A new operator for dynamic mode decomposition of discrete-time control-affine systems

AFOSR FA9550-20-1-0127

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Aug 27, 2024

Could not have done this work without

Collaborators:

- Moad Abudia and Zachary Morrison (Oklahoma State University)
- Efrain Gonzalez, Ladan Avazpour, and Joel Rosenfeld (University of South Florida)
- Benjamin Russo (Riverside Research Institute)

Support:

- Air Force Office of Scientific Research
  - Grant FA9550-20-1-0127
  - Grant FA9550-21-1-0134
- National Science Foundation
  - Collaborative project, grants 2027999 (OSU), 2027976 (USF), and 2028001 (Vanderbilt)

DMD in continuous time using the Liouville operator

System

$$x \longrightarrow \boxed{f} \xrightarrow{\dot{x}}$$

Liouville Operator

$$g \longrightarrow \boxed{A_f} \xrightarrow{A_f g = \frac{\partial g}{\partial x} f}$$

Identity Map

$$\boxed{h_{\mathrm{id}}(x) = x}$$

Componentwise Derivative

$$\dot{x} = \begin{bmatrix} (h_{\mathrm{id}})_1\,(\dot{x}) \\ \vdots \\ (h_{\mathrm{id}})_n\,(\dot{x}) \end{bmatrix} = \begin{bmatrix} [\frac{\partial (h_{\mathrm{id}})_1}{\partial x} f](x) \\ \vdots \\ [\frac{\partial (h_{\mathrm{id}})_n}{\partial x} f](x) \end{bmatrix} = \begin{bmatrix} [A_f\,(h_{\mathrm{id}})_1](x) \\ \vdots \\ [A_f\,(h_{\mathrm{id}})_n](x) \end{bmatrix}$$

Approximate Point Spectrum

$A_f \phi_i \approx \lambda_i \phi_i$ or

$A_f = \sum_{i \in I} \sigma_i \psi_i \langle \cdot, \phi_i \rangle$

Componentwise Spectral Projection

$(h_{\mathrm{id}})_j \approx \sum_{i \in I} (\xi_i)_j \phi_i$ or

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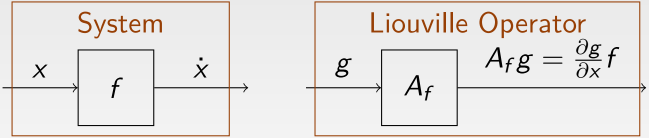

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Computations utilize spectra of finite-rank operators

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
H_d \arrow[r, dashed, "P_{d^M}"] \arrow[d, dashed, "A_f"] & \operatorname{span} d^M \arrow[d, "h"{pos=0.2}, "P_{r^M} A_f|_{d^M}"{description}, "g"{pos=0.8}] & \mathbb{R}^M \arrow[l, dashed, leftrightarrow, "h = \delta^\top d^M"'] \arrow[d, "\delta"{pos=0.2}, "W\Sigma^+ V^\top G_{d^M}"{description}, "a"{pos=0.8}] \\
H_r \arrow[r, dashed, "P_{r^M}"] & \operatorname{span} r^M \arrow[u, dashed, bend right=60, "A_f^* r_j = d_j"'] & \mathbb{R}^M \arrow[l, dashed, leftrightarrow, "g = a^\top r^M"']
\end{tikzcd}
$$

Convergence under appropriate assumptions

Mathematical Constructs

- Trajectories: $\{\gamma_i : [0, T_i] \to \mathbb{R}^n\}_{i=1}^M$
- Domain basis $d^M = \{K_d(\cdot, \gamma_i(T_i)) - K_d(\cdot, \gamma_i(0))\}_{i=1}^M$ and Gram matrix G_{d^M}
- Range basis $r^M = \{\Gamma_{\gamma_i}\}_{i=1}^M$, and Gram matrix G_{r^M}
- SVD of G_{r^M}: (V, Σ, W)
- Singular functions: $\psi_i := w_i^\top r^M$ and $\phi_i := v_i^\top d^M$
- DMD modes: $\xi = DV$
- $\hat{f}^M = \sum_{i=1}^N \frac{1}{\sigma_i} \xi_i \psi_i = DV\Sigma^+ W^\top r^M = DG_r^+ r^M$

Assumptions

- γ continuous
- K_d continuously differentiable
- $A_f : H_d \to H_r$ compact
  - $H_d = F^2_{\rho_r}$, $H_r = F^2_{\rho_r}$
  - $\rho_d < \rho_r$
  - f row-wise polynomial
- $(h_{\mathrm{id}})_j \in H_d$ for all $j = 1, \ldots, n$
- span d^∞ is dense in H_d
- span r^∞ is dense in H_r

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Convergence results

- $\lim_{M\to\infty} \|P_{r^M} A_f P_{d^M} - A_f\| = 0$
- $\lim_{M\to\infty} \left(\sup_{x\in X} \left\|\hat{f}^M(x) - f(x)\right\|_2\right) = 0$

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Linearity of the Liouville operator with respect to its symbol makes it convenient for incorporation of partial knowledge

$$\dot{x} = g(x) + e(x), \text{ trajectories } \{\gamma_i\}_{i=1}^M, \quad A_{g+e}^*\Gamma_{\gamma_i} = K(\cdot,\gamma_i(T_i)) - K(\cdot,\gamma_i(0))$$

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$$A_{g+e}^* h = A_g^* h + A_e^* h \implies A_e^*\Gamma_{\gamma_i} = K(\cdot,\gamma_i(T_i)) - K(\cdot,\gamma_i(0)) - A_g^*\Gamma_{\gamma_i}$$

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Computation of inner products in J

$$\left\langle A_g d_j^M, r_i^M \right\rangle_{H_r} = \langle A_g K(\cdot,\gamma_j(T_j)) - K(\cdot,\gamma_j(0)), \Gamma_{\gamma_i}\rangle_{H_r} =$$
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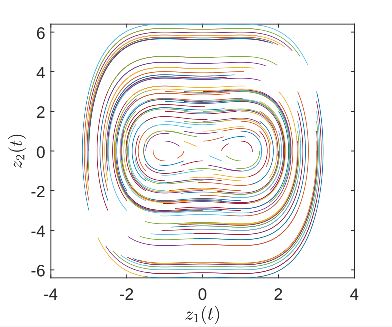
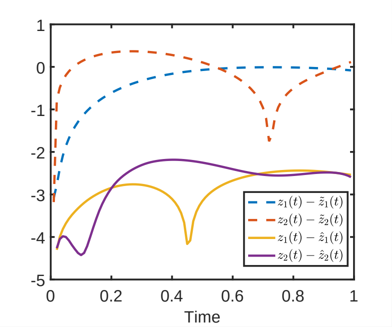
Unsurprisingly, incorporation of partial knowledge improves prediction accuracy

Example: Duffing oscillator

$$(\dot{x})_1 = (x)_2, \quad (\dot{x})_2 = (x)_1 - (x)_1^3, \quad g(x) = \begin{bmatrix} (x)_2 \\ 0 \end{bmatrix}, \quad e(x) = \begin{bmatrix} 0 \\ (x)_1 - (x)_1^3 \end{bmatrix}$$





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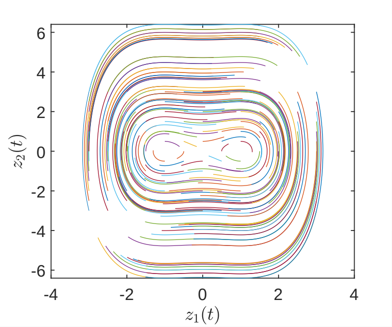

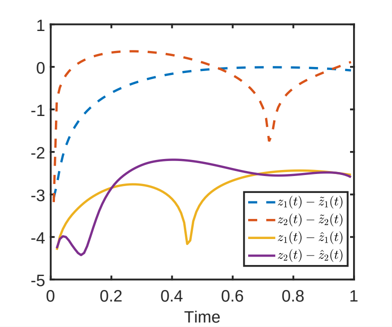

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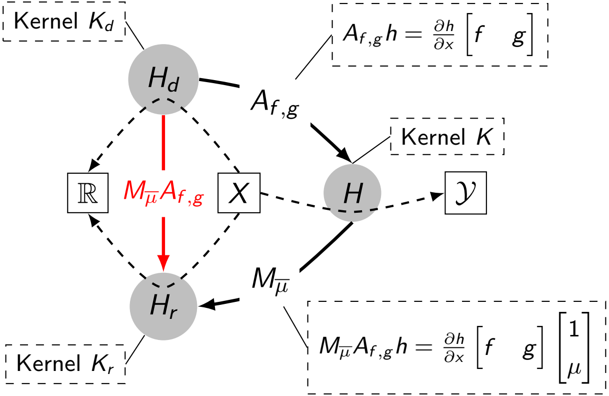
$\dot{x} = F(x,u) := f(x) + g(x)u$, feedback μ



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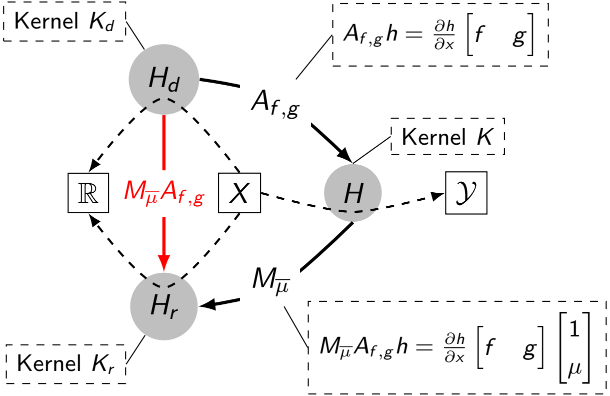

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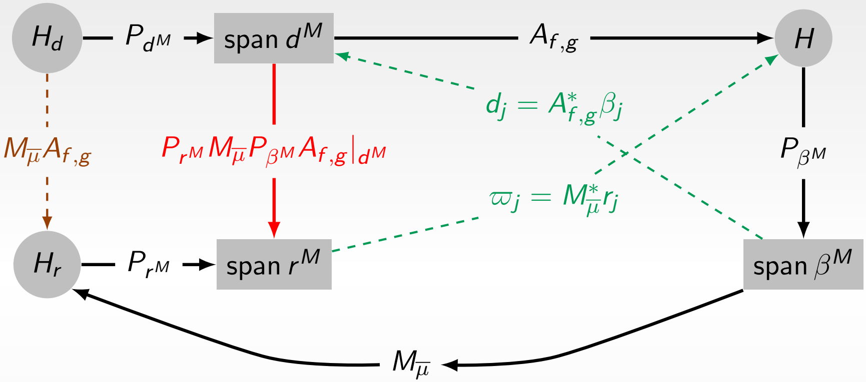
Finite rank representation requires four sets of basis vectors



$d^M = \{K_d(\cdot, \gamma_{u_i}(T_i)) - K_d(\cdot, \gamma_{u_i}(0))\}_{i=1}^M, \beta^M = \left\{\Gamma_{\gamma_{u_i}, u_i}\right\}_{i=1}^M, r^M = \left\{\Gamma_{\gamma_{u_i}}\right\}_{i=1}^M, \varpi^M = \left\{\Gamma_{\gamma_{u_i}, \mu \circ \gamma_{u_i}}\right\}_{i=1}^M,$

$\Gamma_{\gamma,u} \in H$ represents $Tp = \int_0^T p(\gamma(t)) \begin{bmatrix} 1 \\ u(t) \end{bmatrix} \mathrm{d}t$

Convergence guarantees from the uncontrolled case also extend to the controlled case

Proposition

If $M_{\overline{\mu}} : H \to H_r$ is bounded, $A_{f,g} : H_d \to H$ is a compact, and the spans of $\{d_i\}_{i=1}^{\infty}$, $\{r_i\}_{i=1}^{\infty}$, and $\{\beta_i\}_{i=1}^{\infty}$ are dense in H_d, H_r, and H, respectively, then $\lim_{M\to\infty} \left\| M_{\overline{\mu}} A_{f,g} - P_{r^M} M_{\overline{\mu}} P_{\beta^M} A_{f,g} P_{d^M} \right\|_{H_d}^{H_r} = 0$. In particular, with $\hat{F}_{\mu,M} := P_{r^M} M_{\overline{\mu}} P_{\beta^M} A_{f,g}|_{d^M} h_{\mathrm{id}}$, $\lim_{M\to\infty} \left(\sup_{x\in X} \left\| \hat{F}_{\mu,M}(x) - F_{\mu}(x) \right\|_2 \right) = 0$.

J. A. Rosenfeld and R. Kamalapurkar, "Dynamic mode decomposition with control Liouville operators," *IEEE Trans. Autom. Control*, to appear

In discrete time, we rely on the Koopman operator

System

$$x_k \longrightarrow \boxed{F} \longrightarrow x_{k+1}$$

Koopman Operator

$$h \longrightarrow \boxed{\mathcal{K}_F} \xrightarrow[{[\mathcal{K}_F h](x_k) = h(x_{k+1})}]{\mathcal{K}_F h = h \circ F}$$

Identity Map

$$\boxed{h_{\mathrm{id}}(x) = x}$$

Componentwise Propagation

$$x_{k+1} = \begin{pmatrix} (h_{\mathrm{id}})_1\,(x_{k+1}) \\ \vdots \\ (h_{\mathrm{id}})_n\,(x_{k+1}) \end{pmatrix} = \begin{pmatrix} [(h_{\mathrm{id}})_1 \circ F](x_k) \\ \vdots \\ [(h_{\mathrm{id}})_n \circ F](x_k) \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{\mathcal{F}}\,(h_{\mathrm{id}})_1\,(x_k) \\ \vdots \\ \mathcal{K}_{\mathcal{F}}\,(h_{\mathrm{id}})_n\,(x_k) \end{pmatrix}$$

Approximate Point Spectrum

$$\mathcal{K}_{\mathcal{F}}\phi_i \approx \lambda_i \phi_i$$

Componentwise Spectral Projection

$$(h_{\mathrm{id}})_j \approx \textstyle\sum_{i=1}^{N} (\xi_i)_j \phi_i$$

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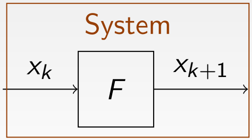
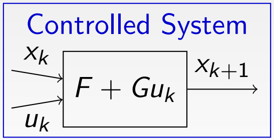
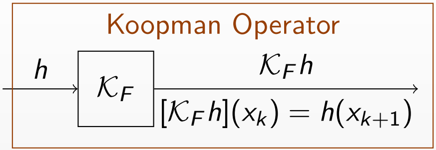
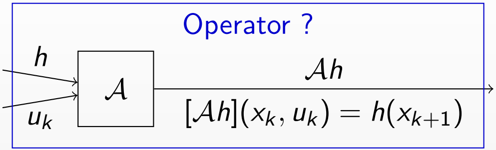
Controllers complicate spectral analysis









Different operators have been used over the years to embed controlled systems

Data: $\{x_k, u_k\}_{k=1}^{N+1}$ from $x_{k+1} = F(x_k, u_k)$, $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$.

DMDc (2014 Proctor et al.):

Find matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ such that $x_{k+1} = Ax_k + Bu_k$.

SINDYc (2016 Brunton et al.):

Find a transformation $\psi : \mathbb{R}^n \to \mathbb{R}^N$ and matrices $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times m}$ such that $\psi(x_{k+1}) = A\psi(x_k) + Bu_k$

eDMDc (2018 M. Korda et al.): $[\mathcal{K}g]\left(\begin{pmatrix} x \\ \{u\} \end{pmatrix}\right) = g\left(\begin{pmatrix} F(x, u_0) \\ \mathcal{S}\{u\} \end{pmatrix}\right)$

Find a transformation $\psi : \mathbb{R}^n \to \mathbb{R}^N$ and matrices $A \in \mathbb{R}^{N\times N}$ and $B_i \in \mathbb{R}^{N\times N}$ for $i = 1, \dots, m$ such that $\psi(x_{k+1}) = A\psi(x_k) + \sum_{i=1}^m B_i\psi(x_k)(u_k)_i$

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KIC (2018 Proctor et al.) $[\mathcal{K}g](x, u) = g(F(x, u), 0)$

Model is similar to DMDc.

KCT (2021 Goswami et al.) Koopman operator of unforced dynamics

Model is similar to eDMDc, transformation constructed from eigenfunctions of the unforced Koopman operator

Taylor's series (2020 Abraham et al.) linearization

$$h(x_{k+1}) \approx h(F(x_k)) + \frac{\partial h}{\partial x}(F(x_k))G(x_k)u_k = [\mathcal{K}_F h]((x_k)) + \frac{\partial h}{\partial x}(F(x_k))G(x_k)u_k$$

A new approach for spectral analysis of controlled system

- Predictors are linear or bilinear, which typically result in short prediction horizons
- Spectra of transfer operators corresponding to the controlled systems are not computed/analyzed

Idea:

If we restrict the prediction to feedback controllers $\mu : \mathbb{R}^n \to \mathbb{R}^m$, then it is possible to compute and analyze the spectrum of the Koopman operator associated with the closed-loop system $x_{k+1} = F(x_k) + G(x_k)\mu(x_k)$ using open-loop data.

If the observables were linear...

$$h \longrightarrow \boxed{A_{F,G}} \xrightarrow[\text{Vector-valued } (\mathbb{C}^{1\times m+1})]{\left(h\circ F, \quad h\circ(G)^1, \quad \ldots, \quad h\circ(G)^m\right)} \boxed{M_{\bar{u}_k}} \xrightarrow{\left(h\circ F, \quad h\circ(G)^1, \quad \ldots, \quad h\circ(G)^m\right)\bar{u}_k}$$

where $\bar{u}_k = \begin{pmatrix} 1 & u_k^\top \end{pmatrix}^\top$. If h is a linear function, then

$$\left(h\circ F, \quad h\circ(G)^1, \quad \ldots, \quad h\circ(G)^m\right)\bar{u}_k = h\circ F + \sum_{i=1}^{m}(u_k)_i h\circ(G)^i = h\circ(F + Gu_k)$$

and as a result,

$$[M_{\bar{u}_k}A_{F,G}h](x_k) = [h\circ(F+Gu_k)](x_k) = h(F(x_k) + G(x_k)u_k) = h(x_{k+1})$$

Idea:

An extension of $A_{F,G}$ from linear functions to reproducing kernels of a RKHS turns out to be sufficient for DMD.

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An extension of $A_{F,G}$ from linear functions to reproducing kernels of a RKHS turns out to be sufficient for DMD.

Propagation of all functions is not possible, but we can propagate reproducing kernels

RKHS $(\tilde{H}, \tilde{K})$, vvRKHS (H,K), both defined on $X \subset \mathbb{R}^n$, and $\mathcal{Y} = \mathbb{C}^{1\times m+1}$

Proposition

For each reproducing kernel $\tilde{K}_z$ centered at $z \in X$, there exists a unique function $L_z \in H$ such that for all tuples (x, u, y) satisfying $y = F(x) + G(x)u$, we have $[M_{\overline{u}} L_z] = \tilde{K}_z(F(\cdot) + G(\cdot)u)$, i.e., $[M_{\overline{u}} L_z](x) = \tilde{K}_z(y)$, where $\overline{u} = \begin{pmatrix} 1 & u^\top \end{pmatrix}^\top$.

$$\xrightarrow{\tilde{K}_{z_i}} \boxed{A_{F,G}} \xrightarrow{L_{z_i}} \boxed{M_{\overline{u}_k}} \xrightarrow{\tilde{K}_{z_i}(F(\cdot)+G(\cdot)u_k)}$$

$$\tilde{K}_{z_i}(x_k) \dashrightarrow \boxed{M_{\overline{u}_k} A_{F,G}} \dashrightarrow \tilde{K}_{z_i}(x_{k+1})$$

Extension to complex span of reproducing kernels

$$\xrightarrow{\sum_{i=1}^{N_r} a_i \tilde{K}_{z_i} + \mathrm{i} \sum_{j=1}^{N_c} b_j \tilde{K}_{z_j}} \boxed{A_{F,G}} \xrightarrow{\sum_{i=1}^{N_r} a_i L_{z_i} + \mathrm{i} \sum_{j=1}^{N_c} b_j L_{z_j}}$$

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If $\varphi_j = \sum_{i=1}^N (v_j)_i \tilde{K}_{x_i}$ is an eigenfunction of $M_{\overline{\mu}} A_{F,G}$ with eigenvalue λ_j, then

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$$h_{\mathrm{id}}(x) \approx \sum_{i=1}^N \xi_i \varphi_i = \xi \varphi$$

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Finite-rank representation results from projections onto spans of reproducing kernels

Data

Open-loop response $\{x_k, u_k, x_{k+1}\}_{k=1}^N$

Bases for projection

$\alpha = \text{span}\left\{\tilde{K}_{x_k}\right\}_{k=1}^N \subset \mathcal{D}(A_{F,G})$ and $\beta = \{K_{x_k,\bar{u}_k}\}_{k=1}^N \subset \mathcal{D}(M_{\overline{\mu}})$ (Assumptions)

Proposition

If v_j is an eigenvector of the matrix $[M_{\overline{\mu}} P_\beta A_{F,G}]_\alpha^\alpha$ with eigenvalue λ_j, then the function $\varphi_j = \sum_{i=1}^N (v_j)_i \tilde{K}_{x_i}$ is an eigenfunction of the operator $P_\alpha M_{\overline{\mu}} P_\beta A_{F,G}|_\alpha$ with eigenvalue λ_j.

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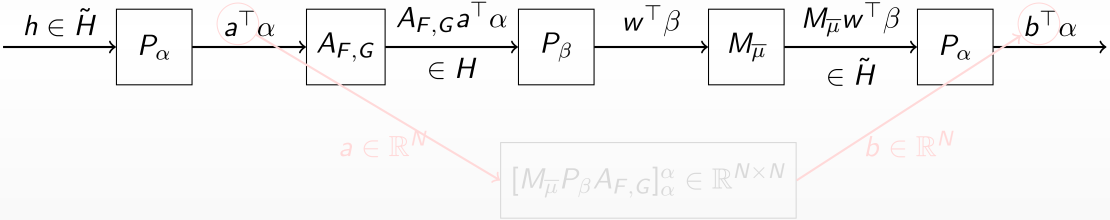


Proposition

If v_j is an eigenvector of the matrix $[M_{\overline{\mu}} P_\beta A_{F,G}]_\alpha^\alpha$ with eigenvalue λ_j, then the function $\varphi_j = \sum_{i=1}^{N} (v_j)_i \tilde{K}_{x_i}$ is an eigenfunction of the operator $P_\alpha M_{\overline{\mu}} P_\beta A_{F,G}|_\alpha$ with eigenvalue λ_j.

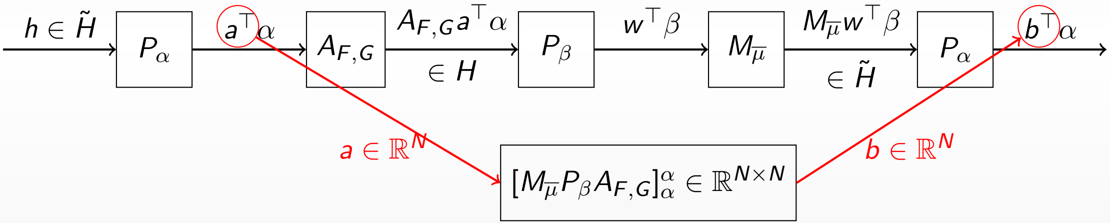
Finite-rank representation results from projections onto spans of reproducing kernels

Data

Open-loop response $\{x_k, u_k, x_{k+1}\}_{k=1}^{N}$

Bases for projection

$\alpha = \text{span}\left\{\tilde{K}_{x_k}\right\}_{k=1}^{N} \subset \mathcal{D}(A_{F,G})$ and $\beta = \{K_{x_k,\bar{u}_k}\}_{k=1}^{N} \subset \mathcal{D}(M_{\overline{\mu}})$ (Assumptions)



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Finite-rank representation results from projections onto spans of reproducing kernels

Finite-rank Representation

$$[M_{\overline{\mu}} P_\beta A_{F,G}]_\alpha^\alpha = \tilde{G}^+ I G^+ \tilde{I}^\top,$$

$$\tilde{G} = \left(\tilde{K}(x_i, x_j)\right)_{i,j=1}^N, \; G = \left(\langle K_{x_i,\bar{u}_i}, K_{x_j,\bar{u}_j}\rangle_H\right)_{i,j=1}^N, \; \tilde{I} = \left(\tilde{K}(x_{i+1}, x_j)\right)_{i,j=1}^N, \; I = \left(\langle K_{x_j,\mu(x_j)}, K_{x_i,\bar{u}_i}\rangle_H\right)_{i,j=1}^N$$

Koopman modes

$$\xi = X(V^\top \tilde{G})^+, \quad V = \begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & \cdots & x_N \end{pmatrix}$$

Z. Morrison, M. Abudia, J. Rosenfeld, and R. Kamalapurkar, "Dynamic mode decomposition of control-affine nonlinear systems using discrete control Liouville operators," *IEEE Control Syst. Lett.*, vol. 8, pp. 79–84, 2024

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The technique shows promise as a tool for equation-free prediction of closed-loop response

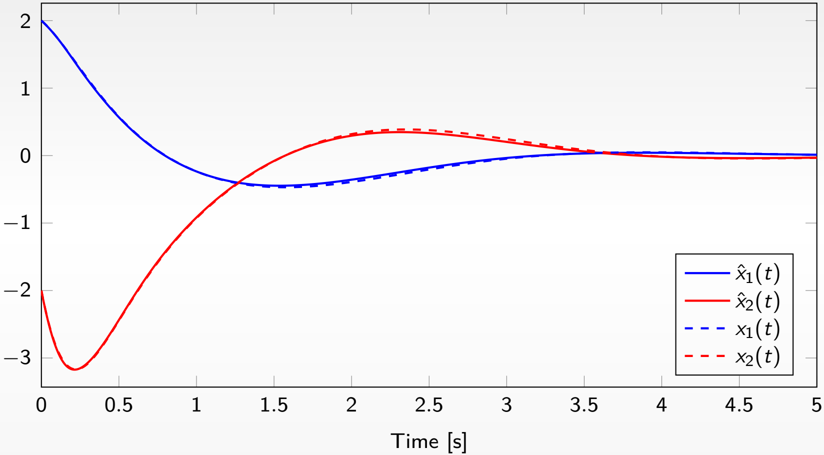
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\delta x_2 - \beta x_1 - \alpha x_1^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 + \sin(x_1) \end{pmatrix} u,$$

Goal:

Predict the response of the system in to two feedback laws: Linear: $\mu(x_k) = -2x_{k,1} - x_{k,2}$ Nonlinear: $\bar{\mu}(x_k) = -2x_{k,1}^3 - x_{k,2}$

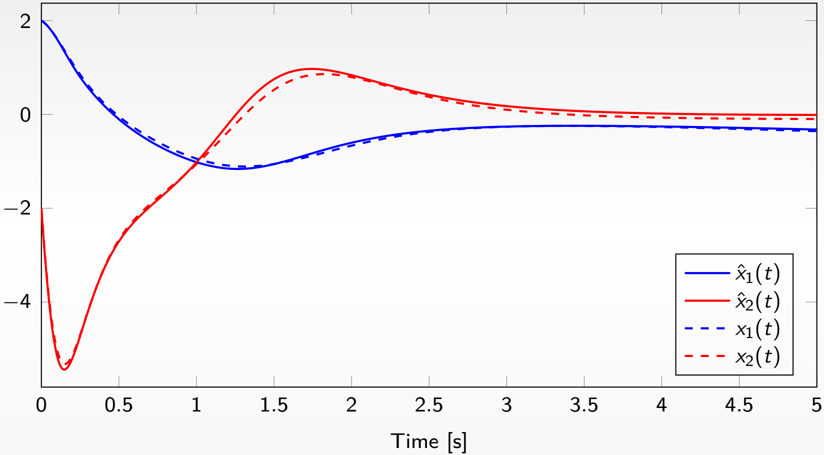
- Data points $\{(x_k, x_{k+1}, u_k)\}_{k=1}^{225}$ with initial conditions sampled from the set $[-3, 3] \times [-3, 3] \subset \mathbb{R}^2$.
- Control inputs are sampled uniformly from the interval $[-2, 2] \subset \mathbb{R}$.
- Gaussian RBF with parameter $\sigma = 10$ for both $\tilde{K}$ and K (diagonal).

The technique shows promise as a tool for equation-free prediction of closed-loop response



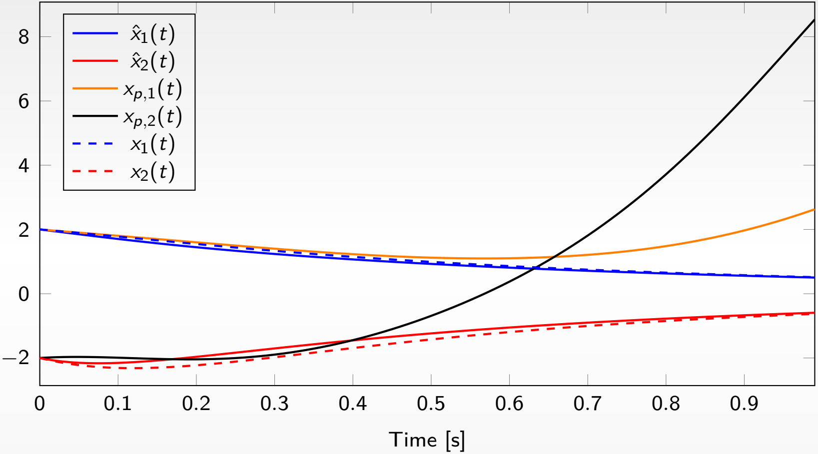
Linear feedback

The technique shows promise as a tool for equation-free prediction of closed-loop response



Nonlinear feedback

The technique shows promise as a tool for equation-free prediction of closed-loop response



Comparison against linear eDMDc predictor[1]

[1] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, 2018.

What next?

- Compactness, finite-data L_∞ error bounds (inspirations from MTNS 2024[2])
- Error bounds open up applications in control, e.g., MPC[3] and SoS[4]
- Operator representations of input-output models (NARMAX, delay embedding, etc.)
- Deep kernel learning

[2] F. Köhne, F. M. Philipp, M. Schaller, A. Schiela, and K. Worthmann, *L^∞-error bounds for approximations of the koopman operator by kernel extended dynamic mode decomposition*, arXiv:2403.18809, 2024.

[3] M. Schaller, K. Worthmann, F. Philipp, S. Peitz, and F. Nüske, "Towards reliable data-based optimal and predictive control using extended dmd," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 169–174, 2023, 12th IFAC Symposium on Nonlinear Control Systems NOLCOS 2022.

[4] R. Strässer, M. Schaller, K. Worthmann, J. Berberich, and F. Allgöwer, "Koopman-based feedback design with stability guarantees," *IEEE Transactions on Automatic Control*, pp. 1–16, 2024.

In summary,

$x_k \approx \xi\varphi(x_k)$

$$\begin{tikzcd}
\text{span}\,\alpha \arrow[r, "A_{F,G}"] \arrow[loop, dashed, "P_\alpha M_{\overline{\mu}} P_\beta A_{f,g}|_\alpha"] & H \arrow[d, "P_\beta"] \\
\tilde{H} \arrow[u, "P_\alpha"] & \text{span}\,\beta^M \arrow[l, "M_{\overline{\mu}}"]
\end{tikzcd}$$

$x_{k+1} \approx \lambda\xi\varphi(x_k)$

Thank you!