# Robust Optimal Estimation and Control for Dynamic Systems with Additive Heavy-Tailed Uncertainty for Aerospace Applications

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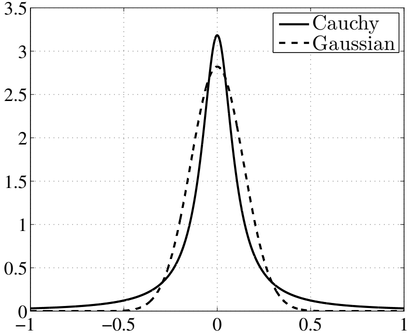
# Estimation of Linear Dynamic Systems

- The Kalman filter assumes a linear system with additive Gaussian process and measurement noise.
  - Gaussian probability density function (pdf) has a *light tail* and rarely takes extreme values
  - The conditional pdf of the state given the measurements is Gaussian, i. e. unimodal and symmetric.
- *Our* robust estimator assumes a linear system with additive Cauchy process and measurement noise.
  - Cauchy pdf has a *heavy-tail* that captures physical phenomena which have a more *impulsive* character.
  - The conditional pdf of the state given the measurements is not symmetric and not always unimodal
  - Denote this nonlinear estimator as the multivariate Cauchy estimator (MCE)

# The Gaussian and Cauchy densities are in a class called symmetric $\alpha$-stable densities.

- A class of heavy-tailed pdf's is the symmetric $\alpha$-stable distribution, represented by its characteristic function (Essentially, the Fourier transform of the pdf):
  $\phi(\nu) = \exp(-\beta^{\alpha}|\nu|^{\alpha})$

- $\alpha = 2 \rightarrow$ Gaussian
  $\alpha = 1 \rightarrow$ Cauchy
- $\alpha \in [1, 2)$, infinite variances.
- No closed form pdf for $\alpha \in (1,\ 2)$, Can Simulate
- Radar and sonar
- Atmospheric turbulence
- Adversarial motion
- Economic systems




- LSQ fit between Gaussian $\sqrt{\frac{1}{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$ & Cauchy $\left(\frac{\beta/\pi}{x^2+\beta^2}\right)$.

# Vector-State Cauchy Estimation: Formulation

- Consider a *vector-state*, linear dynamic system as

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma w_k, \\ z_k &= H x_k + v_k \end{aligned}$$

where

- The state vector is $x_k \in \mathbb{R}^n$,
- The measurement $z_k \in \mathbb{R}^q$,
- The known matrices $\Phi \in \mathbb{R}^{n \times n}$, $\Gamma \in \mathbb{R}^{n \times m}$, and $H \in \mathbb{R}^{q \times n}$.
- The *measurement history* is $\mathbf{y}_k = \{z_1, \cdots, z_k\}$
- The additive noises $w_k$ and $v_k$ are Cauchy distributed.
- For simplicity let $w_k$ and $v_k$ be scalars.

# Vector-State Cauchy Estimation: Formulation

- For scalar state estimation, the conditional pdf given the measurement history $\mathbf{y}_k$ can be propagated analytically.
- For vector state estimation, the conditional pdf cannot be propagated analytically, but the characteristic function of the conditional pdf can be propagated analytically and recursively.

- The initial states, measurement noise, process noise are *Cauchy distributed* as ($\nu \in \mathbb{R}^n,\ \bar{\nu} \in \mathbb{R}$)

$$f_{X_1}(x_1) = \prod_{i=1}^{n} \frac{\alpha_i/\pi}{(x_{1,i})^2 + \alpha_i^2} \overset{\text{characteristic function}}{\Rightarrow} \phi_{X_1}(\nu) = \prod_{i=1}^{n} e^{-\alpha_i|\nu_i|} = e^{-\sum_{i=1}^{n} \alpha_i |a_i^T \nu|},$$

$$f_V(v_k) = \frac{\gamma/\pi}{v_k^2 + \gamma^2} \Rightarrow \phi_V(\bar{\nu}) = e^{-\gamma|\bar{\nu}|};\ f_W(w_k) = \frac{\beta/\pi}{w_k^2 + \beta^2} \Rightarrow \phi_W(\bar{\nu}) = e^{-\beta|\bar{\nu}|}$$

- The $a_i$ above are unit vectors, but will be general directions forming a central arrangement of cells.

# Unnormalized Conditional pdf at $k$

- The *conditional pdf* at $k$ is

$$f_{X_k|Y_k}(x_k|\mathbf{y}_k) \quad = \quad \frac{f_{X_kY_k}(x_k, \mathbf{y}_k)}{f_{Y_k}(\mathbf{y}_k)}$$

- The *unnormalized* conditional pdf (ucpdf) is defined as the joint pdf of $x_k$ and $\mathbf{y}_k$,

$$\bar{f}_{X_k|Y_k}(x_k|\mathbf{y}_k) = f_{X_kY_k}(x_k, \mathbf{y}_k)$$

- The ucpdf can only be obtained *recursively and in closed form for the scalar system.*
- The *characteristic function* (CF) of the ucpdf for the vector state is obtained *recursively and in closed form* and denoted as $\bar{\phi}_{X_k|Y_k}(\nu)$.

# The CF of the ucpdf: Propagation and Update

- The **Characteristic Function (CF) of the ucpdf** is

$$\bar{\phi}_{X_{k-1}|Y_{k-1}}(\nu) = \int_{-\infty}^{\infty} e^{j\nu^T x_{k-1}} \bar{f}_{X_{k-1}|Y_{k-1}}(x_{k-1}|\mathbf{y}_{k-1}) dx_{k-1}.$$

  where $\nu \in \mathbb{R}^n$ is the spectral vector.
- The **propagation** to $t_k$ is

$$\bar{\phi}_{X_k|Y_{k-1}}(\nu) = \bar{\phi}_{X_{k-1}|Y_{k-1}}\left(\Phi^T \nu\right) \phi_W\left(\Gamma^T \nu\right).$$

- The **measurement update** at $k$, using $z_k = Hx_k + v_k$, produces $\bar{\phi}_{X_k|Y_k}(\nu)$. It is determined by using the general convolution integral

$$\bar{\phi}_{X_k|Y_k}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}_{X_k|Y_{k-1}}\left(\nu - H^T \eta\right) \phi_V(-\eta)\, e^{jz_k \eta} d\eta.$$

- The closed-form solution to the convolution integral is the **key** to our estimation approach.

# General form of $\bar{\phi}_{X_k|Y_k}(\nu)$ at Measurement time $k$

- From the convolution integral, $\bar{\phi}_{X_k|Y_k}(\nu)$ is analytic and recursive

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{n^{k|k}} g_i^{k|k}\left(y_{gi}^{k|k}(\nu), \mathbf{y}_k\right) \exp\left(y_{ei}^{k|k}(\nu, \mathbf{y}_k)\right), \quad \begin{cases} n^{k|k} \text{ is \# of terms that grows} \\ \text{with measurement update.} \end{cases}$$

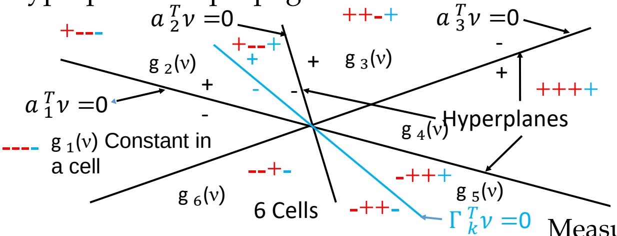
$$y_{gi}^{k|k}(\nu) = \sum_{\ell=1}^{n_i^{k|k}} \rho_{i\ell}^{k|k} \operatorname{sgn}\left(\langle a_{i\ell}^{k|k}, \nu\rangle\right) \in \mathbb{R}^k, \; n_i^{k|k} \sim \text{\# of hyperplanes per term.}$$

$$y_{ei}^{k|k}(\nu, \mathbf{y}_k) = -\sum_{\ell=1}^{n_i^{k|k}} p_{i\ell}^{k|k} |\langle a_{i\ell}^{k|k}, \nu\rangle| + j\langle b_i^{k|k}(\mathbf{y}_k), \nu\rangle,$$

- $\rho_{i\ell}^{k|k} \in \mathbb{R}^k$, $p_{i\ell}^{k|k}$, and $a_{i\ell}^{k|k} \in \mathbb{R}^n$ are all parameters that can be computed recursively.
- The hyperplanes form cells, in a cell $g_i^{k|k}\left(y_{gi}^{k|k}(\nu, \mathbf{y}_k)\right)$ is constant.
- The cell is identified by a sign sequence of the hyperplanes.
- New innovation is the propagation of the enumeration table of sign sequences. High increase in numerical speed.

# Propagation of Cell Enumeration Sign Tables

Consider three hyperplanes in two-dimensions (parent) with added Γ hyperplane for propagation:




Construct table of sign sequences as

$$T^{k-1|k-1}_{\text{Parent}} = \begin{bmatrix} + & + & + \\ - & + & + \\ - & - & + \\ - & - & - \\ + & - & - \\ + & + & - \end{bmatrix} \Rightarrow \overbrace{T^{k|k-1}_{\text{Propagation}}}^{\text{New Parent}} = \begin{bmatrix} + & + & + & + \\ - & + & + & + \\ - & + & + & - \\ - & - & + & - \\ - & - & - & - \\ + & - & - & - \\ + & - & - & + \\ + & + & - & + \end{bmatrix} \overbrace{\Rightarrow T^{k|k}_{t=\text{Children}}}^{\text{Measurement Update: Computation Breakthrough}}$$

Rows: cell sign sequences. Columns: hyperplane sign sequences.

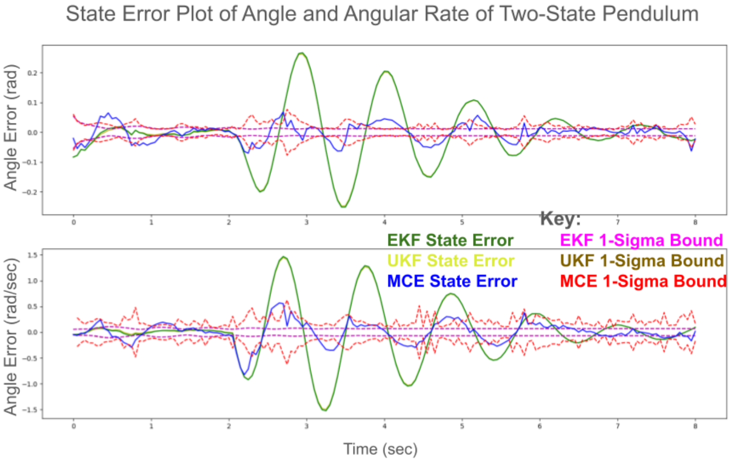
# Partial Differentiation of $\bar{\phi}_{X_k|Y_k}(\nu)$ Produce Conditional Mean and Conditional Variance

- $\bar{\phi}_{X_k|Y_k}(\nu)$ is twice continuously differentiable.
- To construct the conditional mean and variance, choose $\nu = \epsilon\hat{\nu}$ where $\epsilon > 0$ and $\hat{\nu}$ is a fixed direction.
- The normalization variable is $f_{\mathbf{Y}_k}(\mathbf{y}_k) = \bar{\phi}_{X_k|\mathbf{Y}_k}(\epsilon\hat{\nu})\Big|_{\epsilon=0}$.
- $\hat{x}_k = E[x_k|\mathbf{y}_k] = \dfrac{1}{j\,f_{\mathbf{Y}_k}(\mathbf{y}_k)}\left(\dfrac{\partial\bar{\phi}_{X_k|\mathbf{Y}_k}(\epsilon\hat{\nu})}{\partial\nu}\right)^T\Bigg|_{\epsilon=0}$
- $P_k = E[x_k x_k^T|\mathbf{y}_k] - \hat{x}_k\hat{x}_k^T = \dfrac{1}{j^2\,f_{\mathbf{Y}_k}(\mathbf{y}_k)}\dfrac{\partial^2\bar{\phi}_{X_k|\mathbf{Y}_k}(\epsilon\hat{\nu})}{\partial\nu\partial\nu^T}\Bigg|_{\epsilon=0} - \hat{x}_k\hat{x}_k^T$
- To cap growth in the number of terms of $\bar{\phi}_{X_k|Y_k}(\nu)$, a bank of fixed, sliding windows of measurements is constructed.
  - Initialize window by mean and variance of a full window.
  - This initialization, using one measurement, only requires rotating a positive definite matrix into diagonal form.
  - Nonlinearities accomidated by linearization.

# Estimation of Damped Pendulum System: miss-specification: Measure Angle Only.

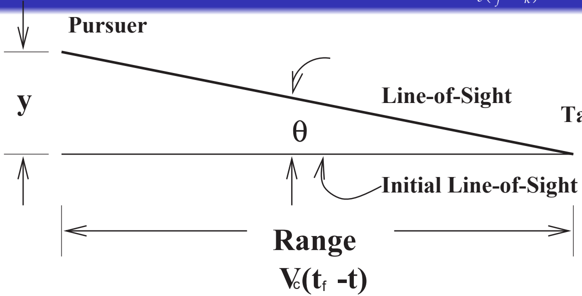
Damping change at 2 sec. is four times nominal in Gaussian simulation.

- The estimation error of the EKF/UKF jumps and then oscillates, but the EMCE (extended MCE) has small transient error.



# Missile State Estimation: Impulsive Meas. Noise

$z_k = \theta_k + n_k$, $n_k$ is glint, clutter, fading noise. $\theta_k = \arctan \frac{y_k}{V_c(t_f - t_k)}$




- Missile dynamics used in estimators

$$
\begin{aligned}
\dot{y} &= v, \quad y \text{ (lateral relative position)} \in \mathbb{R} \\
\dot{v} &= a_p - a_T, \quad v \text{ (relative velocity)}, a_p \text{ (pursuer acc.)} \in \mathbb{R} \\
\dot{a}_T &= -\frac{1}{\tau} a_T + w_{a_T}, \quad a_T \text{ (target acceleration)}, w_{a_T} \in \mathbb{R},
\end{aligned}
$$

$$
\vec{x} = \begin{bmatrix} y & v & a_T \end{bmatrix}^T, \quad \dot{\vec{x}} = F\vec{x} + Ba_p + Gw_{a_T} \Rightarrow \text{Discrete Time}
$$

- Gauss-Markov process used in the estimators match the target's auto-correlation, modeled as a telegraph process in simulation.

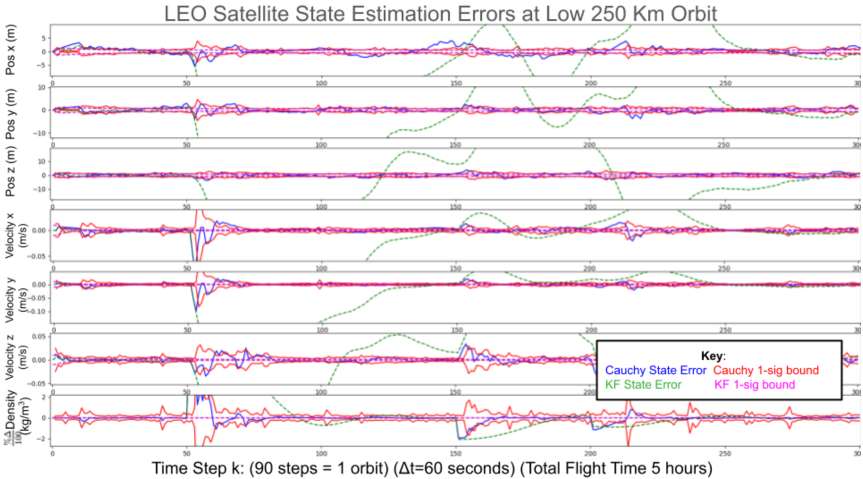
# Robust Performance: Geometric Mean

radar measurements in a clutter environment

Monte-Carlo simulation of 9000 trials for $\alpha = 2,\ 1.7,\ 1.5,\ 1.3,\ 1$.
Geometric Mean Square $G_m$: Let $A_e = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \log\left[(x_k^i - \hat{x}_k^i)^2\right]$.
Plotting $\sqrt{G_m} = \sqrt{e^{A_e}}$. Window size = 7. $a_{p_k} = -N_R V_c \dot{\theta}_k$

# 7 State Estimation with Window Bank of Five for LEO satellites: density dispersion ($\alpha = 1.3,\ \beta = .0013$)

No tuning for either filter.



# Conclusions for the Vector Cauchy Estimator

- Developed a real-time, robust dynamic estimator.
  - Estimator highly competitive with the state of the art.
- Matlab and Python wrappers for the Cauchy estimator C/C++ code are available.
- Future Research
  - Understand the fundamental structure of the MCE.
  - Determine how to embed the CF of the unnormalized conditional pdf into new stochastic prediction and control formulations.
    - From the CF, determine in real-time the conditional pdf, its marginal densities, and the expectation of particular cost criteria.
    - Evaluate performance of the MCE and Cauchy stochastic controllers with measures of robustness, convergence, and stochastic stability.