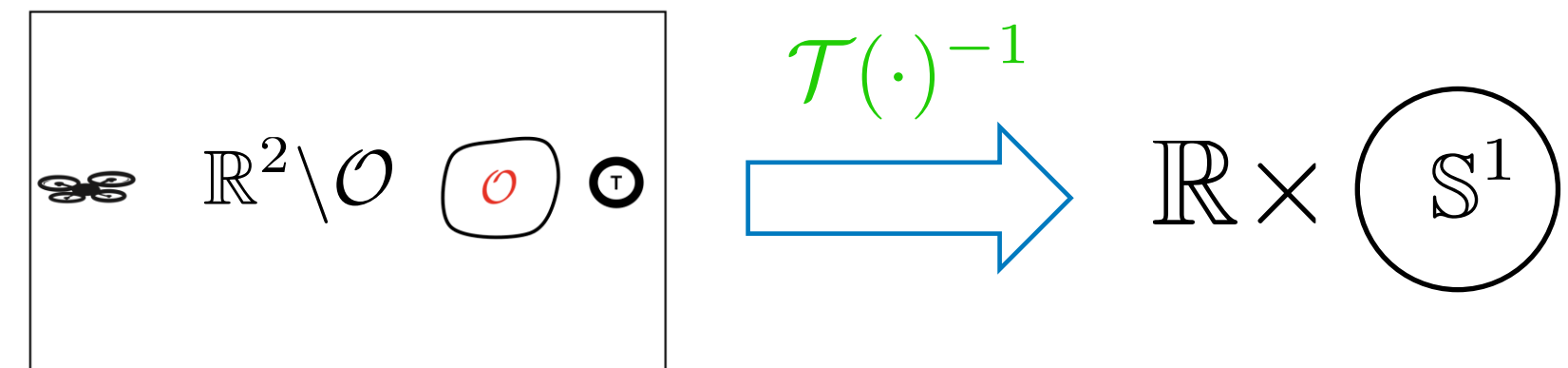


Dynamic **exploration**
constraint:
nonholonomic

Static **exploration**
constraint:
obstacle

□ \mathcal{H}_S incorporates **real-time learning** and **adaptation** with **exploratory** behaviors that satisfy **static and dynamic constraints**, while inducing **robust stability**

□ In practice, we first **transform** the system to coordinates that “**reveal**” the obstruction



□ **Partitioning** and **controller architecture** take place in the new coordinates

Today:

1. Introducing Lie-bracket Averaging for Hybrid Systems



“On Lie-Bracket Averaging for a Class of Hybrid Dynamical Systems with Applications to Model-Free Control and Optimization”, Arxiv 2023

Today:

**LIE BRACKET EXTENSIONS
AND AVERAGING: THE
SINGLE-BRACKET CASE**

Héctor J. Sussmann* and Wensheng Liu
Department of Mathematics
Rutgers University



**Hybrid
Systems**



Today:

1. Introducing Lie-bracket Averaging for Hybrid Systems



“On Lie-Bracket Averaging for a Class of Hybrid Dynamical Systems with Applications to Model-Free Control and Optimization”, Arxiv 2023

2. Introducing new hybrid algorithms for model-free optimization and regulation

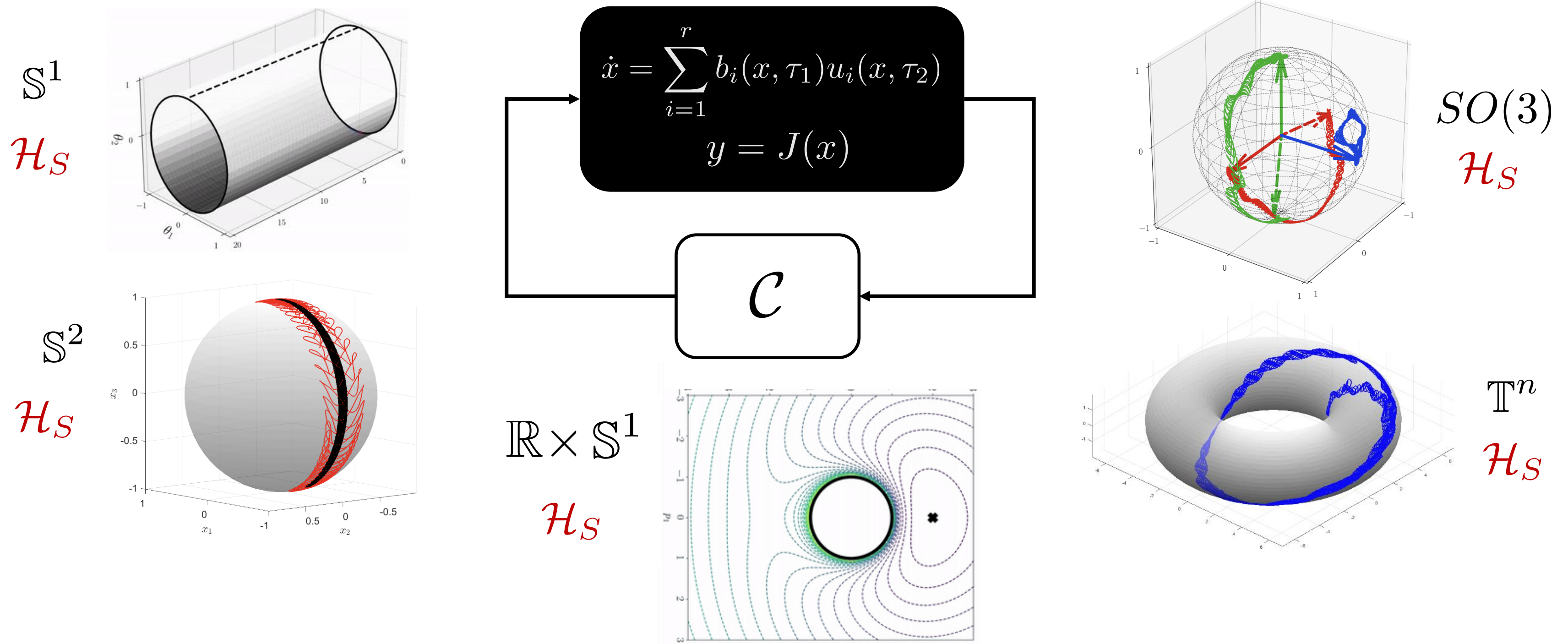
“Hybrid Minimum-Seeking in Synergistic Lyapunov Functions: Robust Global Stabilization under Unknown Control Directions”, Arxiv 2024

Synthesis of Dynamics for Model-Free Optimization:

Zeroth-Order Optimization

Synthesis of Dynamics for Model-Free Optimization

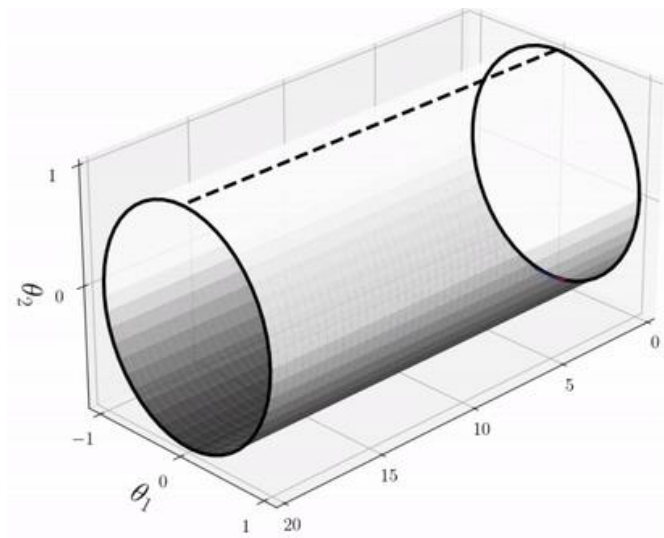
□ **Global** (practical) stability under topological obstructions (based on synergistic control):



Synthesis of Dynamics for Model-Free Optimization

Global (practical) stability under topological obstructions (based on synergistic control):

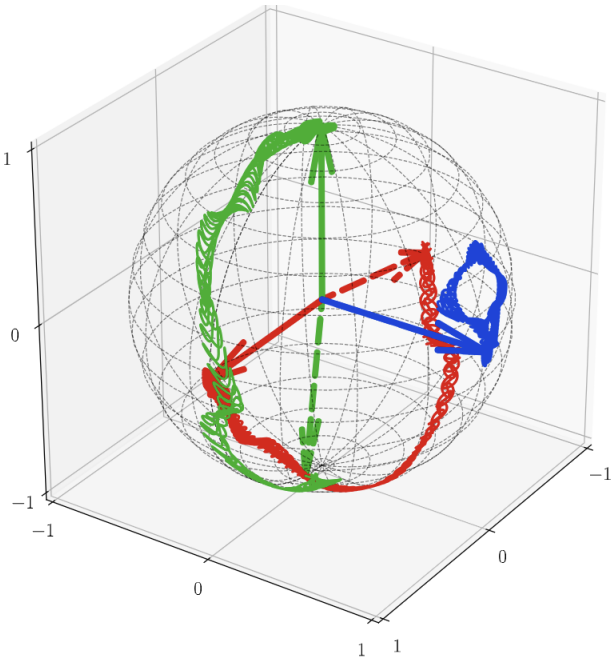
S^1
 \mathcal{H}_S



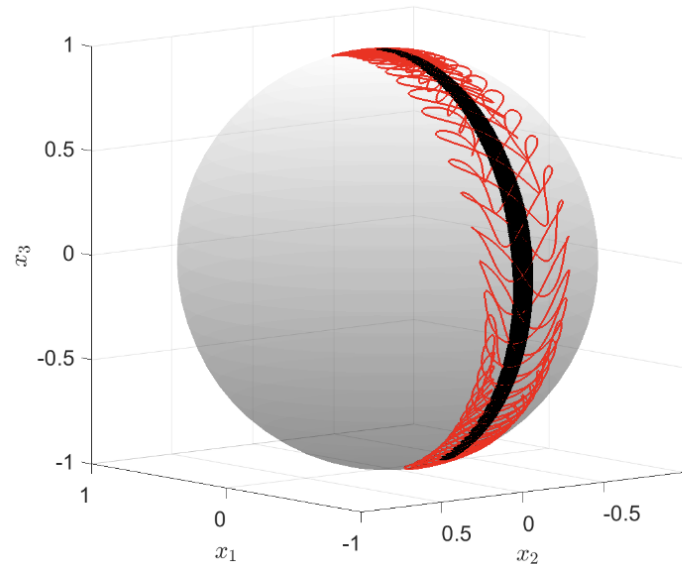
$$\begin{aligned} ((x, z), \tau) \in C \times \mathbb{R}_+^2, \quad & \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \in F_\varepsilon(x, z, \tau_1, \tau_2) \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{cases} \\ \mathcal{H}_S \end{aligned}$$

$$\begin{aligned} ((x, z), \tau) \in D \times \mathbb{R}_+^2, \quad & \begin{cases} \begin{pmatrix} x^+ \\ z^+ \end{pmatrix} \in G(x, z) \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{cases} \end{aligned}$$

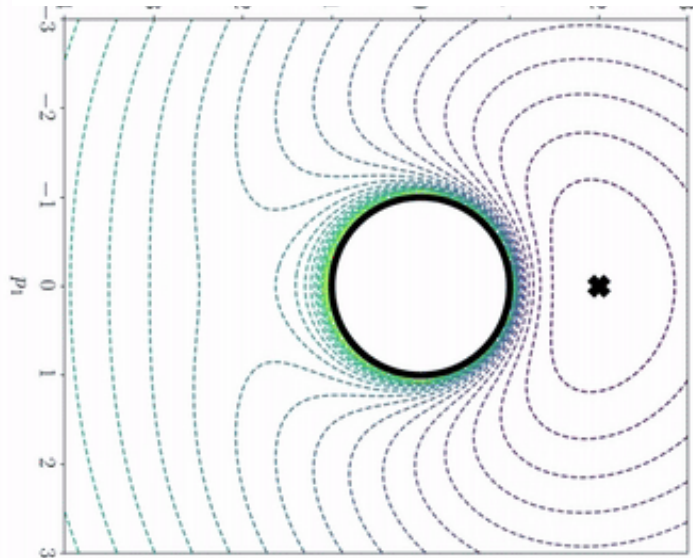
$SO(3)$
 \mathcal{H}_S



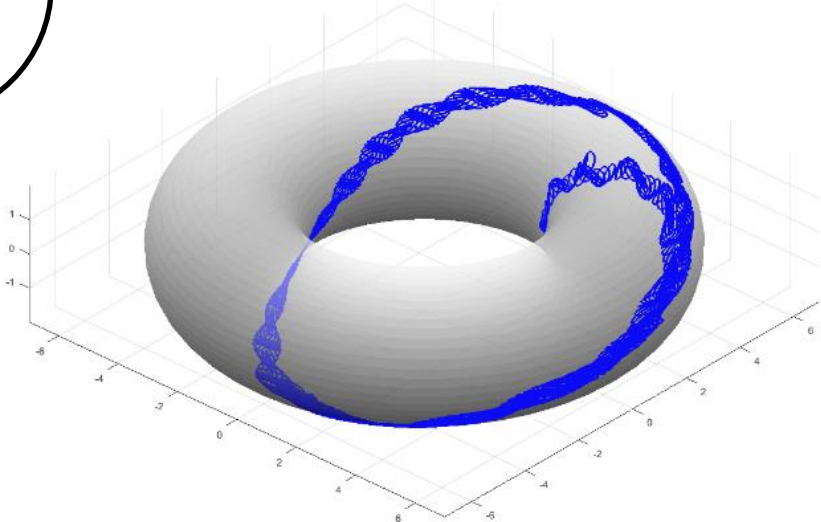
S^2
 \mathcal{H}_S



$\mathbb{R} \times S^1$
 \mathcal{H}_S



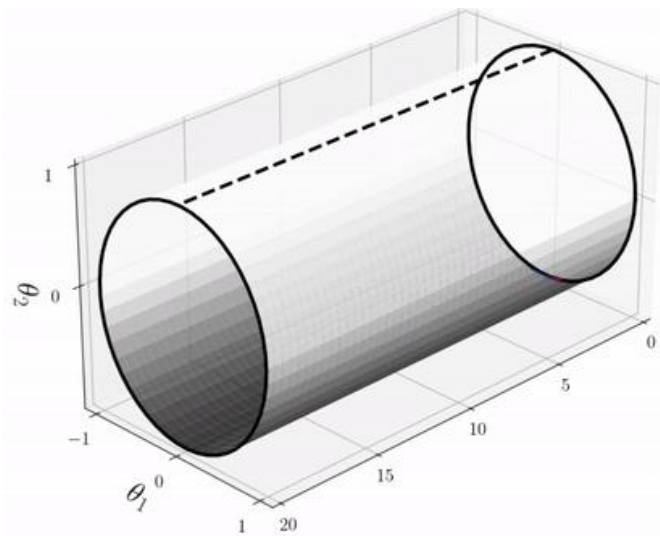
T^n
 \mathcal{H}_S



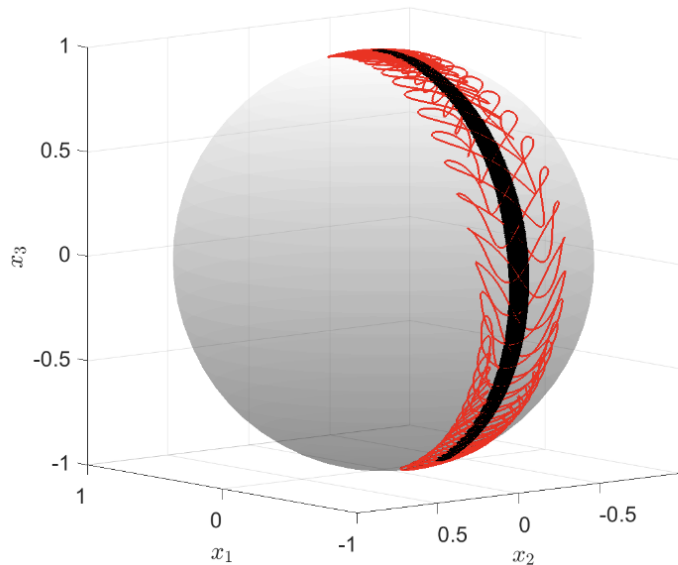
Synthesis of Dynamics for Model-Free Optimization

□ **Global** (practical) stability under topological obstructions (based on synergistic control):

S^1
 \mathcal{H}_S



S^2
 \mathcal{H}_S

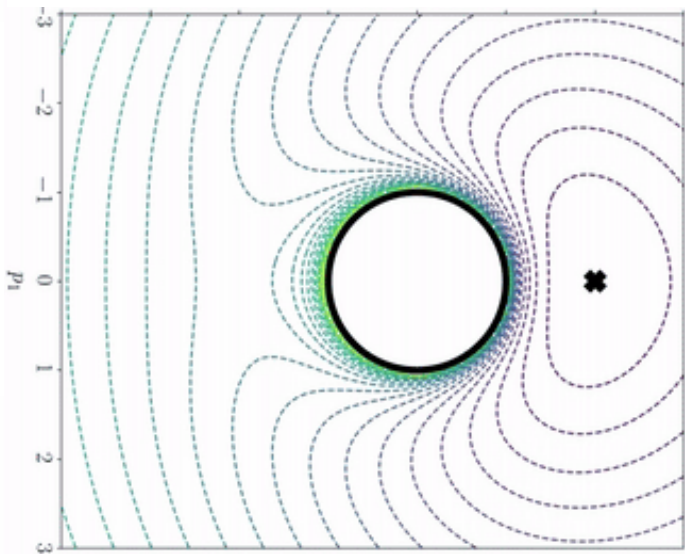


$$(\bar{x}, \bar{z}) \in C, \quad \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} \in \begin{pmatrix} \bar{f}(\bar{x}, \bar{z}) \\ \Phi(\bar{z}) \end{pmatrix}$$

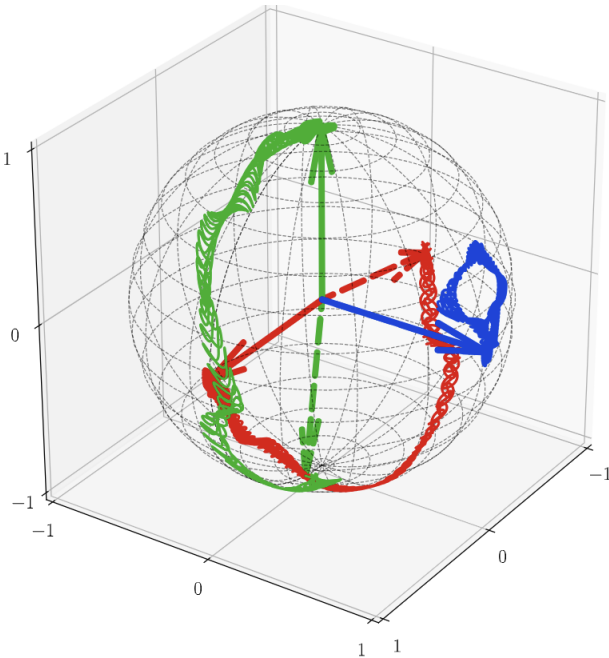
\mathcal{H}_{LB}

$$(\bar{x}, \bar{z}) \in D, \quad \begin{pmatrix} \bar{x}^+ \\ \bar{z}^+ \end{pmatrix} \in G(\bar{x}, \bar{z})$$

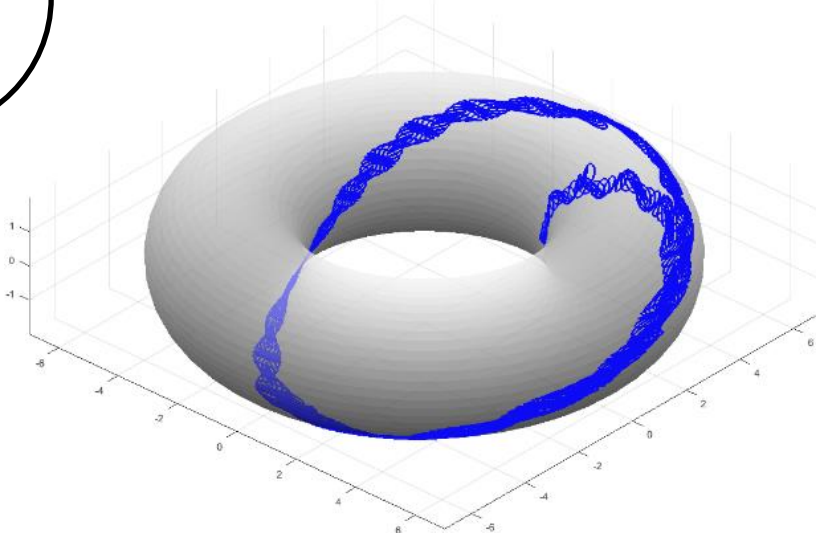
$\mathbb{R} \times S^1$
 \mathcal{H}_S



$SO(3)$
 \mathcal{H}_S

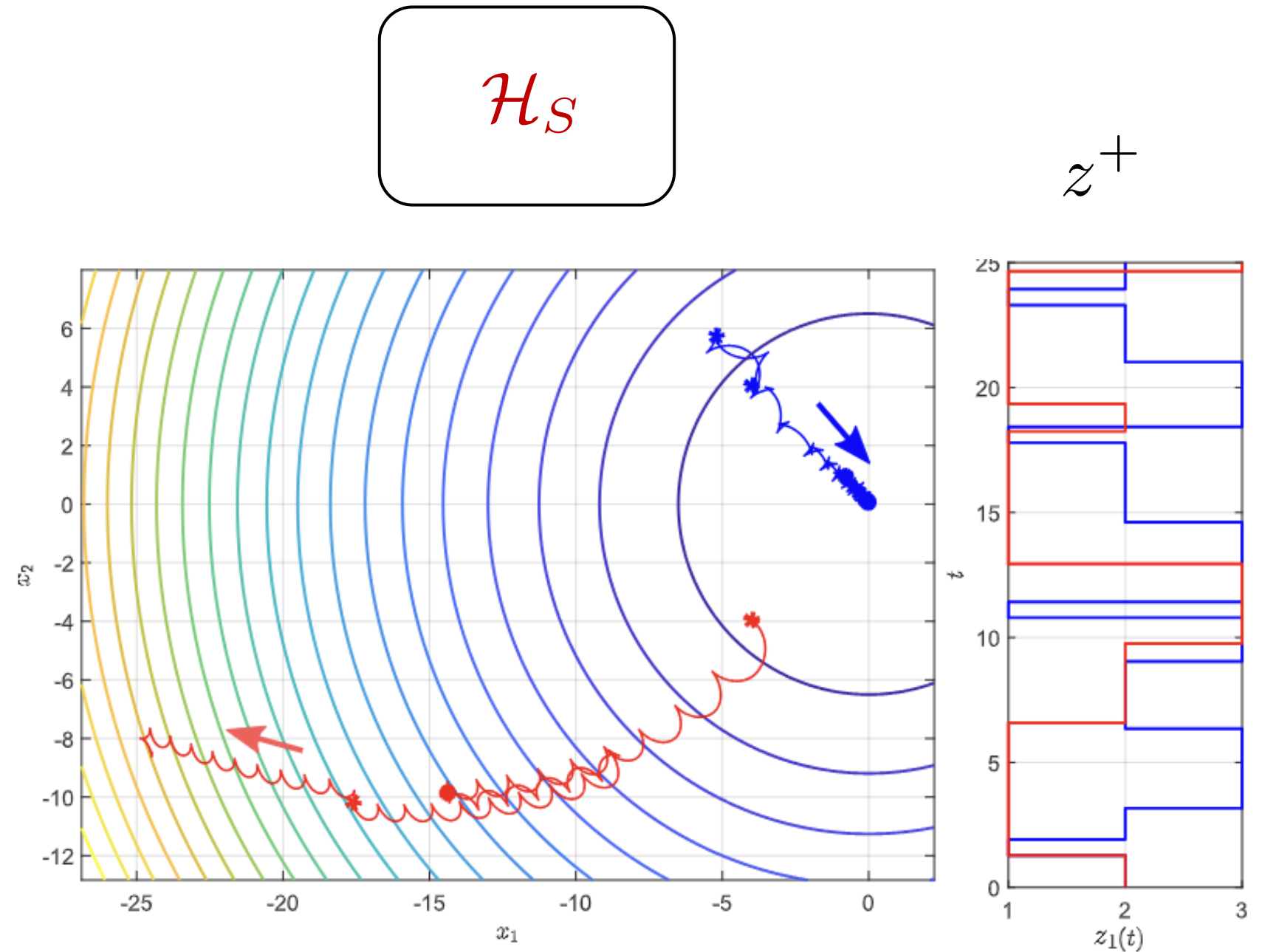
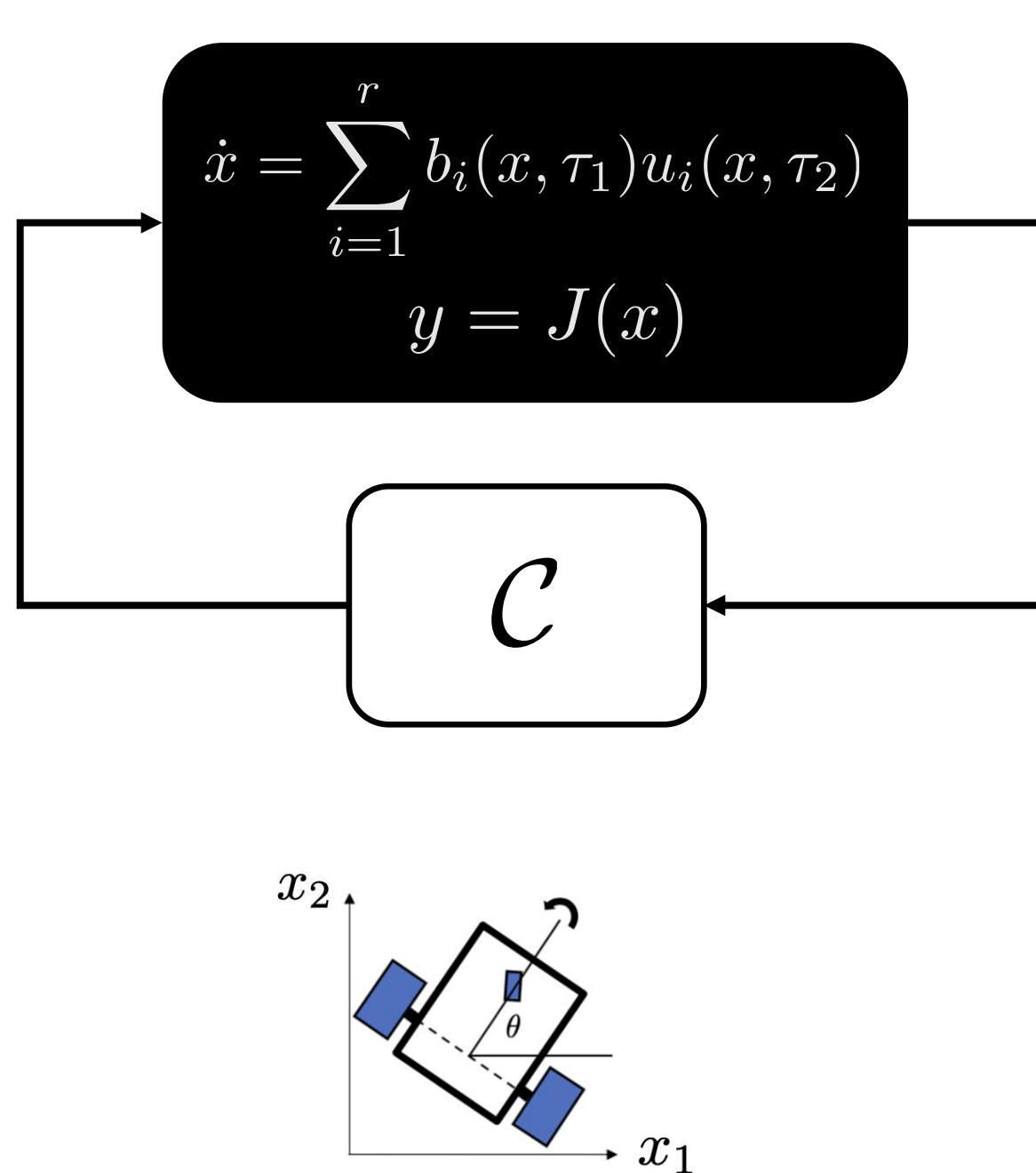


T^n
 \mathcal{H}_S



Synthesis of Dynamics for Model-Free Optimization

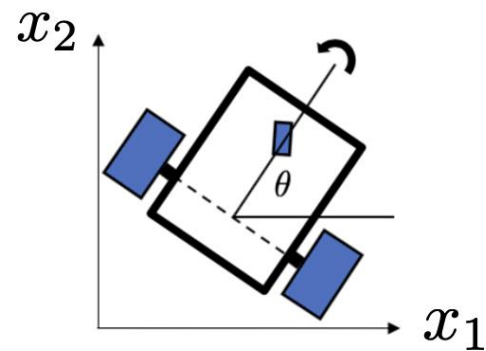
- Stability in seeking systems under **switching (including unstable) modes**:



Synthesis of Dynamics for Model-Free Optimization

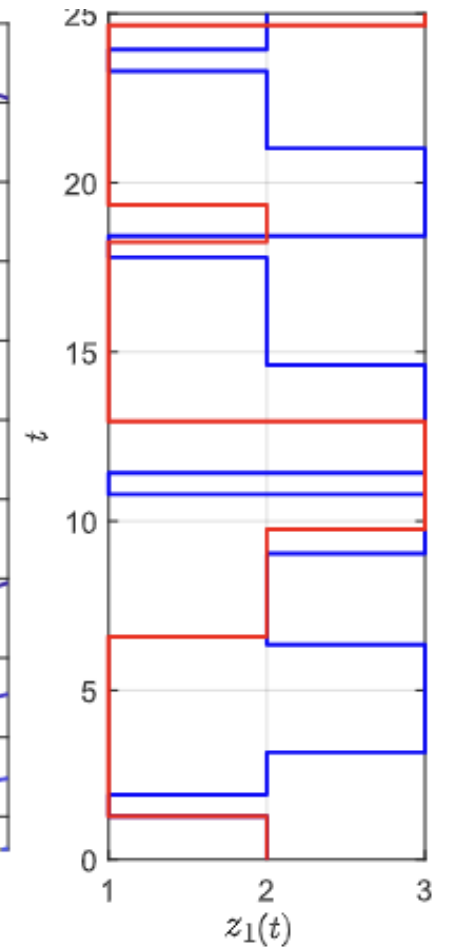
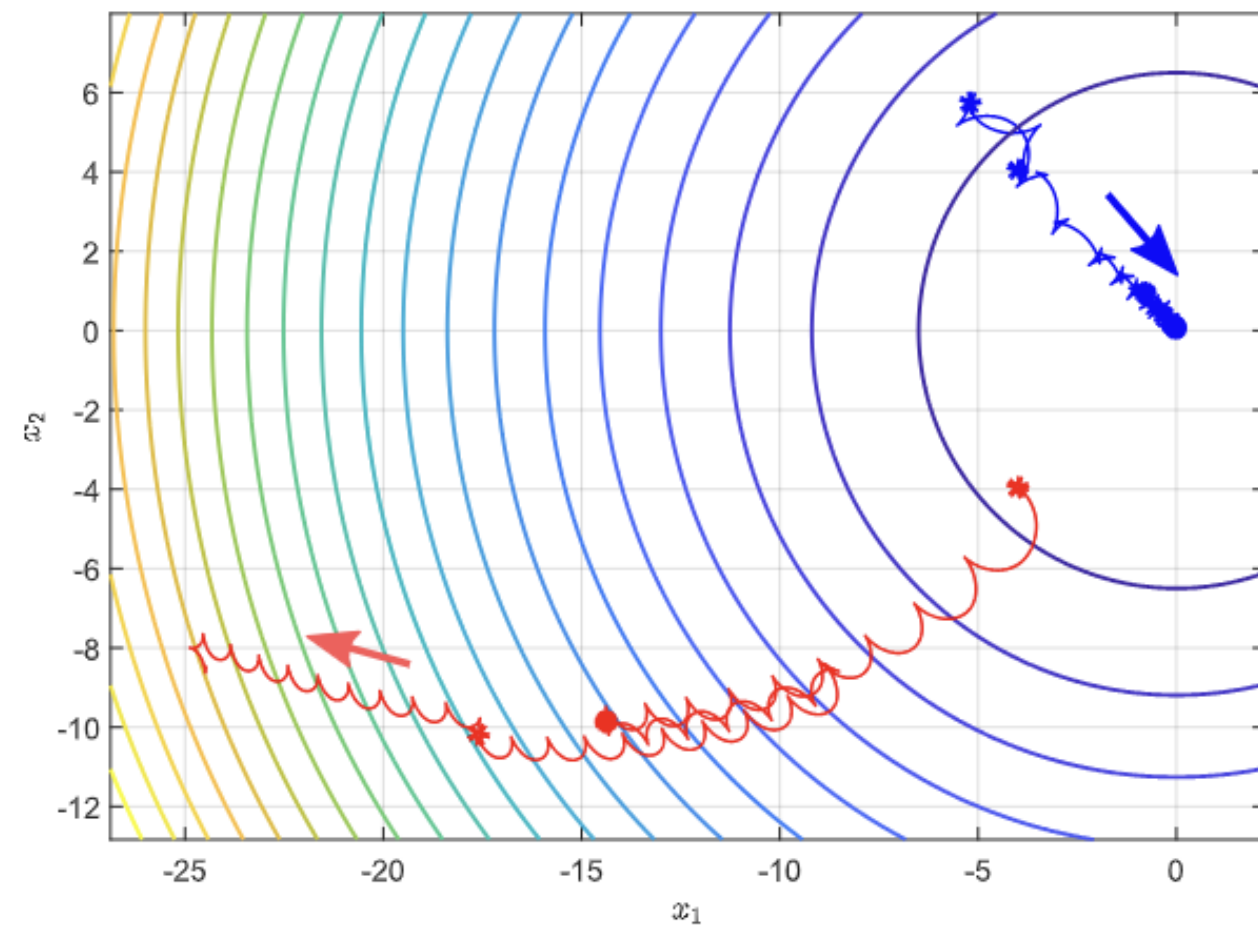
- Stability in seeking systems under **switching (including unstable) modes**:

$$\mathcal{H}_S = \left\{ \begin{array}{l} ((x, z), \tau) \in C \times \mathbb{R}_+^2, \\ \left\{ \begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \in F_\varepsilon(x, z, \tau_1, \tau_2) \\ \dot{\tau}_1 = \varepsilon^{-1} \\ \dot{\tau}_2 = \varepsilon^{-2} \end{array} \right. \end{array} \right. \\ \left\{ \begin{array}{l} ((x, z), \tau) \in D \times \mathbb{R}_+^2, \\ \left\{ \begin{array}{l} \begin{pmatrix} x^+ \\ z^+ \end{pmatrix} \in G(x, z) \\ \tau_1^+ = \tau_1 \\ \tau_2^+ = \tau_2 \end{array} \right. \end{array} \right.$$



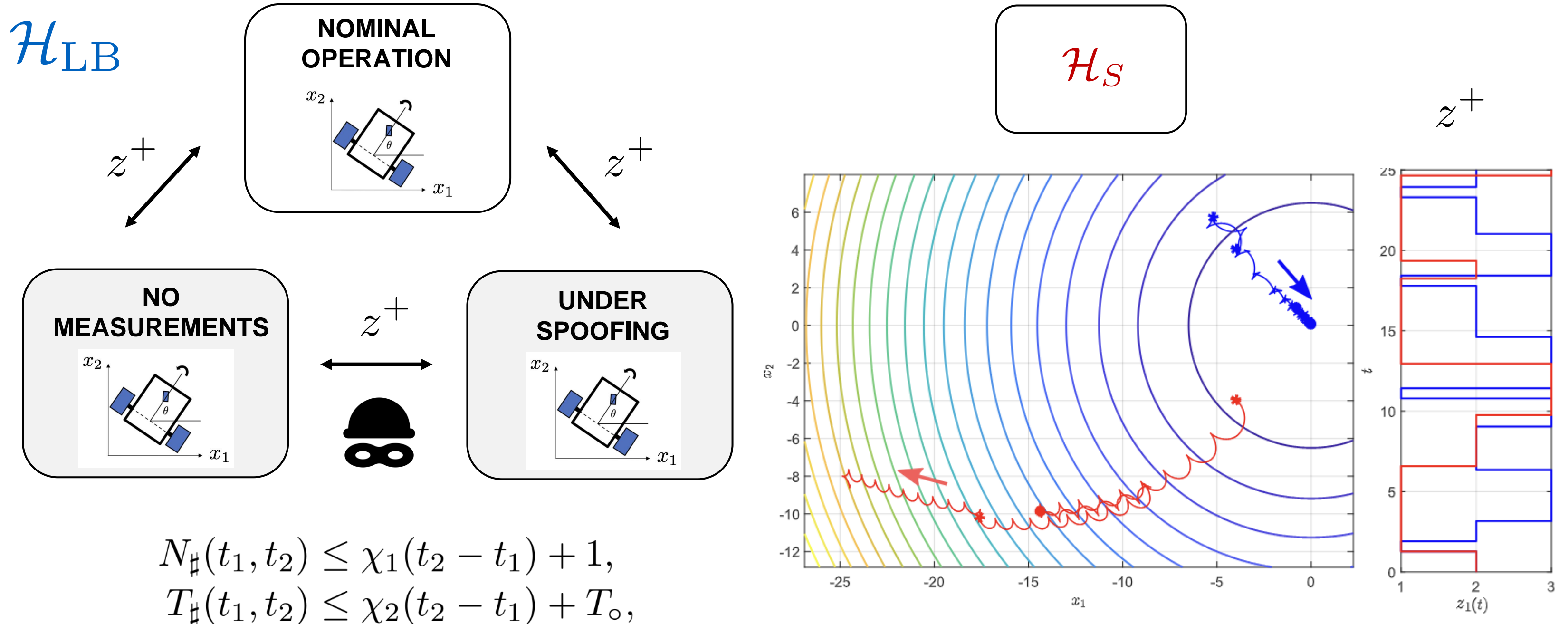
$$\mathcal{H}_S$$

$$z^+$$



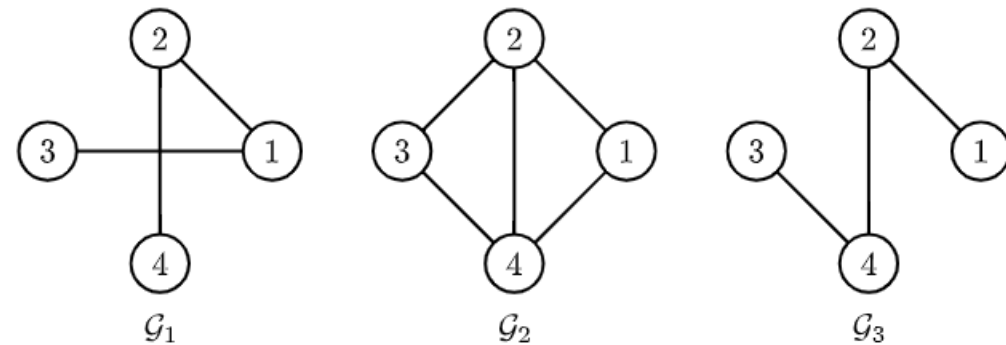
Synthesis of Dynamics for Model-Free Optimization

- Stability in seeking systems under **switching (including unstable) modes**

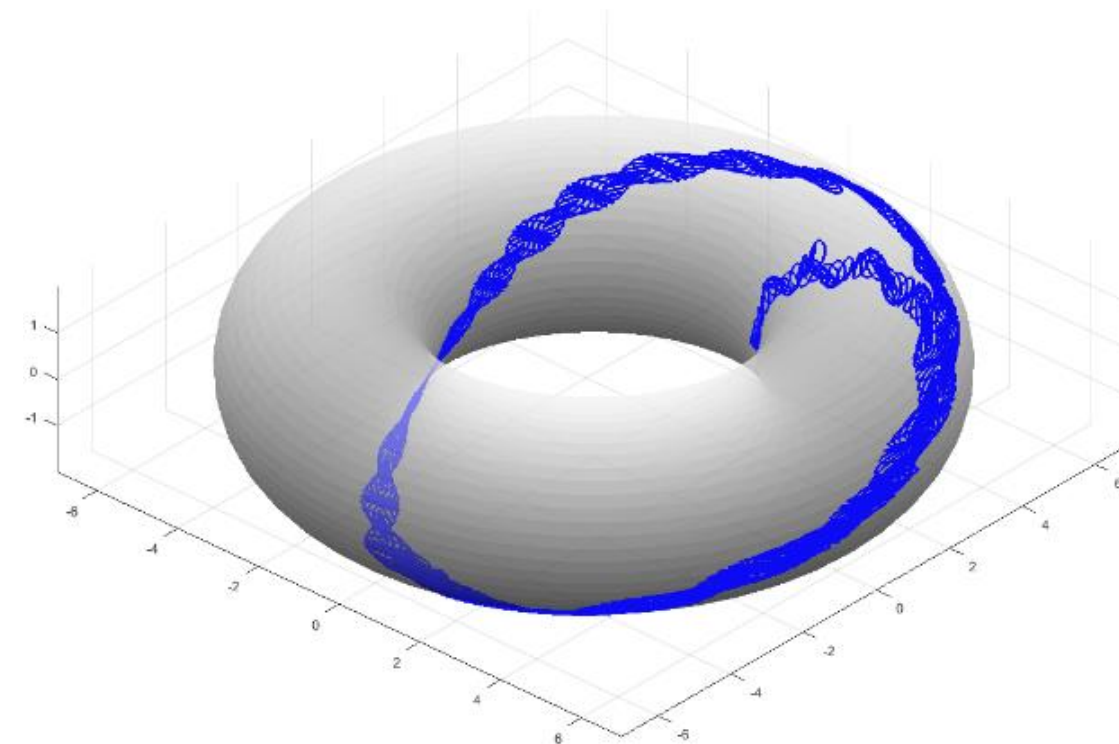
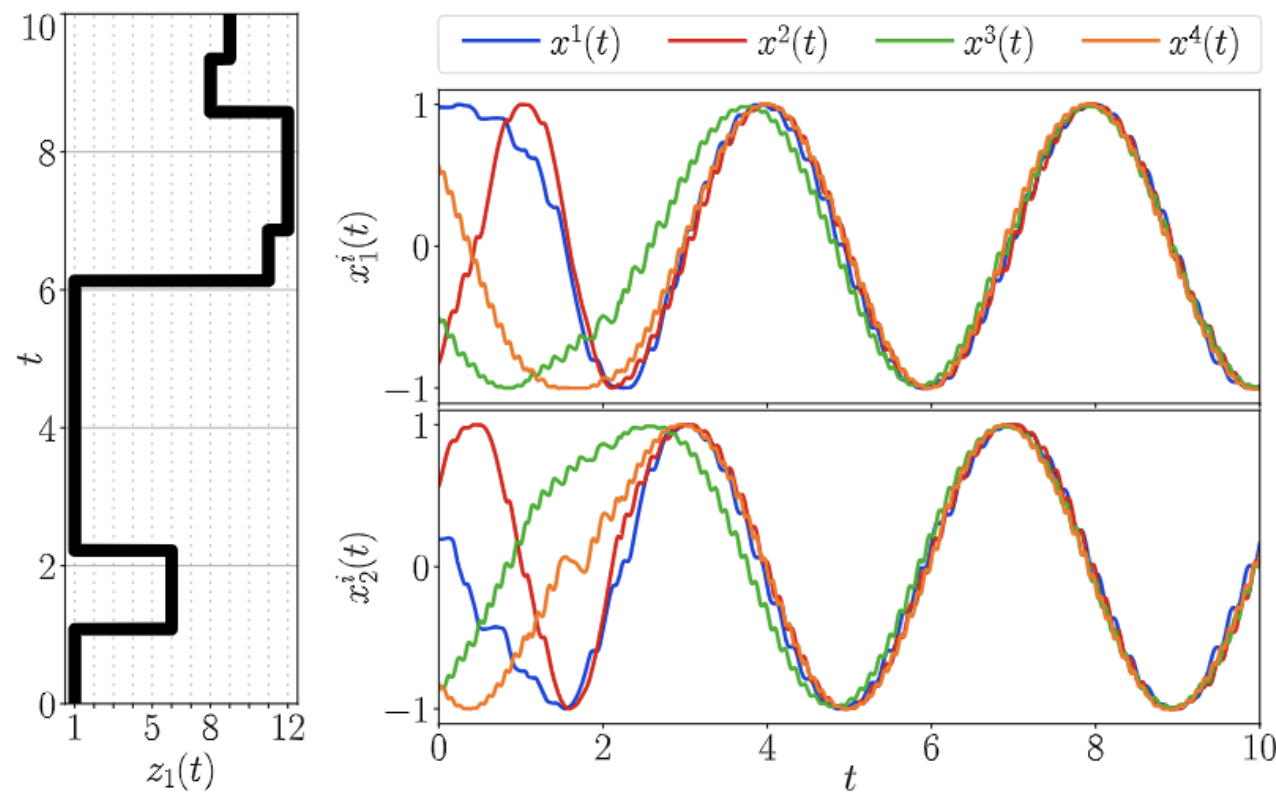


Synthesis of Dynamics for Model-Free Optimization

- Distributed multi-agent optimization problems (based on robustness of well-posed HDS)



$$\begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^r \end{pmatrix} = \begin{pmatrix} (1 + \alpha_1 u_1) S x^1 \\ \vdots \\ (1 + \alpha_r u_r) S x^r \end{pmatrix}$$



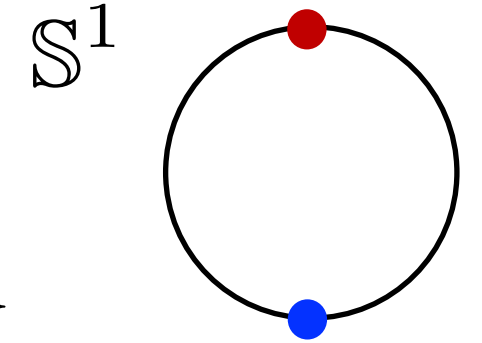
\mathcal{H}_{LB}

Synthesis of Dynamics for Model-Free Regulation:

Unknown Control Directions

Synthesis of Dynamics for Model-Free Regulation

Motivational Example: Global stabilization on the unit circle

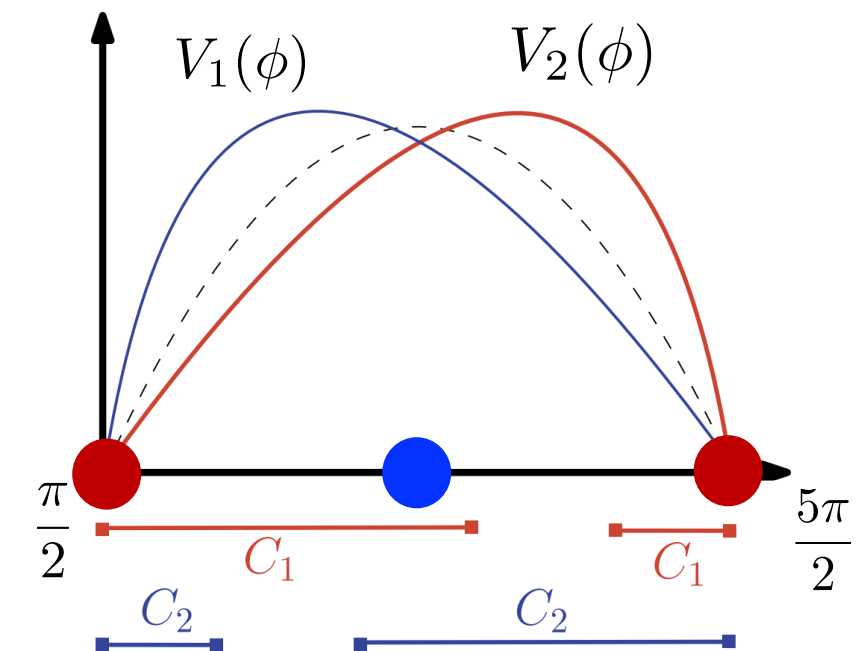


$$\dot{p} = b_1(p)\theta_1 u_1, \quad b_1(p) = Sp \quad \theta_1^+ \in \{-1, 0, 1\}$$

- Requires hybrid feedback for robust **global** stabilization (Mayhew, Teel, 2010)
- When the control direction is **known**, hybrid synergistic control suffices:

$$V(x) = V_q(p) \quad \longrightarrow \quad u = -\gamma \langle \nabla V_q(p), Sp \rangle$$

$$\dot{V} = -\gamma \theta \langle \nabla V_q(p), Sp \rangle^2$$



- Synergistic hybrid control can be **unstable** if the control direction is unknown...

Synthesis of Dynamics for Model-Free Regulation

For **ODEs**, the stabilization problem under **unknown control directions** was addressed by Scheinker and Krstic in 2013:

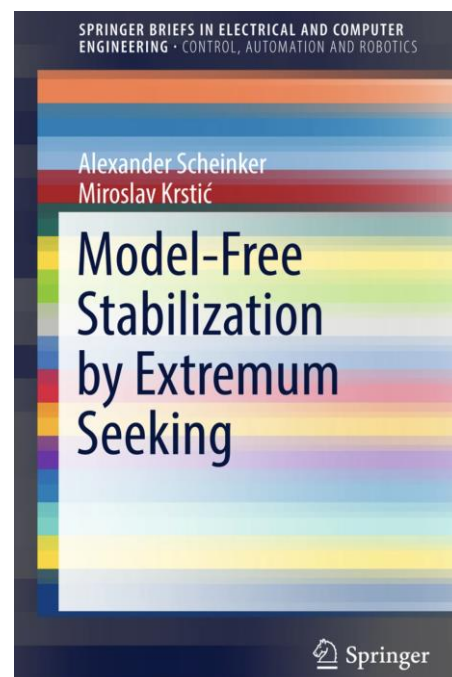
IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 58, NO. 5, MAY 2013

1107

Minimum-Seeking for CLFs: Universal Semiglobally Stabilizing Feedback Under Unknown Control Directions

Alexander Scheinker, *Student Member, IEEE*, and Miroslav Krstić, *Fellow, IEEE*

After that, multiple results were developed for **ODEs**:



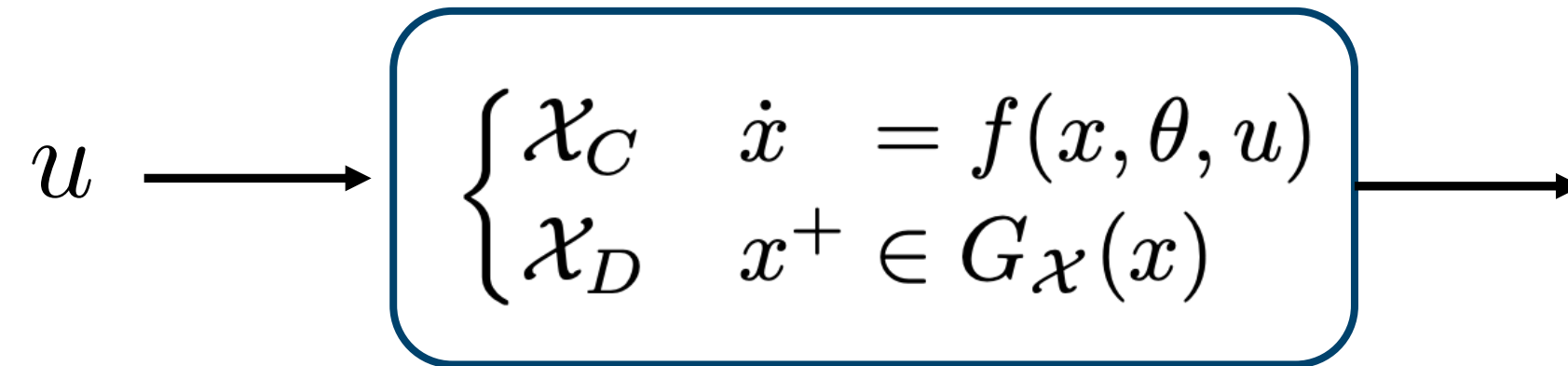
Extremum-Seeking Regulator for a Class of Nonlinear Systems with Unknown Control Direction

Shimin Wang, Martin Guay, Richard D. Braatz

This was a
**longstanding open
problem for hybrid
systems**

Synthesis of Dynamics for Model-Free Regulation

We consider the “open-loop” HDS:



with flow map of the form:

$$f(x, \theta, u) = f_0(x, \theta) + \sum_{i=1}^r f_i(x, \theta) u_i.$$

Dynamic Control Directions

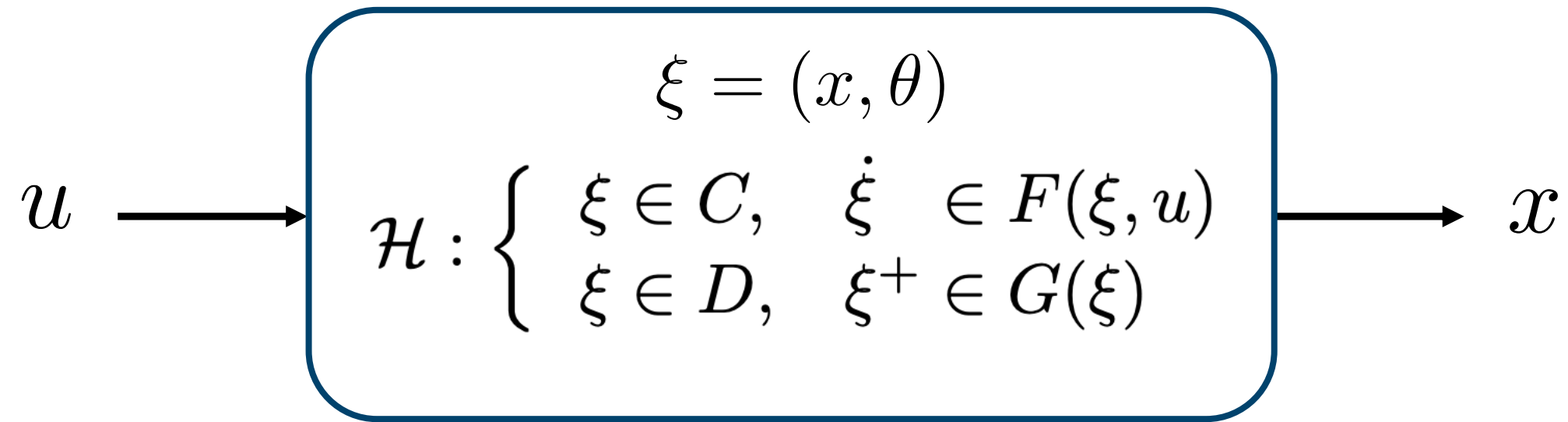
$$\left\{ \begin{array}{ll} \theta \in \Theta_C & \dot{\theta} \in F_e(\theta) \\ \theta \in \Theta_D & \theta^+ \in G_e(\theta) \end{array} \right.$$

Key Assumptions:

- (1) f_0 and f_i are \mathcal{C}^0 , and $f_i(\cdot, \theta)$ is \mathcal{C}^1 ;
- (2) $f_0(x, \theta) \in T_x \mathcal{X}_C$, and $f_i(x, \theta) \in T_x \mathcal{X}_C$;
- (3) $G(\mathcal{X}_D) \subset \mathcal{X}_C$, and $G(\mathcal{X}_D) \cap \mathcal{X}_D = \emptyset$. $G_e(\Theta_D) \subset \Theta_C$, and $G_e(\Theta_D) \cap \Theta_D = \emptyset$.

Synthesis of Dynamics for Model-Free Regulation

Open-loop HDS with dynamic control directions can be written as:



$$C = \mathcal{X}_C \times \Theta_C,$$

$$D = (\mathcal{X}_C \times \Theta_D) \cup (\mathcal{X}_D \times \Theta_C) \cup (\mathcal{X}_D \times \Theta_D),$$

$\nearrow F(\xi, u) = \{f(x, \theta, u)\} \times F_e(\theta),$

$\nearrow G(\xi) = \begin{cases} G_{\mathcal{X}}(x) \times G_e(\theta) & (x, \theta) \in \mathcal{X}_D \times \Theta_D \\ \{x\} \times G_e(\theta) & (x, \theta) \in \mathcal{X}_C \times \Theta_D \\ G_{\mathcal{X}}(x) \times \{\theta\} & (x, \theta) \in \mathcal{X}_D \times \Theta_C \end{cases}$

Synthesis of Dynamics for Model-Free Regulation

Definition 4 A \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be a *Strong Control Lyapunov Function (SCLF)* candidate with respect to \mathcal{A} for \mathcal{H} if there exists $\gamma > 0$, and \mathcal{K}_∞ -functions α_1, α_2 such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{X}, \quad (10)$$

$$\dot{V}(\xi) \leq 0, \quad \forall \xi \in C, \quad (11)$$

$$\Delta V(\xi) \leq 0, \quad \forall \xi \in D. \quad (12)$$

But this controller is **not** implementable

where the Lie derivative is computed using:

$$f(x, \theta, u) = f_0(x, \theta) + \sum_{i=1}^r f_i(x, \theta) u_i. \quad \text{with} \quad u_i(\xi) = -\gamma \langle \nabla V(x), f_i(\xi) \rangle,$$

Namely,

$$\dot{V}(\xi) := \langle \nabla V(x), \bar{f}(\xi) \rangle, \quad \bar{f}(\xi) := f_0(x, \theta) - \gamma \sum_{i=1}^r \langle \nabla V(x), f_i(x, \theta) \rangle f_i(x, \theta),$$

Similar to the notion of "strong $L_g V$ - stabilizability" for ODEs

Synthesis of Dynamics for Model-Free Regulation

Definition 4 A \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be a *Strong Control Lyapunov Function (SCLF)* candidate with respect to \mathcal{A} for \mathcal{H} if there exists $\gamma > 0$, and \mathcal{K}_∞ -functions α_1, α_2 such that

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$$f(x, \theta, u) = f_0(x, \theta) + \sum_{i=1}^r f_i(x, \theta) u_i. \quad \text{with} \quad u_i(\xi) = -\gamma \langle \nabla V(x), f_i(\xi) \rangle,$$

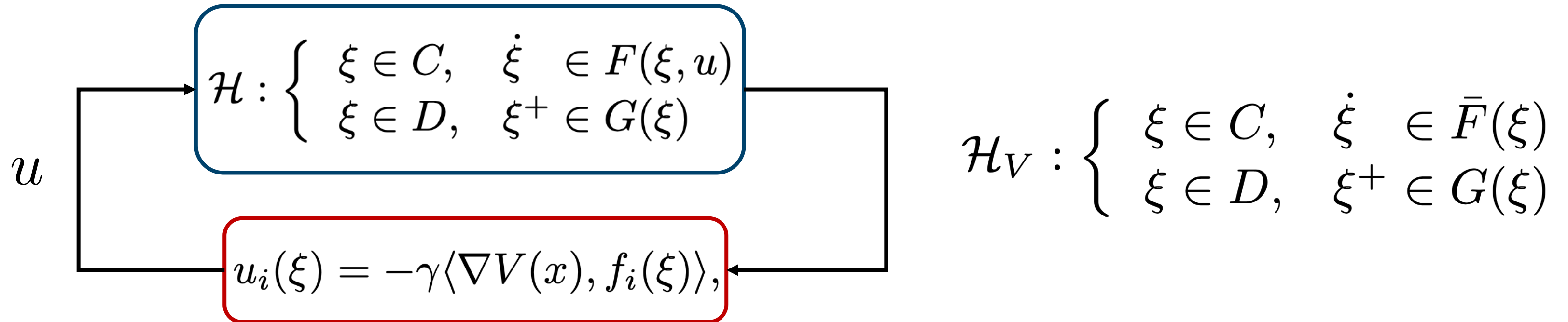
Namely,

$$\dot{V}(\xi) := \langle \nabla V(x), f_0(x, \theta) \rangle - \gamma \sum_{i=1}^r \langle \nabla V(x), f_i(x, \theta) \rangle^2 \leq 0,$$

Similar to the notion of "strong $L_g V$ - stabilizability" for ODEs

Synthesis of Dynamics for Model-Free Regulation

Consider the following “ideal” (not implementable) closed-loop system:

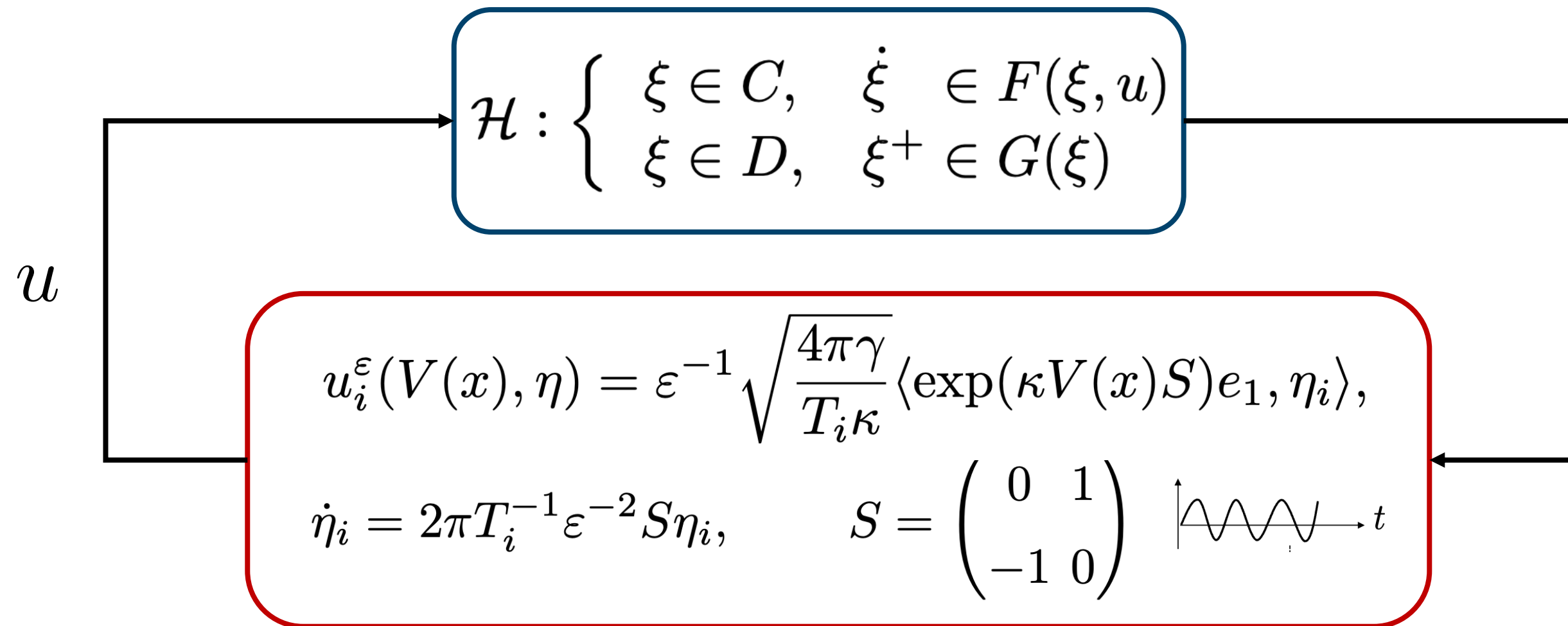


Definition: The open-loop HDS is said to be ∇V – stabilizable if it admits a SCLF V such that no solution of the “ideal” closed-loop system keeps V in a non-zero level set.

 $\mathcal{A} \times (\Theta_C \cup \Theta_D)$ is UGAS (via hybrid invariance principle)

Main Result 2: Model-Free Regulation of HDS

Suppose the open-loop HDS is ∇V – stabilizable. Consider the feedback:

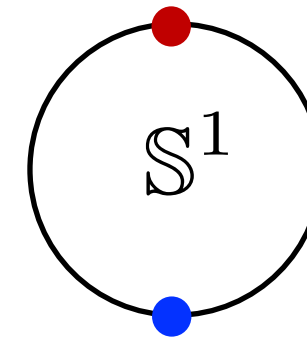


Then, the set $\bar{\mathcal{A}} := \mathcal{A} \times (\Theta_C \cup \Theta_D) \times \mathbb{T}^r$ is SGpAS as $\varepsilon \rightarrow 0^+$

“Hybrid Minimum-Seeking in Synergistic Lyapunov Functions: Robust Global Stabilization under Unknown Control Directions”, *Arxiv 2024*

Synthesis of Dynamics for Model-Free Stabilization

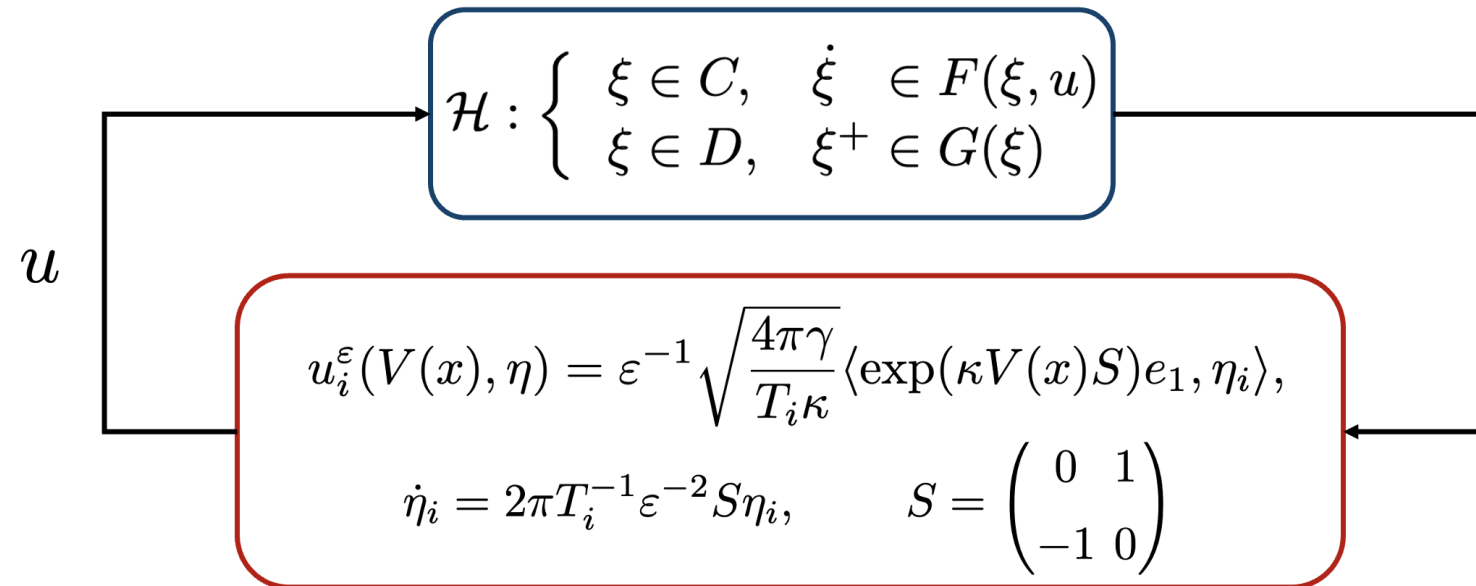
Example: Global Stabilization on the Unit Circle



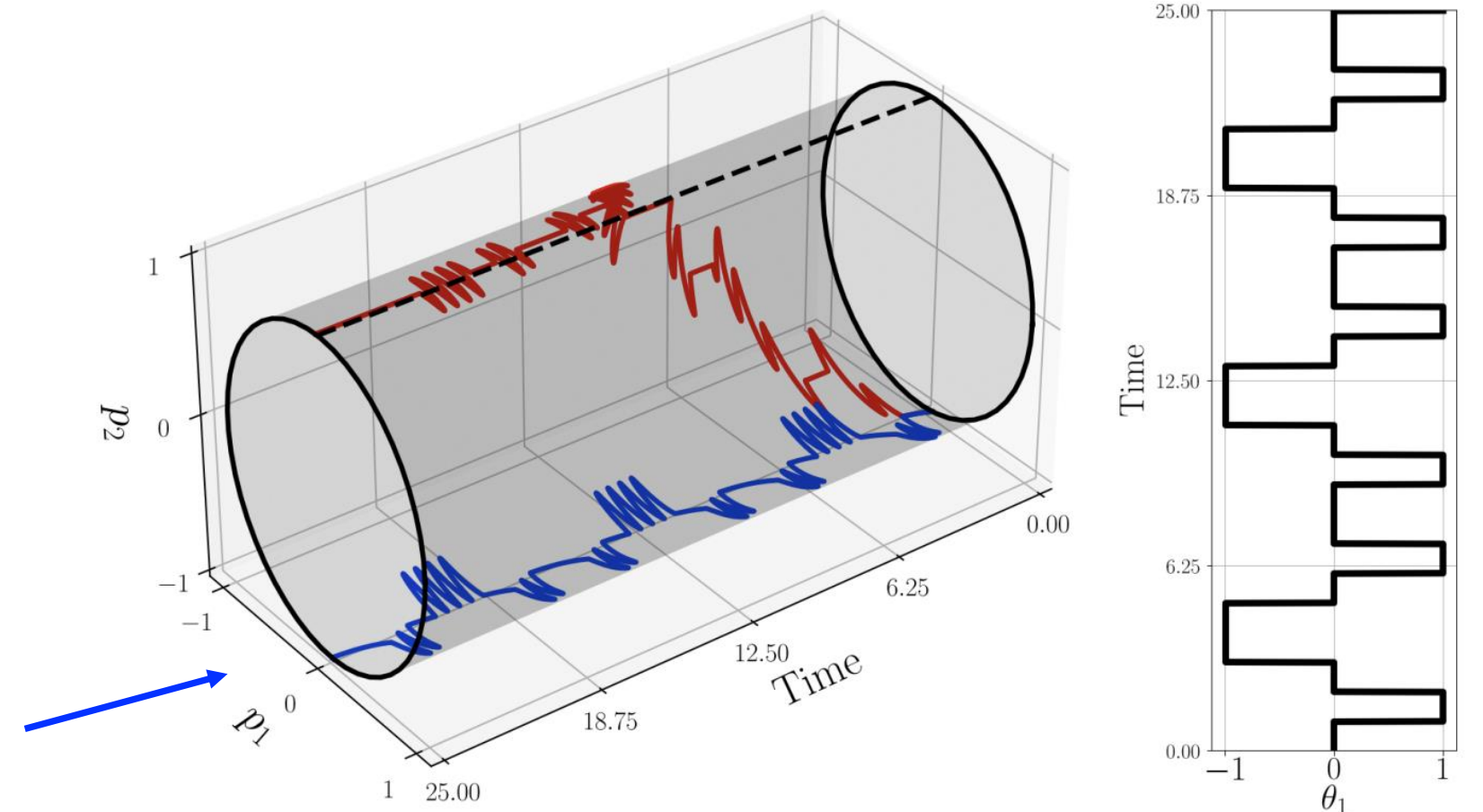
$$\dot{p} = b_1(p)\theta_1 u_1,$$

$$b_1(p) = Sp$$

$$\theta_i^+ \in \{-1, 0, 1\}$$



Adversarial perturbations are problematic in non-hybrid controllers

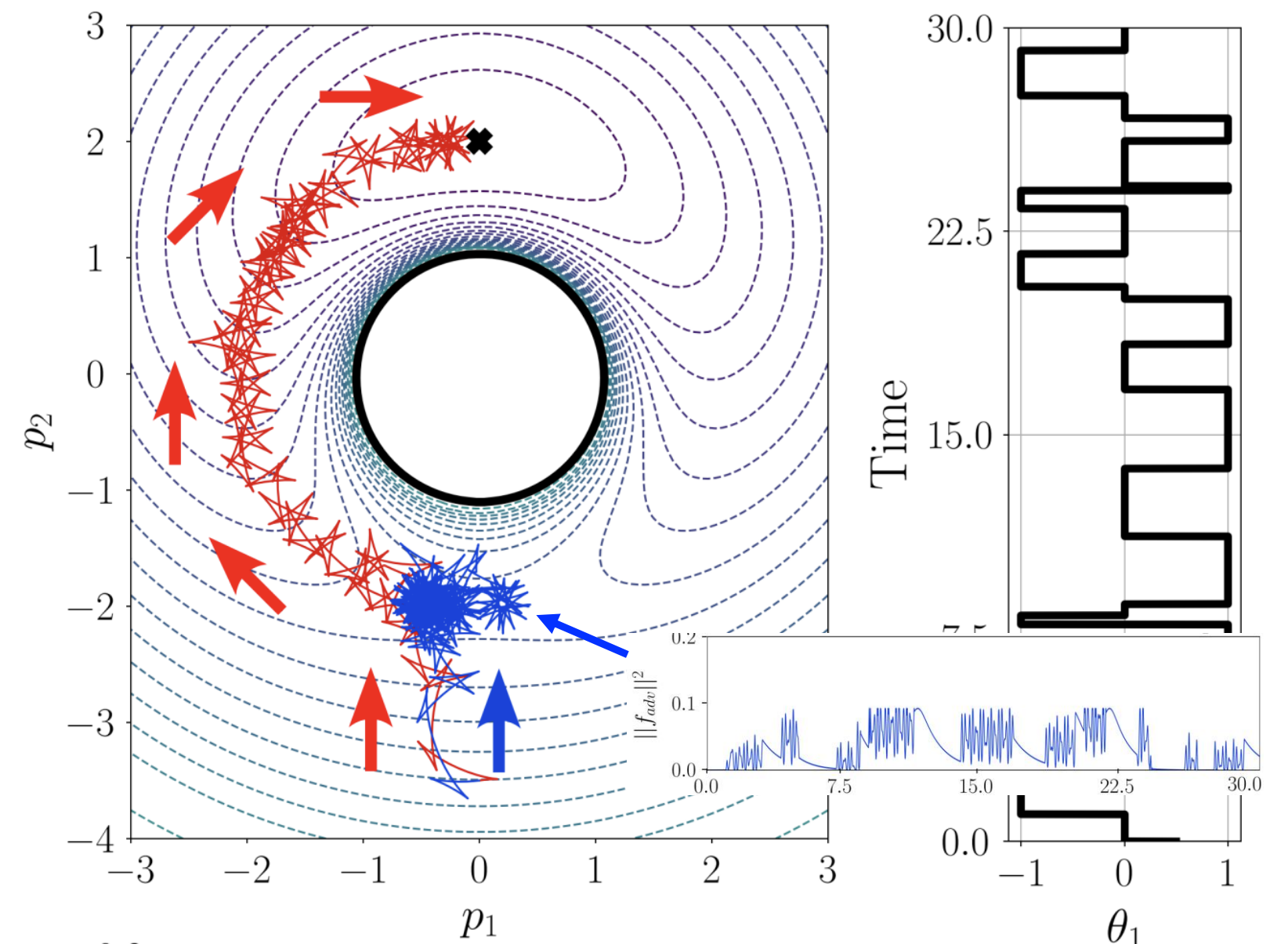
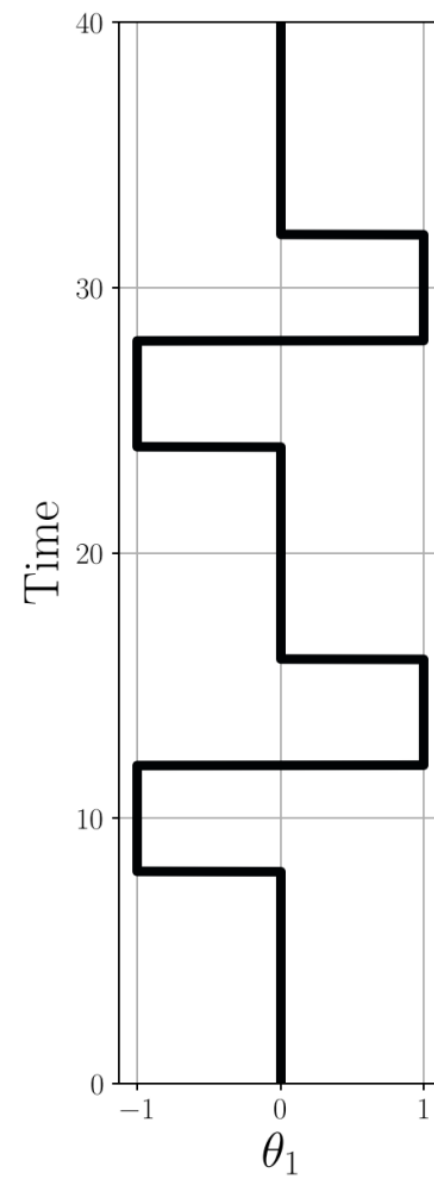
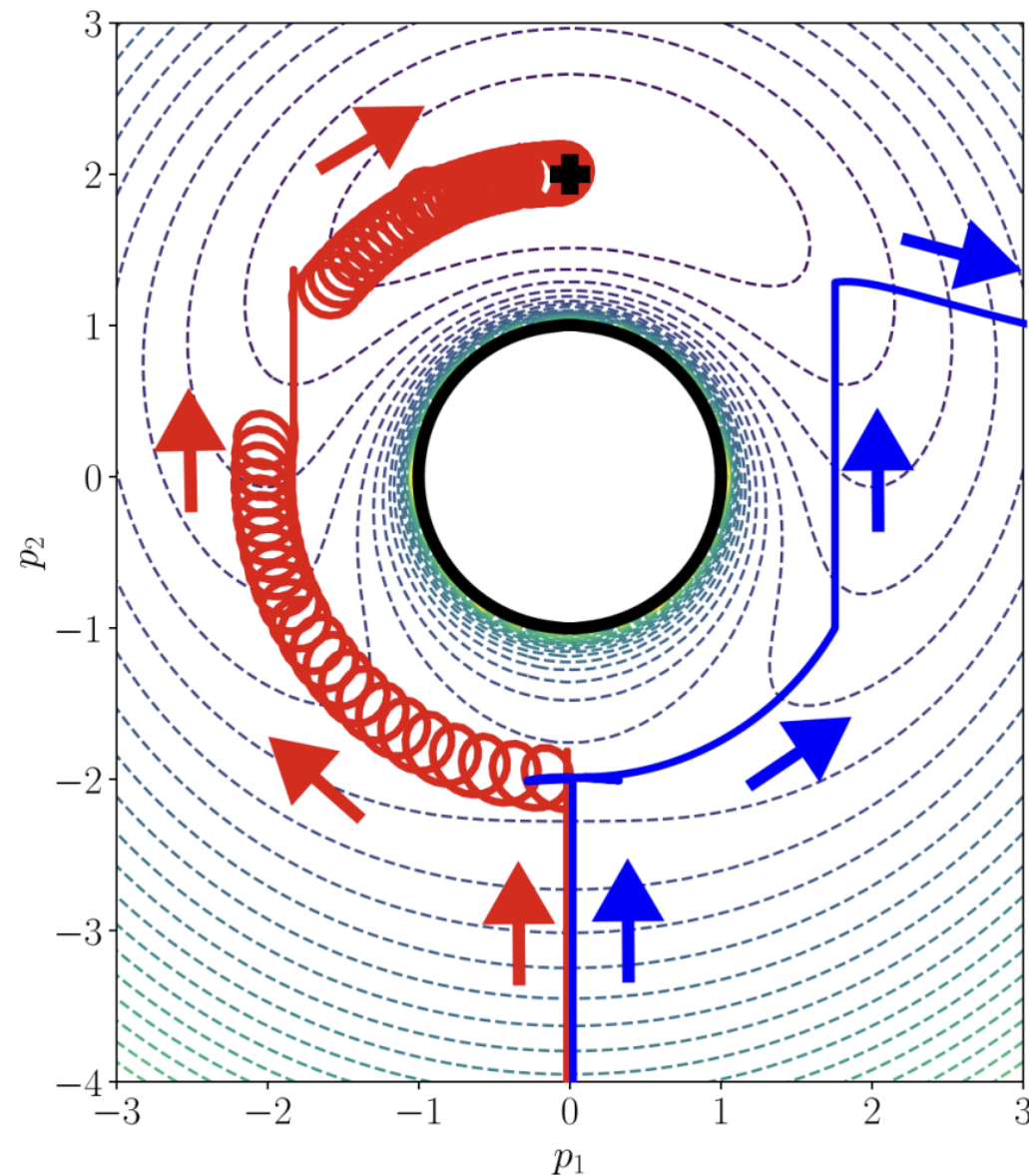


Synthesis of Dynamics for Model-Free Regulation

Example: Global Regulation with Obstacle Avoidance

$$\dot{z} = \sum_{i=1}^2 e_i \theta_i u_i, \quad \theta_i^+ \in \{-1, 0, 1\}$$

$$\dot{z} = \theta_1 u_1 \psi, \quad \dot{\psi} = u_2 S \psi$$



Synthesis of Dynamics for Model-Free Regulation

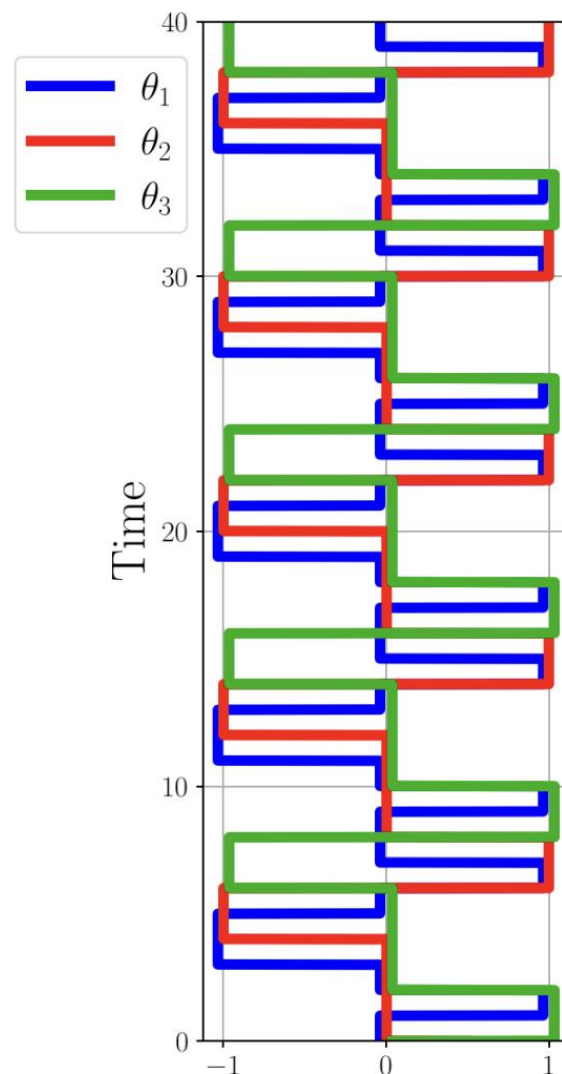
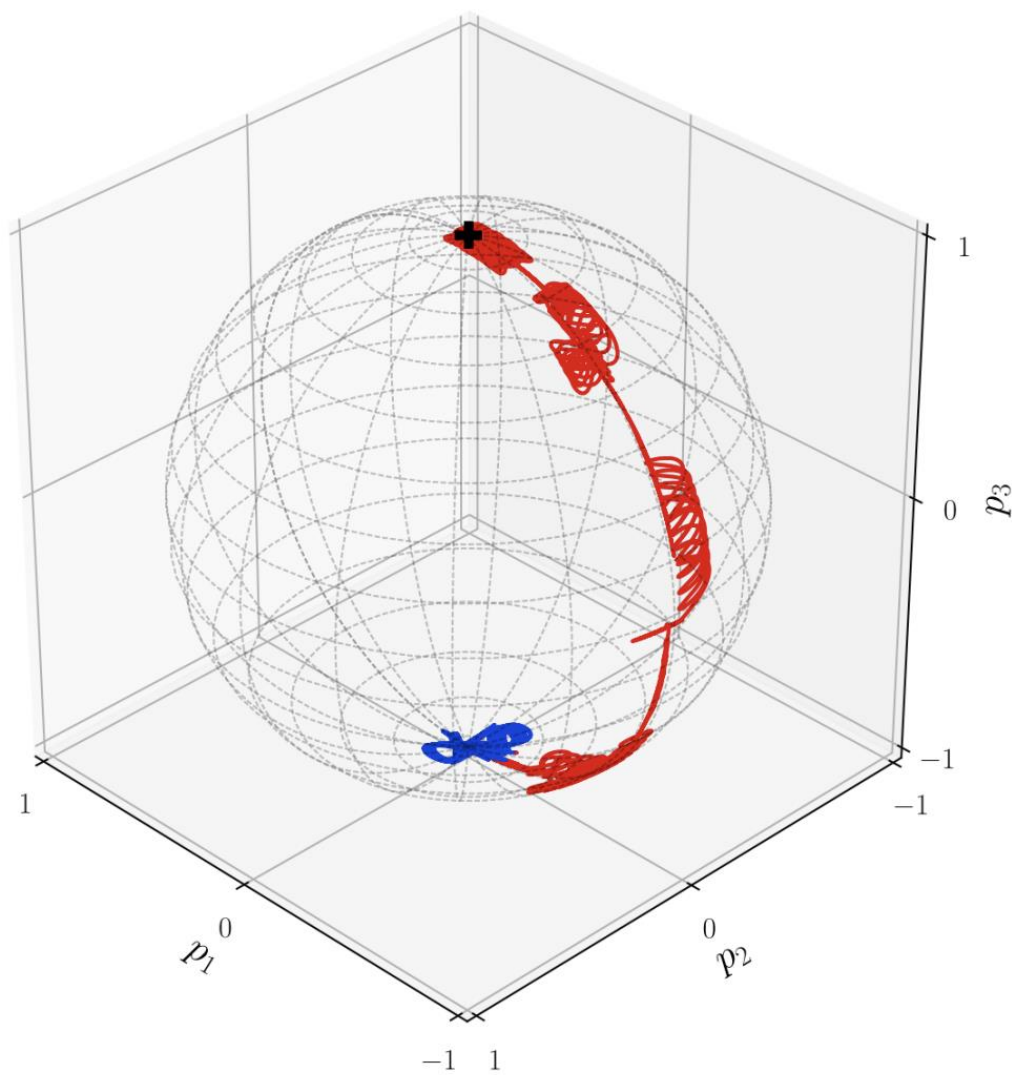
Example: Global Regulation in \mathbb{S}^2

$$\dot{p} = \sum_{i=1}^3 b_i(p) \theta_i u_i, \quad b_i(p) = e_i - \langle p, e_i \rangle p,$$

$$\theta_i^+ \in \{-1, 0, 1\}$$

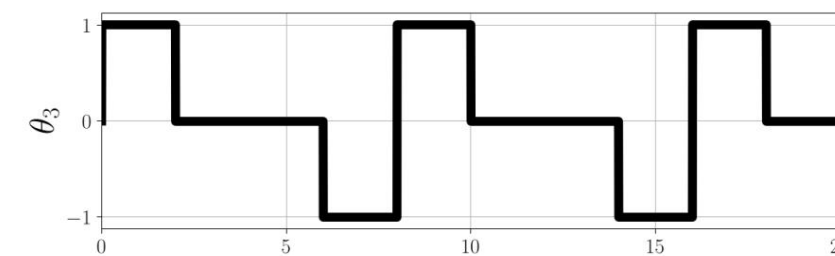
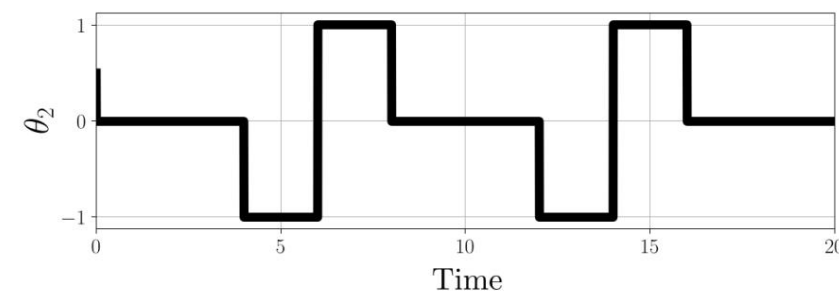
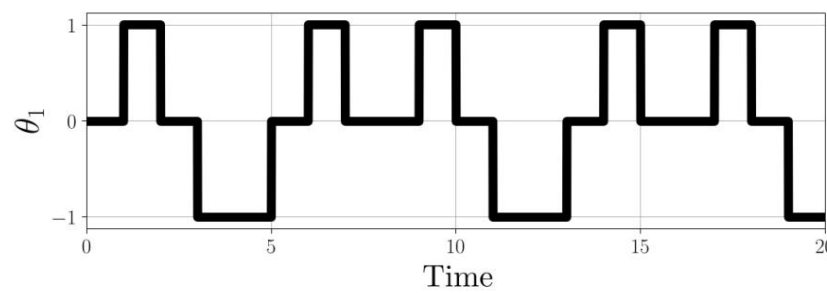
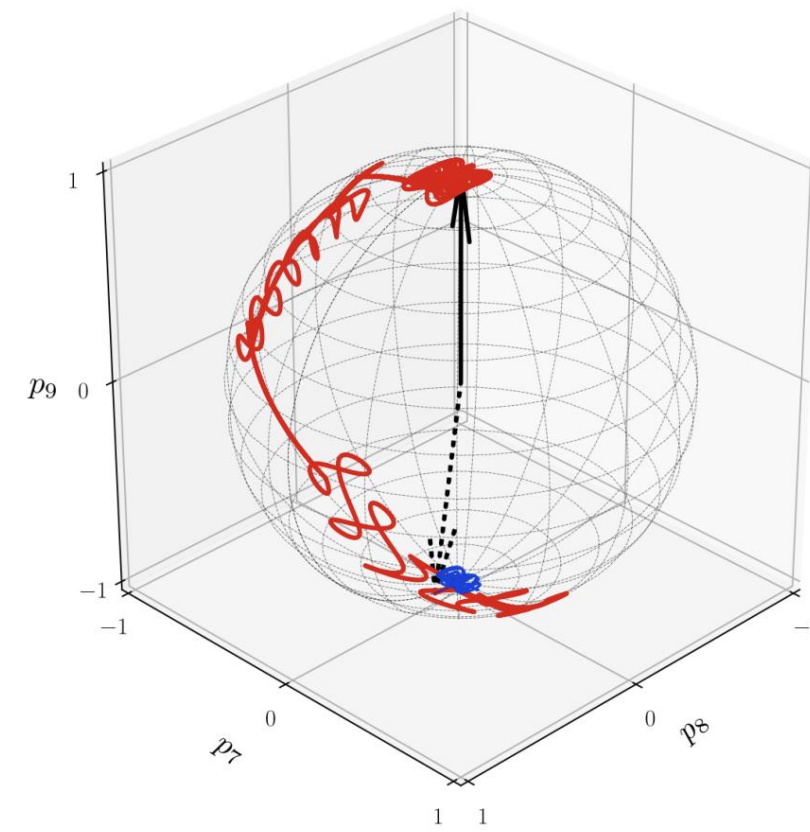
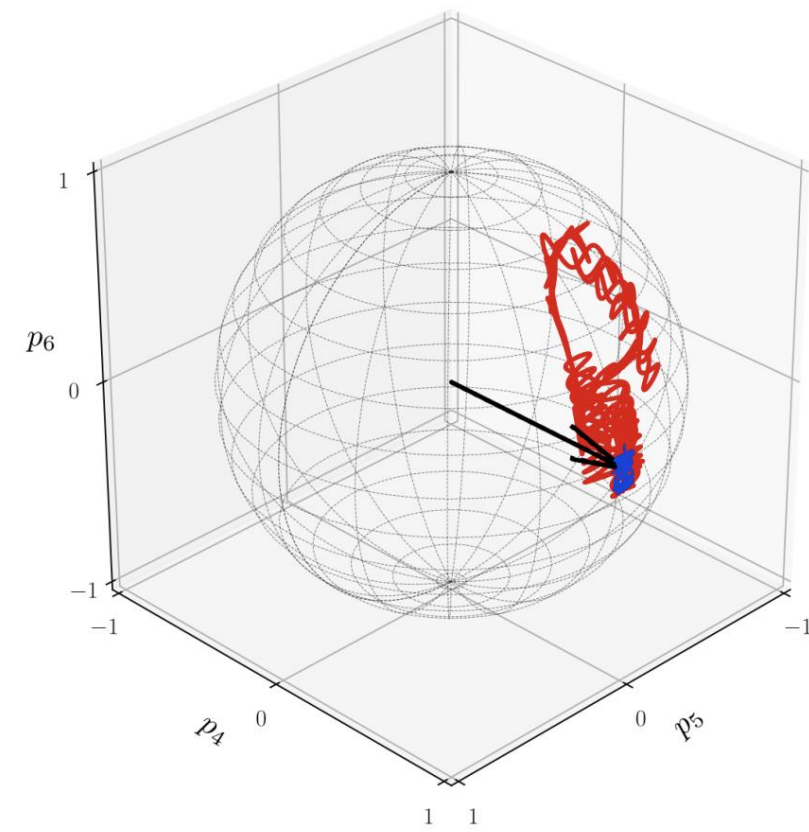
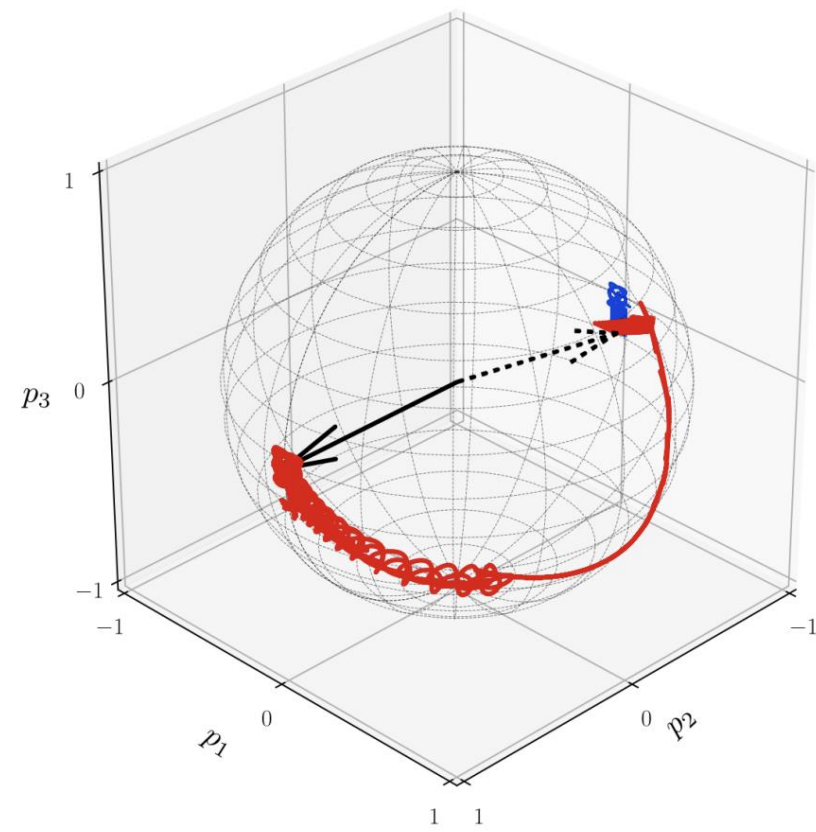
$$V_q(p) := W \circ \Phi_q(p), \quad W(p) := 1 - \langle p, p^* \rangle,$$

$$\Phi_q(p) := \exp((3/2 - q)W(p)[p_\perp^*]_\times) p,$$



Synthesis of Dynamics for Model-Free Stabilization

Example: Global Stabilization in Lie groups, e.g., rigid bodies $SO(3)$



$$\dot{R} = \sum_{i=1}^3 R \hat{e}_i \theta_i u_i,$$

$$\theta_i^+ \in \{-1, 0, 1\}$$

Today:

LIE BRACKET EXTENSIONS
AND AVERAGING: THE
SINGLE-BRACKET CASE

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Hybrid
Systems

2. Introducing new hybrid algorithms for model-free optimization and regulation



“Hybrid Minimum-Seeking in Synergistic Lyapunov Functions: Robust Global Stabilization under Unknown Control Directions”, *Arxiv* 2024

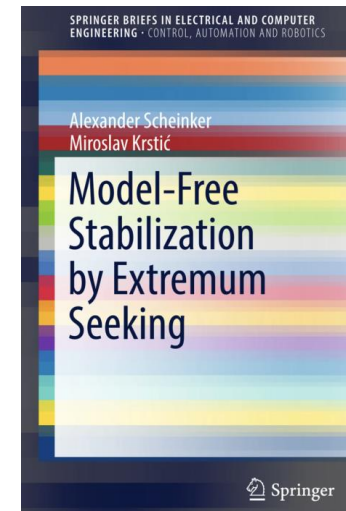
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Rutgers University



Hybrid
Systems



Hybrid
Systems

3. Introducing novel **global (practical) stability** properties for Lie-bracket averaging systems and algorithms

“Initialization-free Lie-bracket Extremum Seeking”, *Systems and Control Letters*, Vol. 191, pp. 105881

Global Stability via Lie-bracket Averaging

- Previous results were of **semi-global** (practical) nature:
 - They become **global** if the state space is **compact**
 - But if the state space is **unbounded**, we need to pre-specify the compact set of initial conditions *a priori*
 - This is a well-known limitation of averaging-based seeking systems
 - **Can we overcome this limitation using Lie-bracket averaging?**
 - A positive answer to this question would be very useful later for the incorporation of **stochastic phenomena**

Global (Practical) Stability via Lie-bracket Averaging

$$\dot{x} = f_\varepsilon(x, \tau) = \varepsilon^{-1} f_1(x, \tau) + f_2(x, \tau), \quad \dot{\tau} = \varepsilon^{-2},$$

Assumption 1. There exists $\delta_1 \in [0, \infty)$ such that, for all $k \in \{1, 2\}$, the following conditions hold

(a) The map f_k is C^0 in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, and there exist a positive constant L_k such that

$$|f_k(x_1, \tau) - f_k(x_2, \tau)| \leq L_k |x_1 - x_2|,$$

for all $x_1, x_2 \in \{x \in \mathbb{R}^n : |x| \geq \delta_1\}$ and all $\tau \in \mathbb{R}_{\geq 0}$.

(b) There exists $T \in \mathbb{R}_{>0}$ such that

$$f_k(x, \tau + T) = f_k(x, \tau), \quad \int_0^T f_1(x, \tau) d\tau = 0,$$

for all $(x, \tau) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

(c) The map f_k is C^{3-k} with respect to x in the domain $\{x \in \mathbb{R}^n : |x| \geq \delta_1\}$.

(d) There exists $L_3 > 0$ such that

$$|D_x f_1(x_1, \tau_1) f_1(x_1, \tau_2) - D_x f_1(x_2, \tau_1) f_1(x_2, \tau_2)| \leq L_3 |x_1 - x_2|,$$

for all $x_1, x_2 \in \{x \in \mathbb{R}^n : |x| \geq \delta_1\}$ and all $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$. \square

$$\Psi(\cdot)^{-1} \quad \Psi(\cdot)$$

$$\dot{x} = \bar{f}(x) + \varepsilon g(x, \tau, \varepsilon),$$

$$|g(x, \tau, \varepsilon)| \leq L_g(|x| + 1),$$

Global (Practical) Stability via Lie-bracket Averaging

$$\dot{x} = \bar{f}(x) + \varepsilon g(x, \tau, \varepsilon),$$

Assumption 2. There exists a vector δ satisfying (7) with the same δ_1 generated by Assumption 1, a C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\alpha_i \in \mathcal{K}_\infty$, $c_i > 0$, for $i \in \{1, 2\}$, and a positive definite function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, all independent of δ_1 , such that the following holds:

(a) For all $x \in \mathbb{R}^n$, we have that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad (15a)$$

$$|\nabla V(x)| \leq c_2 \phi(x). \quad (15b)$$

(b) For all $x \in \mathcal{M}_3$, we have that

$$\langle \nabla V(x), \bar{f}(x) \rangle \leq -c_1 \phi(x)^2. \quad (15c)$$

(c) At least one of the following statements holds:

(i) There exists $\bar{L}_g > 0$, such that

$$|g(x, \tau, \varepsilon)| \leq \bar{L}_g(\phi(x) + 1),$$

for all $(x, \tau, \varepsilon) \in \mathcal{M}_3 \times \mathbb{R}_{\geq 0} \times [0, \varepsilon_0]$, where the map g and the constant ε_0 are generated by Proposition 1.

(ii) There exists $\alpha_3 \in \mathcal{K}$, such that $\alpha_3(|x|)|x| \leq \phi(x)$, for all $x \in \mathbb{R}^n$. \square

Theorem 1. Suppose that Assumptions 1–2 hold. Then, there exists $\Delta_\delta > 0$ such that system (6) is Δ_δ -UGUB.

$$\dot{x} = f_\varepsilon(x, \tau) = \varepsilon^{-1} f_1(x, \tau) + f_2(x, \tau), \quad \dot{\tau} = \varepsilon^{-2},$$

Corollary 1. Suppose the assumptions of Theorem 1 hold. Then, there exists $\tilde{\Delta}_\delta > 0$ such that system (4) is $\tilde{\Delta}_\delta$ -UGUB.

Corollary 2. Suppose that Assumptions 1 and 2 are satisfied for all δ such that $\delta_1 > 0$. Then, system (4) is UGpAS as $\varepsilon \rightarrow 0^+$.

Corollary 3. Suppose that Assumptions 1 and 2 are satisfied for all δ such that $\delta_2 = \delta_1 = 0$. Then, system (4) is UGpAS as $\varepsilon \rightarrow 0^+$.

Global (Practical) Stability via Lie-bracket Averaging

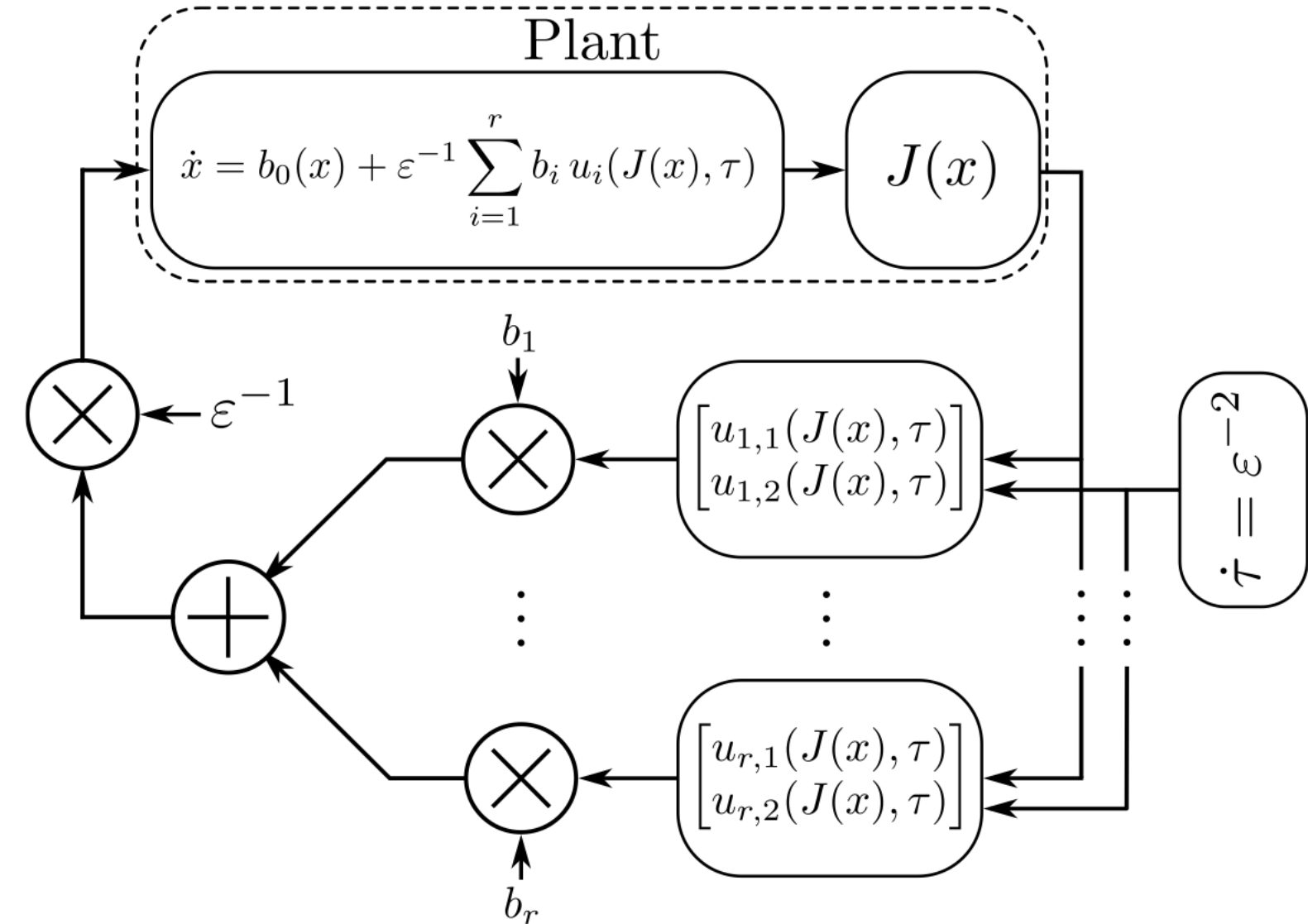
$$\dot{x} = f_\varepsilon(x, \tau) = \varepsilon^{-1} f_1(x, \tau) + f_2(x, \tau), \quad \dot{\tau} = \varepsilon^{-2},$$

$$\dot{x} = \varepsilon^{-1} \left(\sum_{i=1}^r \sum_{j=1}^2 b_{i,j} u_{i,j}(J(x), \tau) \right) + b_0(x), \quad \dot{\tau} = \varepsilon^{-2},$$

Assumption 3. There exists $\gamma > 0$, such that the vectors $b_{i,j}$ satisfy $\sum_{i=1}^r \sum_{j=1}^2 (b_{i,j}^\top v)^2 \geq \gamma |v|^2$, for all $v \in \mathbb{R}^n$. \square

Assumption 4. The following holds:

- (a) $J(x) > J(0)$, for all $x \neq 0$.
- (b) $\nabla J(x) = 0$ if and only if $x = 0$.
- (c) There exists $L_J > 0$ such that $|\nabla^2 J(x)| \leq L_J$, for all $x \in \mathbb{R}^n$.
- (d) There exists $\kappa_3 > 0$ such that $|b_0(x)| \leq \kappa_3 |\nabla J(x)|$, for all $x \in \mathbb{R}^n$.
- (e) There exists $L_0 > 0$ such that $|b_0(x_1) - b_0(x_2)| \leq L_0 |x_1 - x_2|$, for all $x_1, x_2 \in \mathbb{R}^n$. \square



Main Result 3: Global Stability Properties via Lie-bracket Averaging

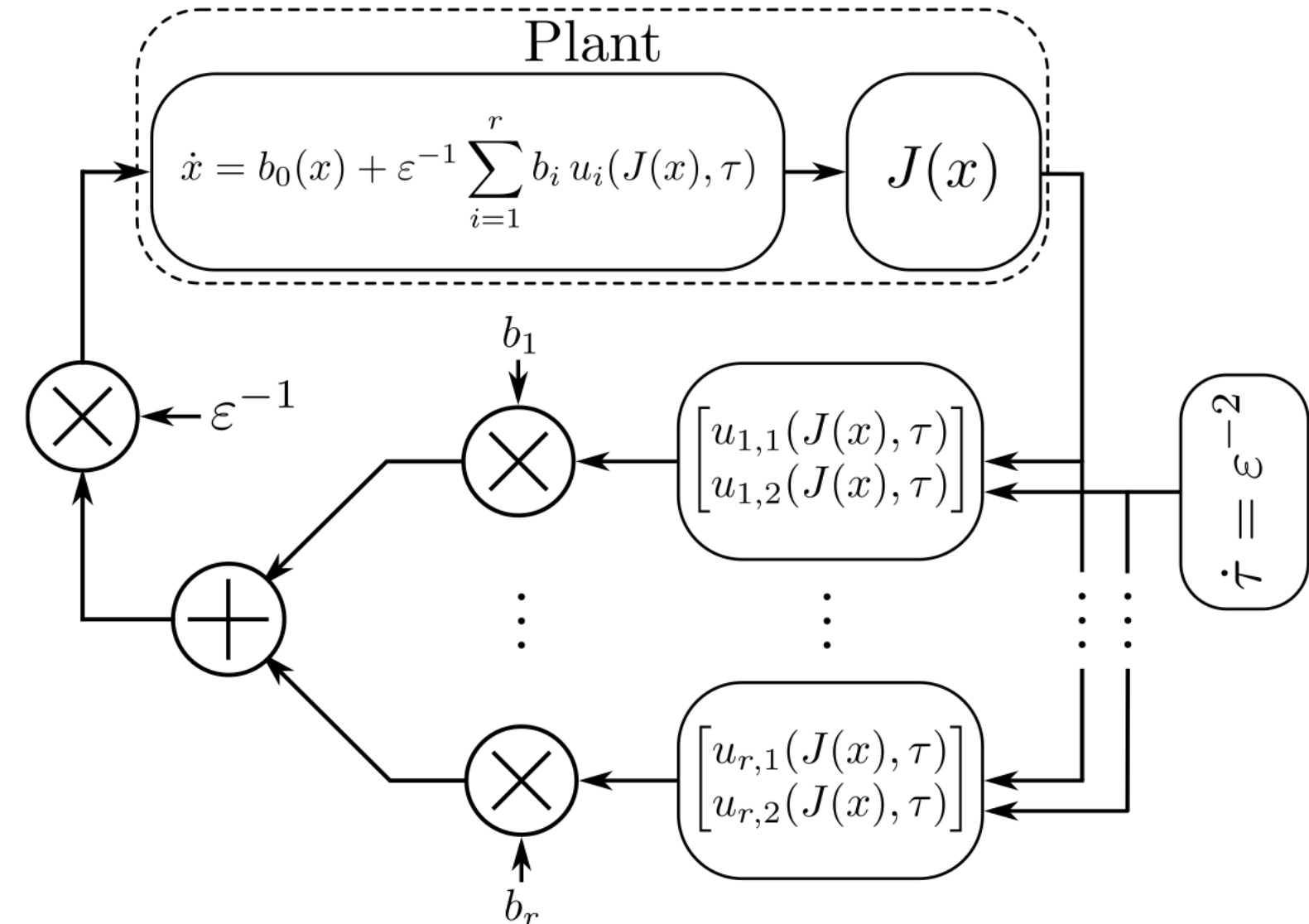
We consider two different feedback laws:

$$u_{i,1}(y, \tau) := \begin{cases} \sqrt{2\omega_i} y \cos(\log(y) + \omega_i \tau) & y > 0 \\ 0 & y \leq 0, \end{cases}$$
$$u_{i,2}(y, \tau) := \begin{cases} \sqrt{2\omega_i} y \sin(\log(y) + \omega_i \tau) & y > 0 \\ 0 & y \leq 0, \end{cases}$$

Suttner, Dashkovskiy, Exponential stability for Extremum Seeking Control, IFAC-PapersOnLine, 50 (1) (2017) 15464–15470.

$$u_{i,1}(y, \tau) := \sqrt{2\omega_i} \cos(y + \omega_i \tau),$$
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A. Scheinker, M. Krstić, Extremum seeking with bounded update rates, Systems Control Lett. 63 (2014) 25–31.



Main Result 3: Global Stability Properties via Lie-bracket Averaging

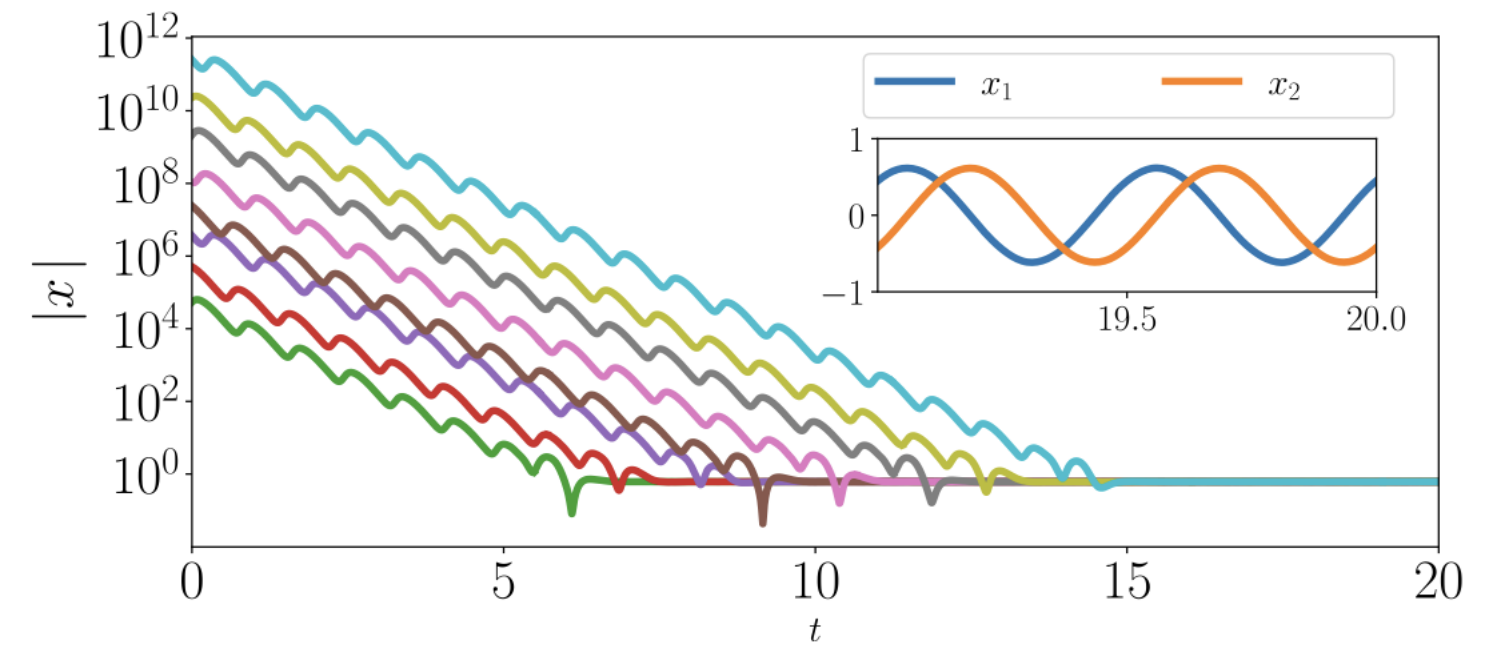
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$$\alpha_J(|x|)|x| \leq |\nabla J(x)|.$$

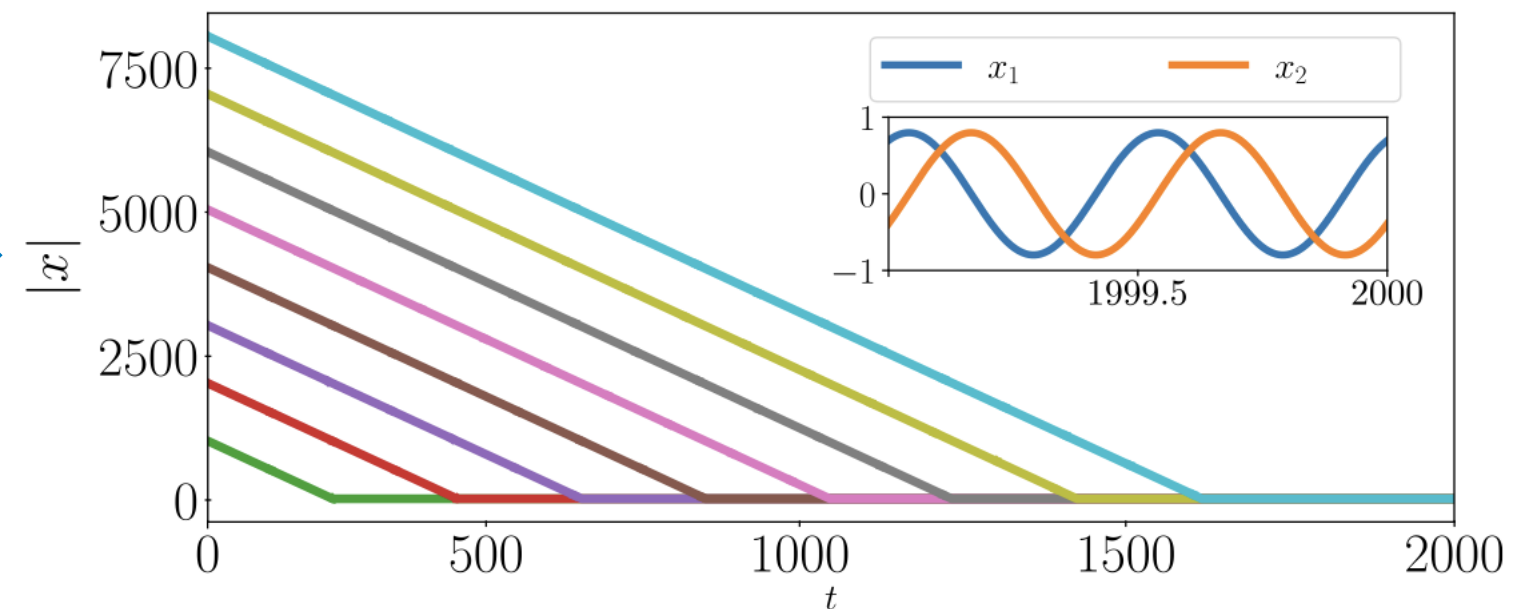


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$$\alpha_J(|x|) \leq |\nabla J(x)| \leq M_J,$$



Today:

1. Introducing Lie-bracket Averaging for Hybrid Systems



“On Lie-Bracket Averaging for a Class of Hybrid Dynamical Systems with Applications to Model-Free Control and Optimization”, Arxiv 2023

2. Introducing new hybrid algorithms for model-free optimization and regulation



“Hybrid Minimum-Seeking in Synergistic Lyapunov Functions: Robust Global Stabilization under Unknown Control Directions”, Arxiv 2024

3. Introducing novel global (practical) stability properties for Lie-bracket averaging systems and algorithms



“Initialization-free Lie-bracket Extremum Seeking”, Systems and Control Letters, Vol. 191, pp. 105881

Some Current and Future Research Directions

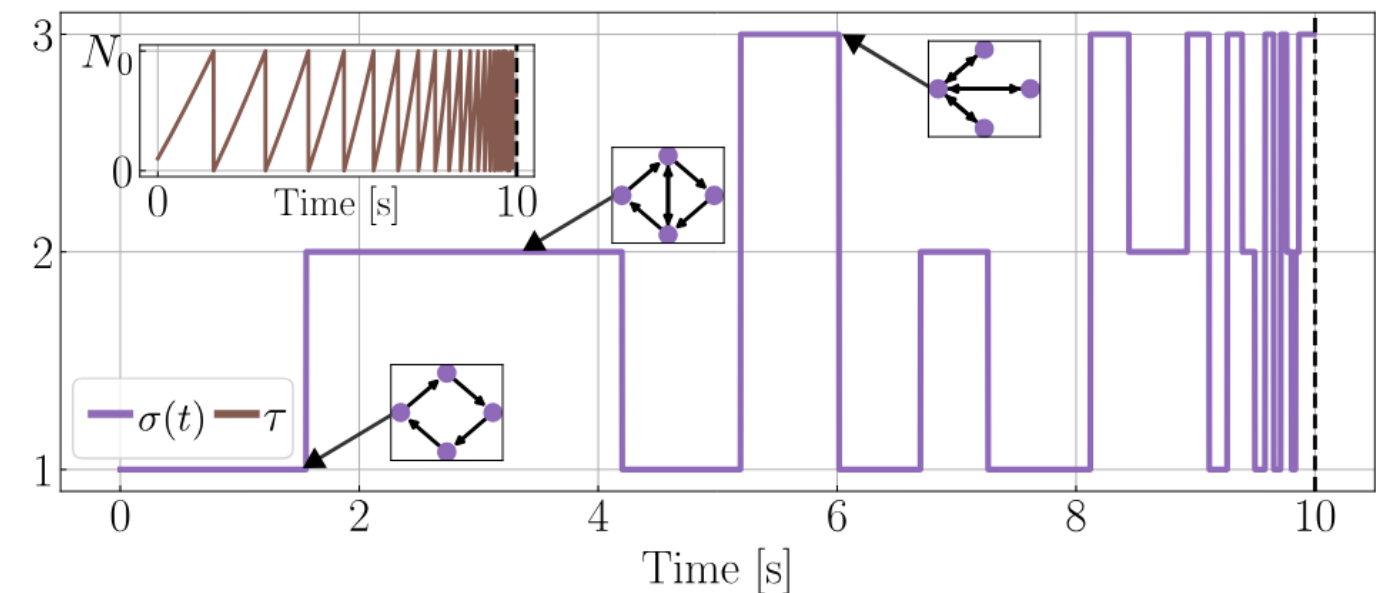
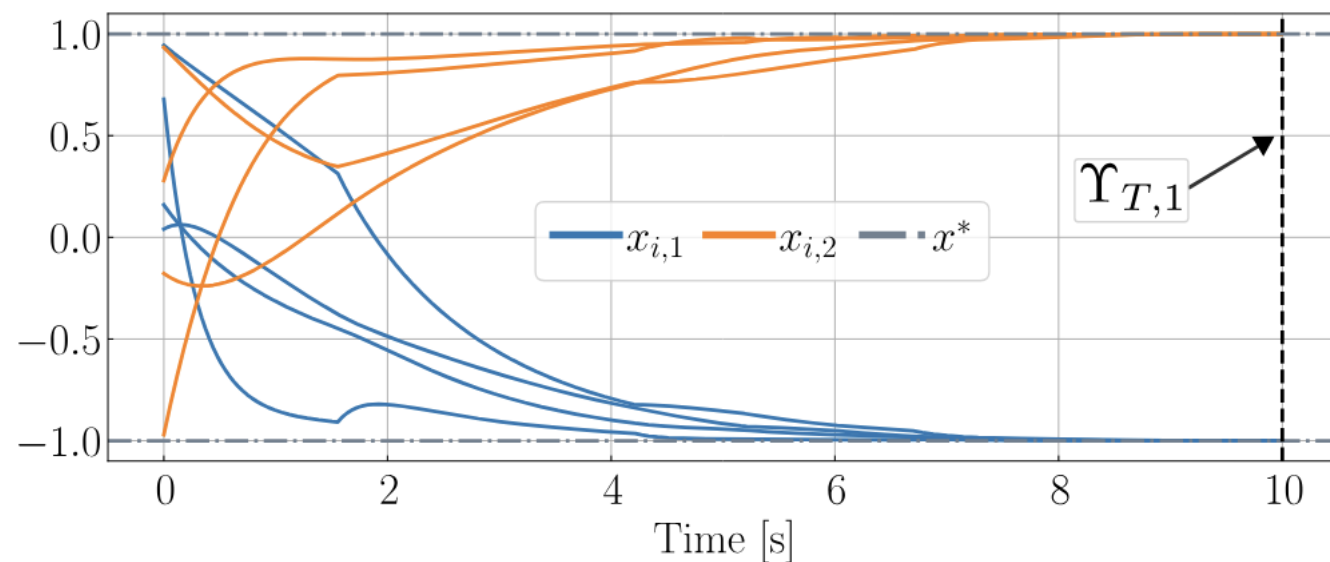
- Incorporating **probabilistic** models and **stochastic** stability guarantees:

$$\mathbb{P}\left(\left(\text{graph}(\mathbf{y}_\omega) \subset (\Gamma_{<\tau} \times \mathbb{R}^n)\right) \vee \left(\text{graph}(\mathbf{y}_\omega) \cap (\Gamma_{\leq\tau} \times \mathcal{O})\right)\right) \geq 1 - \rho,$$

"Averaging for a Class of Stochastic Hybrid Dynamical Systems with Time-Varying Flow Maps", 2023;

"Singularly Perturbed Stochastic Hybrid Systems: Stability and Recurrence via Composite Nonsmooth Lyapunov Functions", 2023

- Studying **interconnected hybrid seeking** systems:



"Dynamic Gains for Asymptotic-Behavior Shaping in Hybrid Dynamic Inclusions", IEEE CDC 2024.

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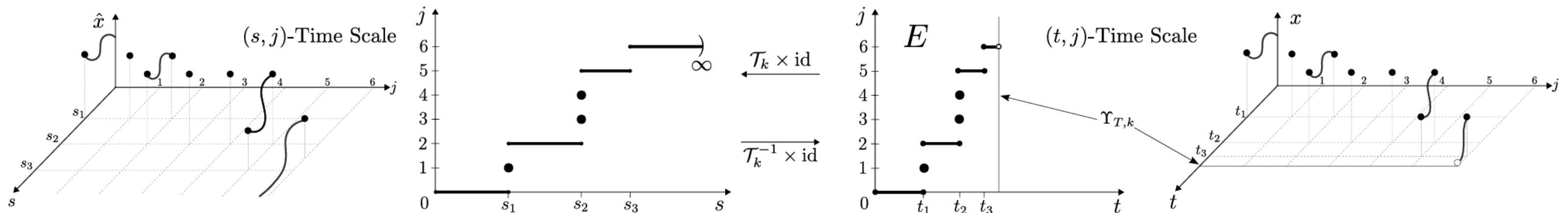
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Thank you for your time