Converse theorems for strong forward invariance with applications to interconnections

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One motivation for studying *strong forward invariance* is found in the context of safety:

One way to show that the solutions of

$$x \in C \qquad \dot{x} \in F(x)$$

do not reach X_b when starting in X_g

is to find a set K satisfying $X_g \subset K \subset \mathbb{R}^n \backslash X_b$

that is strongly forward invariant (SFI):

A set is SFI if every solution starting in the set remains in the set for all positive times in the solution's domain.

The first goal of this talk is to introduce new *converse Lyapunov-like theorems for SFI* for constrained differential inclusions

$$x \in C \qquad \dot{x} \in F(x)$$

The literature contains converse theorems for $\dot{x} = f(x)$ regarding "Barrier certificates," which can be used to establish safety.

E.g.: Prajna & Rantzer, *IFAC WC* '05, Wisniewski & Sloth, *IEEE TAC* '16, S. Ratschan, *IEEE TAC*, 2018 (robust safety)

And a converse theorem for SFI for Lipschitz ODEs and Lipschitz hybrid systems, concluding with functions that are only lower semicontinuous or assuming *robust* SFI to get smooth functions.

E.g.: Maghenem & Sanfelice, *HSCC* 2019 (hybrid), *IEEE TAC* '23 (ODEs), J. Liu. *IEEE TAC* 2022 (ODEs), Y. Meng and J. Liu. *NAHS*, 2023 (hybrid).

Endpoint for today's talk: a set of Lyapunov-like sufficient conditions for SFI for interconnections

Subsystems: $(x_i, w_i) \in C_i \qquad \dot{x}_i \in F_i(x_i, w_i)$

$i \in \{1, \ldots, N\}$

Interconnection condition:

$$x := (x_1, \ldots, x_N), \quad w := (w_1, \ldots, w_N) \qquad (x, w) \in H$$

Interested in SFI for a set of the form:

$$K := K_1 \times \cdots \times K_N$$

Sufficient conditions motivated by necessary conditions for SFI for Lipschitz inclusions, where interconnection results of this type have already been established.

(Saoud/Girard/Fribourg, *Automatica*, 2021, "Assume-guarantee contracts for CT systems")

New converse Lyapunov-like theorems are available for SFI for constrained differential inclusions

$$x \in C \qquad \dot{x} \in F(x)$$

Assumptions

1) $C \subset \mathbb{R}^n$ is closed.

2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded with values that are nonempty and convex on C.

3) The SFI set $K \subset C$ is nonempty and compact.

Converse theorem 1:

K is SFI if and only if

For any bounded and open neighborhood $U \subset \mathbb{R}^n$ of K there exists a continuous function $V : U \to \mathbb{R}_{\geq 0}$ that is smooth on $U \setminus K$ such that $V(x) = 0 \iff x \in K$ and

$$\nabla V(x) \cdot f \leq V(x) \quad \forall x \in (U \cap C) \setminus K, \ \forall f \in F(x).$$

Question:
Can a function be found that is smooth *everywhere*?

Answer: currently unknown.

Converse theorem 2:

K is SFI if and only if

For any bounded and open neighborhood $U \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$ there exists a smooth function $V : U \to \mathbb{R}_{\geq 0}$ such that $V(x,\tau) = 0 \iff (x,\tau) \in K \times [0,1]$ and

$$\nabla V(x,\tau) \cdot (f,1) \leq -V(x,\tau) \quad \forall (x,\tau) \in U \cap (C \times [0,1]),\ \forall f \in F(x).$$

Key differences relative to converse theorem 1:

1) V is smooth *everywhere*

2) V is exponentially *decreasing* rather than potentially exponentially *increasing*.

Sketch of necessity proof in converse theorem 2:

Since K is SFI for $x \in C$, $\dot{x} \in F(x)$, the set $K \times [0,1]$

is locally asymptotically stable (LAS) for the hybrid system

$$(x,\tau) \in C \times [0,1] \qquad \begin{bmatrix} \dot{x} \\ \dot{\tau} \end{bmatrix} \in \begin{bmatrix} F(x) \\ 1 \end{bmatrix}$$

$$(x,\tau) \in C \times [0,1] \qquad \begin{bmatrix} x^+ \\ \tau^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \end{bmatrix}$$

since it is SFI and locally uniformly attractive.

Now invoke converse Lyapunov theorem
for LAS for hybrid systems. $\blacksquare$

Robustness corollary of converse theorem 2, which is used in the proof of converse theorem 1:

If K is SFI then

there exists $\rho \in \mathcal{K}$ such that K is SFI for

$$x \in C_\rho, \quad \dot{x} \in F_\rho(x)$$

where

$$C_\rho := \left\{ x \in \mathbb{R}^n : \Big(x + \rho\,(|x|_K)\,\mathbb{B} \Big) \cap C \neq \emptyset \right\}$$

$$F_\rho(x) := \overline{\mathrm{co}} F \Big((x + \rho\,(|x|_K)\,\mathbb{B}) \cap C \Big) + \rho(|x|_K)\mathbb{B}$$

Sketch of proof: local asymptotic stability is robust. ■

Stronger robust SFI equals Lyapunov function

Converse theorem 3:

There exists $\rho \in \mathcal{K}^+$ such that K is SFI for

$$x \in C_\rho, \quad \dot{x} \in F_\rho(x)$$

if and only if

there exist $\rho' \in \mathcal{K}^+$, a bounded, open neighborhood $U \subset \mathbb{R}^n$ of K and a smooth function $V : U \to \mathbb{R}_{\geq 0}$ such that $V(x) = 0 \iff x \in K$ and

$$\nabla V(x) \cdot f \leq -V(x) \quad \forall x \in U \cap C_{\rho'}, \ \forall f \in F_{\rho'}(x).$$

Inspired by: J. Liu. "Converse barrier functions via Lyapunov functions" *EEE TAC 2022*

New converse Lyapunov-like theorems are available for SFI for constrained differential inclusions

$$x \in C \qquad \dot{x} \in F(x)$$

Assumptions

1) $C \subset \mathbb{R}^n$ is closed. And it is SFI for $\dot{x} \in F(x)$.

2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded with values that are nonempty ~~and convex on C~~.

3) The SFI set $K \subset C$ is nonempty and compact.

4) F is L-Lipschitz on a neighborhood of K.

$$\exists \delta > 0 : \quad F(y) \subset F(x) + L|x - y|\mathbb{B} \quad \forall x, y \in K + \delta\mathbb{B}$$

Converse theorem 4:

K is SFI if and only if

For every $0 < \alpha_1 < 1 < \alpha_2$, $\lambda > 2L$, and $\alpha_3 > 2$ there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}^n \backslash K$ such that, for every $x \in K + \delta' \mathbb{B}$,

$$\begin{aligned} \alpha_1 |x|_K^2 \leq V(x) &\leq \alpha_2 |x|_K^2 \\ |\nabla V(x)| &\leq \alpha_3 |x|_K \\ \nabla V(x) \cdot f \leq \lambda V(x) & \qquad \forall f \in F(x) \end{aligned}$$

We turn to interconnections:

Subsystems: $(x_i, w_i) \in C_i \qquad \dot{x}_i \in F_i(x_i, w_i)$

$$i \in \{1, \ldots, N\}$$

Interconnection condition:

$$x := (x_1, \ldots, x_N), \quad w := (w_1, \ldots, w_N) \qquad (x, w) \in H$$

Interested in SFI for a set of the form:

$$K := K_1 \times \cdots \times K_N$$

Assumption:

There exist compact sets $W_i \subset \mathbb{R}^{m_i}$, $i = 1, \ldots, N$, and constants $\delta > 0$ and $\mu > 0$ such that, with the definition $W := W_1 \times \cdots \times W_N$,

1. the following implication holds:

$$\left.\begin{array}{rcl} x & \in & K + \delta\mathbb{B} \\ (x, w) & \in & H \end{array}\right\} \quad \Longrightarrow \quad |w|_W \leq \mu |x|_K,$$

   and

2. for $i = 1, \ldots, N$,

   (a) the mapping F_i is Lipschitz on $(K_i \times W_i) + \delta\mathbb{B}$ and

   (b) K_i is strongly forward invariant for $\dot{x}_i \in F_i(x_i, W_i)$.

Theorem:

> Under this assumption, and if $C_i = \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}$ the set K is SFI for the interconnection.

Cf. Saoud/Girard/Fribourg, *Automatica*, 2021, "Assume-guarantee contracts for CT systems"

Sketch of proof:

1) For each i, invoke Converse theorem 3 for SFI of K_i for $\dot{x}_i \in F_i(x_i, W_i)$ to get V_i:

$$\alpha_{1i}|x_i|_K^2 \le V_i(x_i) \le \alpha_{2i}|x_i|_{K_i}^2$$

$$|\nabla V_i(x_i)| \le \alpha_{3i}|x|_{K_i}$$

$$\nabla V_i(x_i) \cdot f_i \le \lambda_i V_i(x_i) \qquad \forall f_i \in F_i(x_i, W_i)$$

2) Show that $V(x) := \sum_{i=1}^{N} V_i(x_i)$ satisfies sufficient conditions for SFI of K for interconnection:

$$\begin{aligned} f_i \in F_i(x_i, w_i) &\Longrightarrow \\ \nabla V_i(x_i) \cdot f_i &\le \lambda_i V_i(x_i) + |\nabla V_i(x_i)| L_i |w_i|_{W_i} \\ &\le \lambda_i V_i(x_i) + \alpha_{3i}|x_i|_{K_i} L_i \mu |x|_K \\ &\le \lambda_i V_i(x_i) + M_i V(x) \\ &\le \gamma_i V(x) \end{aligned}$$

Assumption that relaxes the Lipschitz condition

There exist compact sets $W_i \subset \mathbb{R}^{m_i}$, $i = 1, \ldots, N$, and constants $\delta > 0$ and $\mu > 0$ such that, with the definition $W := W_1 \times \cdots \times W_N$,

1. the following implication holds:

$$\left.\begin{array}{rcl} x & \in & K + \delta\mathbb{B} \\ (x, w) & \in & H \end{array}\right\} \quad \Longrightarrow \quad |w|_W \leq \mu |x|_K,$$

   and

2. for $i = 1, \ldots, N$,
   (C_i, F_i) satisfies a linear growth condition in $|w_i|_{W_i}$ away from W_i.
   (a) ~~the mapping F_i is Lipschitz on $(K_i \times W_i) + \delta\mathbb{B}$ and~~
   (b) K_i ~~is strongly forward invariant for $\dot{x}_i \in F_i(x_i, W_i)$~~.
   admits Lyapunov conditions that follow from SFI in the Lipschitz case for
   $w_i \in W_i, \; (x_i, w_i) \in C_i, \quad \dot{x}_i \in F_i(x_i, w_i)$

Theorem:

Under this assumption, the set K is SFI for the interconnection.

Example: (generalized an example in Saoud/Girard/Fribourg, *Automatica*, 2021)

$$\dot{x}_i \in -\Gamma_i(x_i) + \Psi_i(w_i) =: F_i(x_i, w_i) \quad i \in \{1,2\}$$

$$(x,w) \in H := \{(x,w) : w_1 = x_2, w_2 = x_1\}$$

$$K_i = W_i := [0,b], \quad b \geq 0$$

$$(x,w) \in H \quad \Longrightarrow \quad |w|_W = |x|_K, \text{ i.e., } \mu = 1$$

Assume:

1) Sector growth of $\Psi_i(\cdot)$ away from $[0,b]$.

2) $\exists \lambda_i \geq 0$ such that, for $\alpha \in K_i + \delta\mathbb{B}$ with $\alpha > \beta = b$ or $\alpha < \beta = 0$, $\gamma_i \in \Gamma_i(\alpha)$ and $\psi_i \in \Psi_i([0,b])$, we have

$$2(\alpha - \beta)(\gamma_i - \psi_i) \geq -\lambda_i(\alpha - \beta)^2.$$

E.g. $\Gamma_i(s) = \Phi_i(s) = a_i s, \quad a_i \geq 0$

New converse theorems for strong forward invariance (SFI) have been developed for constrained differential inclusions.

These theorems can be used to interpret and extend results on SFI for interconnected systems, including interconnected hybrid systems (not covered here).