Duality for Estimation AND Control

AFOSR Dynamic Systems and Control
Program Review Meeting
Washington, DC

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ILLINOIS

Duality in control and estimation

1. **Solving control problem with an estimation algorithm (5 mins):**

2. **Solving estimation problems with control techniques (25 mins):**

Papers related to Part 1:

1. Joshi, Taghvaei, M., Meyn. Controlled interacting particle algorithms for simulation-based reinforcement learning. SCL 2022.
2. Joshi, Taghvaei, M., Meyn. Dual Ensemble Kalman Filter for Stochastic Optimal Control. CDC 2024.
3. Joshi, Chang, Taghvaei, M., Meyn. Design of Interacting Particle Systems for Fast and Efficient Reinforcement Learning. arxiv preprint.

Linear quadratic (LQ) optimal control problem:

$$\min_u \quad J_T(u) = \int_0^T \left(\tfrac{1}{2}|Cx_t|^2 + \tfrac{1}{2}|u_t|_R^2 \right) \mathrm{d}t + \tfrac{1}{2}|x_T|_{P_T}^2$$

$$\text{Subj. to} \quad \dot{x}_t = Ax_t + Bu_t =: f(x_t, u_t)$$

LQR problem: represents the $[T = \infty]$ where additional conditions are necessary to relate the asymptotic solution of the DRE to the p.d. solution of the ARE.

Policy optimization algorithms: Starting with a stability control gain K^0, obtain

$$K^0 \mapsto K^1 \mapsto K^2 \mapsto \ldots \mapsto K^j \mapsto \ldots \mapsto K^N$$

by designing a suitable gradient-descent algorithm to reduce the value $J_T(u)$ with $u_t = Kx_t$.

Literature: [Fazel, 2018], [Mohammadi, 2022] [Zhang, 2021] [Cui, 2023], [Krauth, 2019], Abbasi-Yadkori, 2019] and others. At the j-th step, simulations are used to evaluate the gradient of the value function

$$\dot{X}_t^i = AX_t^i + BK^j X_t^i, \qquad i = 1, 2, \ldots, N$$

Dual EnKF (the details)

Filtering (data assimilation) algorithm to solve the LQ problem

dual EnKF [recall running cost = $|cx|^2 + |u|_R^2$; terminal cost = $x^{\mathrm{T}} P_T x$]:

$$\mathrm{d}Y_t^i = \underbrace{AY_t^i \,\mathrm{d}t + B\,\mathrm{d}\overleftarrow{\eta}_t^i}_{\text{i-th copy of model}} + \underbrace{K_t^{(N)} \left(\frac{CY_t^i + \hat{C}_t^{(N)}}{2} \right) \mathrm{d}t}_{\text{coupling process}}, \qquad i = 1, 2, \ldots, N$$

$$Y_T^i \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, P_T^{-1}), \quad 1 \le i \le N$$

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1. SDE is run backward in time (starting from time $t = T$ to $t = 0$).
2. η_t^i is a W.P. with cov. R^{-1} (so cheap control directions are explored more).
3. Mean-field terms:
   - $\hat{C}_t^{(N)} := \frac{1}{N}\sum_j CY_t^j$ is the empirical mean.
   - $K_t^{(N)} := \frac{1}{N}\sum_j Y_t^j (CY_t^j)^{\mathrm{T}}$ is the gain.

Connection to LQ:

4. The empirical covariance of the ensemble $S_t^N \approx P_t^{-1}$. (duality: $-\log p$ is the value function of an optimal control problem).

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Error bound [Joshi, Taghvaei, M., Meyn, S&CL (2022)]

Under some technical conditions (controllability of (A,B) and a condition stronger than observability of (A,C) and P_T invertible)

$$\mathsf{E}[\|S_t^{(N)} - P_t^{-1}\|_F] \leq \frac{C_1}{\sqrt{N}} + C_2 e^{-2\lambda(T-t)}\mathsf{E}[\|S_T^N - P_T^{-1}\|_F],$$

where S_t^N is the empirical covariance of the ensemble $\{Y_t^i : 1 \leq i \leq N\}$ and P_t is the solution of the DRE at time t (λ is inherited from the theory of DRE).

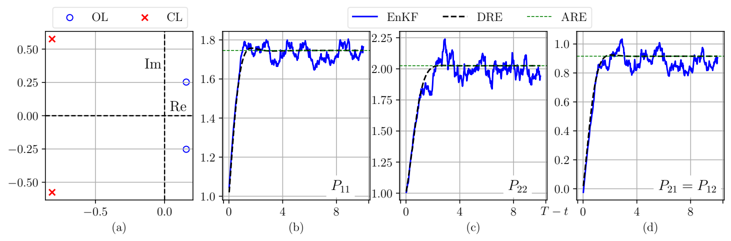


Figure: Performance on a 2 dimensional system ($N = 100$ particles).

Error analysis

Performance of dual EnKF

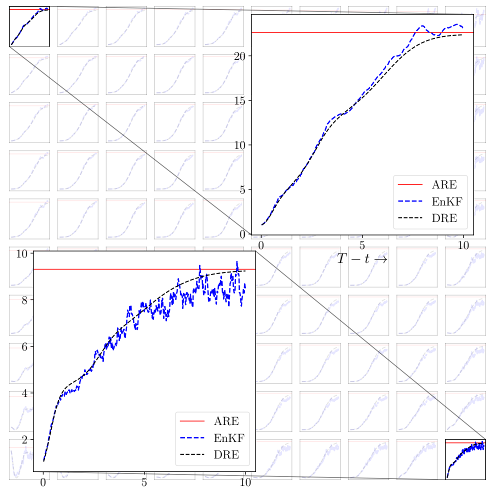
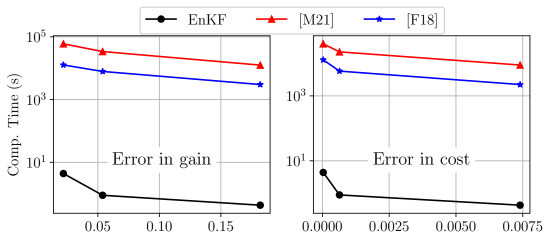


Figure: Performance on a 10 dimensional system ($N = 1000$ particles).

Numerical experiments (benchmark)
Comparison with policy optimization algorithms.



Algorithm	particles/samples	simulation time	iterations
EnKF	$O(1/\varepsilon^2)$	$O(\log(1/\varepsilon))$	1
Fazel [F18]	poly $(1/\varepsilon)$	poly $(1/\varepsilon)$	$O(\log(1/\varepsilon))$
Jovanovic [M21]	$O(1)$	$O(\log(1/\varepsilon))$	$O(\log(1/\varepsilon))$

Table: Computational complexity comparison of the algorithms to achieve ε error in approximating the infinite-horizon LQR optimal gain.

Numerical Experiments for a nonlinear example

Inverted Pendulum on Cart

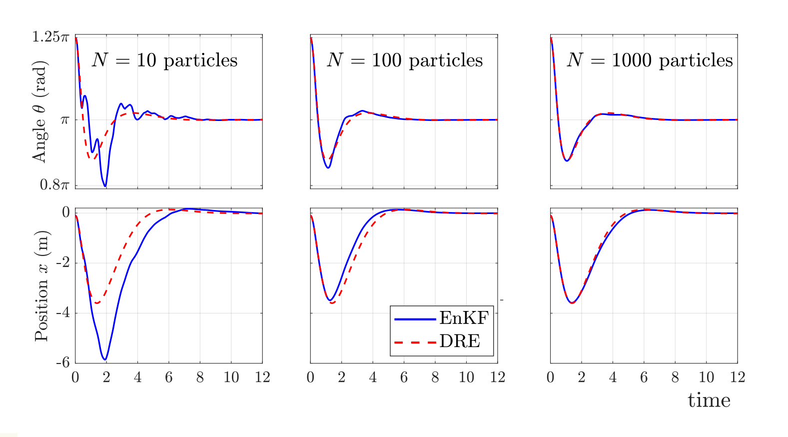


Figure: Nonlinear EnKF on Inverted Pendulum on Cart

Guiding philosophy (of this talk)

Faced with a control problem, do estimation

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1. **Solving control problem with an estimation algorithm:**

2. **Solving estimation problems with control techniques:**

Background
Dissipation in the study of stability of M.P.

Markov process (M.P.): Suppose $X = \{X_t : t \geq 0\}$ is a M.P.

1. taking values in state-space $\mathbb{S}$ (e.g., $\mathbb{S} = \{1, 2, \ldots, d\}$ or $\mathbb{S} = \mathbb{R}^d$);
2. with a given invariant measure $\bar{\mu}$.

Stochastic stability (definition):

$$\text{(stochastic stability)} \qquad \mathsf{E}(f(X_T)|X_0 = x) \to \bar{\mu}(f), \text{ as } T \to \infty \qquad (\text{in } L^2(\bar{\mu}) \text{ or some suitable sense})$$

Dissipation (or variance decay) in the study of M.P.:

$$\text{(Markov operator)} \qquad (P_T f)(x) := \mathsf{E}(f(X_T)|X_0 = x), \quad x \in \mathbb{S} \qquad \text{(linear deterministic opera}$$

$$\text{(variance)} \qquad \mathcal{V}^{\bar{\mu}}(f) := \mathsf{var}^{\bar{\mu}}(f(X_T)) = \bar{\mu}(f^2) - \bar{\mu}(f)^2$$

$$\text{(candidate Lyapunov function)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{\bar{\mu}}(P_t f) = -\bar{\mu}(\Gamma(P_t f)) \leq 0 \quad (\Gamma \text{ is the carre du champ operator}$$

$$\text{(Poincare inequality (PI))} \qquad \bar{\mu}(\Gamma(f)) \geq c\,\mathcal{V}^{\bar{\mu}}(f) \implies \mathcal{V}^{\bar{\mu}}(P_T f) \leq e^{-cT}\mathcal{V}^{\bar{\mu}}(f) \quad \text{(stability)}$$

D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2013).

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Aim of this talk: Generalize this (PI $\implies$ variance decay $\implies$ stability) story to nonlinear filter.

Hidden Markov model (HMM):

(hidden state process) $X = \{X_t : t \geq 0\} = \text{Markov}(\mathcal{A}, \mu)$ on state-space $\mathbb{S}$

(observation) $Z_t = \int_0^t h(X_s)\,\mathrm{d}s + W_t$ (additive white noise observations)

Nonlinear filter:

(cond. expect.) $\pi_T(f) := \mathsf{E}(f(X_T)|\mathcal{Z}_T)$, where $f \in C_b(\mathbb{S})$ and $\mathcal{Z}_T = \sigma(Z_t : 0 \leq t \leq T)$

(nonlinear filter) $\mathrm{d}\pi_t(f) = \pi_t(\mathcal{A}f)\,\mathrm{d}t + \big(\pi_t(hf) - \pi_t(h)\pi_t(f)\big)(\mathrm{d}Z_t - \pi_t(h)\,\mathrm{d}t)$, $\pi_0 = \mu$

(superscript notation) $\pi_T^\mu(f)$ (resp., $\pi_T^\nu(f)$) with prior μ (resp., ν)

(filter stability) $\mathsf{E}(|\pi_T^\mu(f) - \pi_T^\nu(f)|^2) \to 0$, as $T \to \infty$ (asymptotic forgetting of prior)

Questions:

1. Q1. Model properties (e.g., detectability) that are necessary and sufficient for filter stability.
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(cond. expect.) $\pi_T(f) := \mathsf{E}(f(X_T)|\mathcal{Z}_T)$, where $f \in C_b(\mathbb{S})$ and $\mathcal{Z}_T = \sigma(Z_t : 0 \leq t \leq T)$

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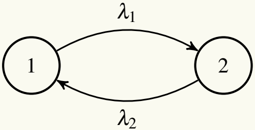
HMM and filter stability

HMM (A,H) on a finite state-space $\mathbb{S}=\{1,2,\ldots,d\}$:



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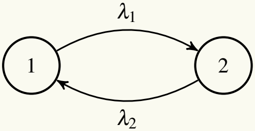

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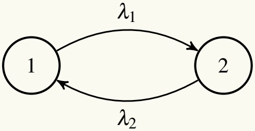

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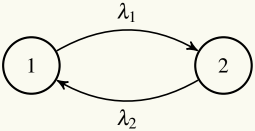

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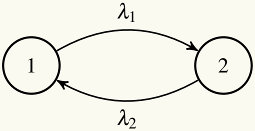

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Part 1: Backward map (This is the important contribution)

1. What is variance decay for nonlinear filter?
2. How is it related to filter stability?

Part 2: Model properties

1. How to define observability for HMM? 1 slide.
2. How to define Poincare constant for HMM? 1 slide.
3. What is the relationship between these? 1 slide.

Part 3: (combines parts 1 and 2.) 2 slides.

1. Main results on filter stability.

Papers related to Part 1:

1. Kim and M. Variance decay property for filter stability, TAC (To appear).
2. Joshi, Kim, and M. Backward map for filter stability, IEEE CDC (2024).

Definitions for filter stability:

$$(L^2 \text{ stability}) \qquad \mathsf{E}^{\mu}(|\pi_T^{\mu}(f) - \pi_T^{\nu}(f)|^2) \to 0, \text{ as } T \to \infty, \quad f \in C_b(\mathbb{S})$$

$$(\text{likelihood ratio}) \qquad \gamma_T(x) = \frac{\mathrm{d}\pi_T^{\mu}}{\mathrm{d}\pi_T^{\nu}}(x), \; x \in \mathbb{S} \qquad (\text{well-defined for } \mu \ll \nu)$$

(The problem of filter stability is to show that the ratio $\gamma_T \to 1$ as $T \to \infty$ (in a suitable sense).)

$$(\chi^2 - \text{divergence}) \qquad \chi^2(\pi_T^{\mu} | \pi_T^{\nu}) = \pi_T^{\nu}(|\gamma_T - 1|^2) \qquad (\text{is a divergence metric})$$

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Follows from elementary inequalities [Prop. 1 in Kim and M. (TAC, To appear)]

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(backward map) $\quad y_0(x) := \mathsf{E}^\nu(\gamma_T(X_T)|X_0 = x), \quad x \in \mathbb{S}$ (this is the key definition)

(why is it useful?) $\quad |\mathsf{E}^\mu\big(\chi^2(\pi_T^\mu \mid \pi_T^\nu)\big)|^2 \leq \mathsf{var}^\nu(y_0(X_0))\,\chi^2(\mu|\nu)$ (this is the key estimate)

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Q. *What is the appropriate notion of variance decay for a nonlinear filter?*
And how is it related to filter stability?

Answer. [Prop. 2 in Kim and M. (TAC, To appear)]

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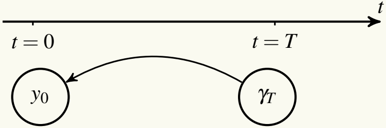
Filter stability

Can we also get rates?



(definition) $\quad y_0(X_0) - 1 = \mathsf{E}^\nu(\gamma_T(X_T) - 1 | X_0)$

(Jensen's inequality) $\quad \mathsf{E}^\nu(|y_0(X_0) - 1|^2) \leq \mathsf{E}^\nu(|\gamma_T(X_T) - 1|^2)$ $\quad$ (backward map is non-expansive)

(I am feeling lucky!) $\quad \mathsf{E}^\nu(|y_0(X_0) - 1|^2) \leq \mathsf{E}^\nu(e^{-cT}|\gamma_T(X_T) - 1|^2)$ $\quad$ (trivially holds with $c = 0$)

Filter stability

Suppose $\mu \ll \nu$. Then

$$\text{(filter stability)} \qquad \mathsf{E}^\mu\left(\chi^2(\pi_T^\mu \mid \pi_T^\nu)\right) \leq \frac{1}{\underline{a}} e^{-cT} \chi^2(\mu \mid \nu)$$

where $\underline{a} = \operatorname*{essinf}_{x \in \mathbb{S}} \gamma_0(x)$.

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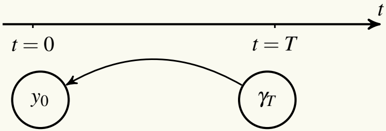

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Filter stability

Suppose $\mu \ll \nu$. Then

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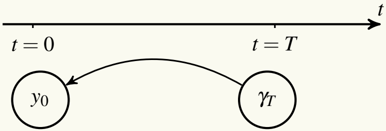

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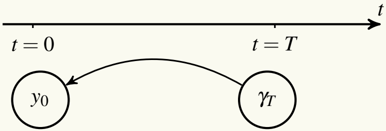

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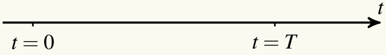
Embedding the backward map

BSDE (this is the only part that requires white noise observations)



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Q. *What is the appropriate generalization of the dissipation equation for the nonlinear filter?*

Answer. [Prop. 2 in Joshi, Kim, M. [CDC 2024]]

Consider the BSDE. Then at time $t = 0$,

$$Y_0(x) = y_0(x), \quad x \in \mathbb{S}$$

and

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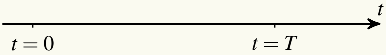

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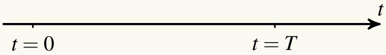

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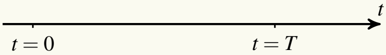

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Rate bounds

Rate bounds for finite state HMM (A is the rate matrix and $\bar{\mu}$ is the invariant measure):

#	Bound	Literature	Our work
(1)	$\min_{i\neq j} \sqrt{A(i,j)A(j,i)}$	[Atar and Zetouni (1997)]	Ex. 4 [CDC '21]
(2)	$\sum_{i\in\mathbb{S}} \bar{\mu}(i) \min_{j\neq i} A(i,j)$	[Baxendale *et. al.* (2004)]	Ex. 2 [CDC '21]
(3)	$\sum_{j} \min_{i\neq j} A(i,j)$	[Moulines 2006]	Ex. 3 [CDC '21]

Part 1: Backward map

1. What is variance decay for nonlinear filter?
2. How is it related to filter stability?

Part 2: Model properties for filter stability (white noise observations)

1. How to define observability for HMM?
2. How to define Poincare constant for HMM?

Part 3: Combines parts 1 and 2

1. Main results on filter stability.

Papers related to Part 1:

1. Kim and M. Duality for nonlinear filtering: I. Observability., TAC (2024).
2. Kim and M. Duality for nonlinear filtering: II. Optimal Control., TAC (2024).
3. Kim and M. Variance decay property for filter stability. TAC (To appear).

Math problem

Filter stability and key questions

Hidden Markov model (HMM):

(hidden state process) $\quad X = \{X_t : t \geq 0\} = \text{Markov}(\mathcal{A}, \mu)$ on state-space $\mathbb{S}$

(observation) $\quad Z_t = \int_0^t h(X_s)\,\mathrm{d}s + W_t \quad$ (additive white noise observations)

Nonlinear filter:

(cond. expect.) $\quad \pi_T(f) := \mathsf{E}(f(X_T)|\mathcal{Z}_T)$, where $f \in C_b(\mathbb{S})$ and $\mathcal{Z}_T = \sigma(Z_t : 0 \leq t \leq T)$

(nonlinear filter) $\quad \mathrm{d}\pi_t(f) = \pi_t(\mathcal{A}f)\,\mathrm{d}t + \big(\pi_t(hf) - \pi_t(h)\pi_t(f)\big)(\mathrm{d}Z_t - \pi_t(h)\,\mathrm{d}t), \quad \pi_0 = \mu$

(superscript notation) $\quad \pi_T^\mu(f)$ (resp., $\pi_T^\nu(f)$) with prior μ (resp., ν)

(filter stability) $\quad \mathsf{E}(|\pi_T^\mu(f) - \pi_T^\nu(f)|^2) \to 0$, as $T \to \infty$ (asymptotic forgetting of prior)

Questions:

1. Q1. Model properties (e.g., detectability) that are necessary and sufficient for filter stability.
2. Q2. Bounds on (exponential) rate of convergence.

J. Xiong. An Introduction to Stochastic Filtering Theory. Oxford Press (2008).

Dual optimal control problem for HMM $(\mathcal{A},h)$ [Kim and M. (TAC 2004)]

$$\underset{U\in L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)}{\text{minimize}} \quad \mathsf{J}(U) = \mathsf{E}^{\rho}\left(\mathsf{var}^{\rho}(Y_0(X_0)) + \int_0^{\tau} \ell(Y_t,V_t,U_t;X_t)\,\mathrm{d}t\right)$$

$$\text{subj. to} \quad -\mathrm{d}Y_t = \big((\mathcal{A}Y_t)(x) + h(x)(U_t + V_t(x))\big)\,\mathrm{d}t - V_t(x)\,\mathrm{d}Z_t, \quad x\in\mathbb{S}, \quad 0\le t\le\tau$$

$$Y_\tau(x) = F(x), \quad x\in\mathbb{S}$$

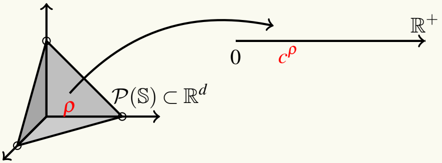
The cost function is: $\ell(y,v,u;x) = \underbrace{(\Gamma y)(x)}_{\text{carre du champ}} + |u+v(x)|_R^2.$

HMM on $\mathbb{S} = \{1,2,\ldots,d\}$**:**

(dual control system) $\quad -\mathrm{d}Y_t = \big((\mathcal{A}Y_t)(x) + h(x)(U_t + V_t(x))\big)\,\mathrm{d}t - V_t(x)\,\mathrm{d}Z_t, \quad Y_T = c1$

(solution operator) $\quad \mathcal{L} : U \mapsto Y_0 \in \mathbb{R}^d$

(controllability defn.) $\quad \mathcal{C}_\tau := \mathsf{Range}(\mathcal{L}) = \mathbb{R}^d$



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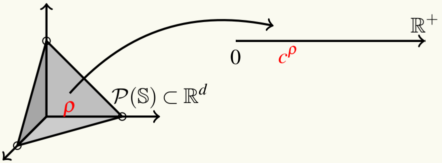
Dual optimal control problem

Answer to Q2. Poincare constant for the filter



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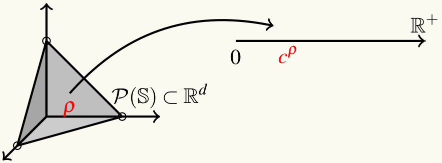

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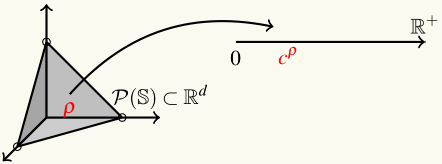

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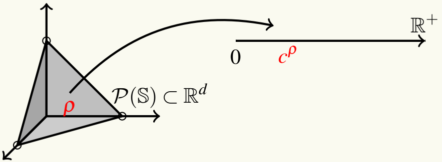

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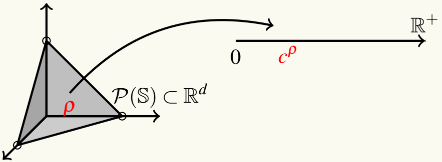

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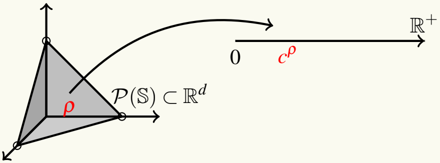


For $\mathbb{S} = \{1,2,\ldots,d\}$, we show that

1. c^ρ is attained as a minimum (i.e., a minimizer F^ρ exists in $\mathbb{H}^\rho_\tau$).
2. $\rho \mapsto c^\rho$ is continuous w.r.t the separating norm topology on $\mathcal{P}(\mathbb{S}) \setminus \mathcal{N}$.

Definition of Poincare constant:

(optimal control system) $\quad -\,\mathrm{d}Y_t(x) = \big((\mathcal{A}Y_t)(x) + h^{\mathrm{T}}(x)(U_t^{\text{opt}} + V_t(x))\big)\,\mathrm{d}t - V_t^{\mathrm{T}}(x)\,\mathrm{d}Z_t, \;\; x \in \mathbb{S},\; 0 \le t \le \tau$

$U_t^{\text{opt}} = -\mathcal{V}_t^\rho(h, Y_t) - \pi_t^\rho(V_t), \quad 0 \le t \le \tau$ (linear feedback control law)

$Y_\tau(x) = F(x),\;\; x \in \mathbb{S}$

(energy) $\quad \mathcal{E}^\rho(F) := \mathsf{E}^\rho\left(\int_0^\tau \ell(Y_t, V_t, U_t^{\text{opt}}; X_t)\,\mathrm{d}t\right)$

(optimality equation) $\quad \mathsf{var}^\rho(Y_0(X_0)) + \mathcal{E}^\rho(F) = \mathsf{E}^\rho(\mathcal{V}_\tau^\rho(F))$ (value function)

(Poincare constant) $\quad c^\rho := \dfrac{1}{\tau}\log(1+\beta^\rho),$ where $\beta^\rho = \inf\{\mathcal{E}^\rho(F) : F \in \mathbb{H}^\rho_\tau \;\&\; \mathsf{var}^\rho(Y_0(X_0)) = 1\}$

(incremental decay) $\quad \mathsf{var}^\rho(Y_0(X_0)) \le e^{-\tau c^\rho}\,\mathsf{E}^\rho\big(\mathcal{V}_\tau^\rho(F)\big), \quad \forall F \in \mathbb{H}^\rho_\tau$

Prop 1 in [Kim, M. (TAC 2004)]

The controllable subspace $\mathcal{C}_\tau$ is the smallest such subspace $\mathcal{C} \subset \mathbb{R}^d$ that satisfies two properties:

(i) The constant function $1 \in \mathcal{C}$; and

(ii) If $g \in \mathcal{C}$ then $\mathcal{A}g \in \mathcal{C}$ and $gh \in \mathcal{C}$.

Definition (of HMM model properties (from duality))

1. HMM $(\mathcal{A}, h)$ is *observable* if $\mathcal{C} = \mathbb{R}^d$.
2. HMM $(\mathcal{A}, h)$ is *detectable* if $\mathcal{C} \subset S_0 := \{f \in \mathbb{R}^d \mid \Gamma f(x) = 0 \; \forall \, x \in \mathbb{S}\}$.

Relationship to positivity of c^ρ [Prop. 6 in [Kim and M., TAC 2025]

Suppose $\mathbb{S} = \{1, 2, \ldots, d\}$ and any of the following conditions holds:

(i) $\mathcal{A}$ is ergodic.

(ii) $(\mathcal{A}, h)$ is observable.

(iii) $(\mathcal{A}, h)$ is detectable.

Then $c^\rho > 0$.

Part 1: Backward map

1. What is variance decay for nonlinear filter?
2. How is it related to filter stability?

Part 2: Model properties for filter stability

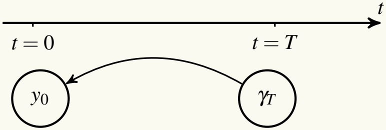
1. How to define observability for HMM?
2. How to define Poincare constant for HMM?

Part 3: Combines parts 1 and 2

1. Main results on filter stability.

Recap of parts 1 and 2

Backward map, BSDE, and Poincare constants

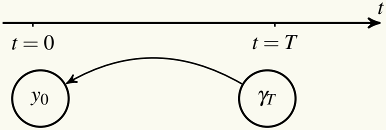


(Jensen's for backward map) $\quad \mathsf{E}^\nu(|y_0(X_0)-1|^2) \le \mathsf{E}^\nu(|\gamma_T(X_T)-1|^2)$

(dissipation equation) $\quad \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{var}^\nu(Y_t(X_t)) = \mathsf{E}^\nu\big(\pi_t^\nu(\Gamma Y_t) + \pi_t^\nu(|V_t|^2)\big), \quad 0 \le t \le T$

(cummulative) $\quad C_N := \sum_{k=0}^{N-1} c^{\pi_{k\tau}^\nu}$

(variance decay) $\quad \mathsf{var}^\nu(y_0(X_0)) \le \mathsf{E}^\nu\left(e^{-\tau C_N}\chi^2(\pi_T^\mu|\pi_T^\nu)\right)$



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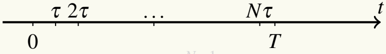

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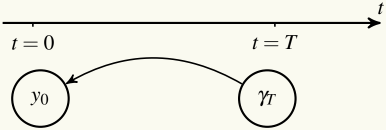

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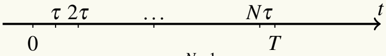

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Because $\{C_N : N = 1,2,\ldots\}$ is non-negative and monotone, define

$$C_\infty(\omega) := \lim_{N\to\infty} \uparrow C_N(\omega), \quad \omega \in \Omega$$

Theorem (2 in Kim and M. TAC (To appear))

Suppose $\{\mathcal{V}_T^\nu(\gamma_T) : T \geq 0\}$ *is* P^ν*-u.i. and* $c^\rho : \mathcal{P}(\mathbb{S}) \setminus \mathcal{N} \to \mathbb{R}$ *is continuous. Then*

(i) *Either* $\mathsf{P}^\nu([C_\infty = \infty]) = 1$*, in which case the variance decay property holds and the filter is stable in* χ^2*-divergence; or*

(ii) $\mathsf{P}^\nu([C_\infty = \infty]) < 1$*, in which case*

$$c^{\pi_T^\nu(\omega)} \stackrel{(T\to\infty)}{\longrightarrow} 0, \quad \mathsf{P}^\nu\text{-a.e. } \omega \in [C_\infty < \infty]$$

Theorem (3 in Kim and M. TAC (To appear))

Suppose $\mathbb{S} = \{1,2,\ldots,\mathbb{S}\}$*,* $\min_{x\in\mathbb{S}} \gamma_0(x) > 0$*, and* $(\mathcal{A},h)$ *is detectable. Then the filter is stable in* χ^2*-divergence.*

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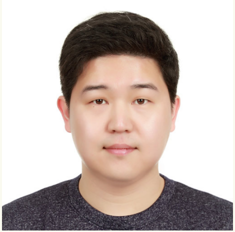

Prof. Jin Won Kim
Hongik University, South Korea

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Anant Joshi

Conclusions

1. "*establishing duality [of the optimal estimator] with the optimal regulator is a favorite technique for establishing estimator stability*" – Rawlings, Mayne, and Diehl (MPC textbook).
2. For MHE and MEE, there have been a number of important contributions: [Krener, 2003], [Rao's PhD Thesis (2000)], [Ch. 4 of Rawlings et. al.], [van Handel's PhD thesis (2006)].
3. Yet these attempts are somewhat dis-connected from stochastic filtering theory – both in terms of model properties and rate estimates for convergence.
4. Minimum variance duality appears to be better suited for this purpose.

Backup

Math preliminaries

Some technicalities (everything in Part 1 applies to general HMMs)

Probability space: $(\Omega, \mathcal{F}_T, \mathsf{P}^\mu)$ where $\mu \in \mathcal{P}(\mathbb{S})$ is the true prior. HMM is (X, Z) with $X_0 \sim \mu$.

Let $\rho \in \mathcal{P}(\mathbb{S})$. On the common measurable space $(\Omega, \mathcal{F}_T)$, P^ρ is used to denote another prob. measure such that the transition law of (X, Z) is identical but $X_0 \sim \rho$. Examples:

- $\rho = \mu$. Then $\pi_T^\mu(f) = \mathsf{E}^\mu(f(X_T)|\mathcal{Z}_T)$. (The measure μ has meaning of the true prior.)
- $\rho = \nu$. Then $\pi_T^\nu(f) = \mathsf{E}^\nu(f(X_T)|\mathcal{Z}_T)$. ($\nu$ is the incorrect prior used to compute the filter.)

Lemma 2.1 in Clark, Ocone, Coumarbatch (1999).

Suppose $\mu \ll \nu$. Then

- $\mathsf{P}^\mu \ll \mathsf{P}^\nu$, and the change of measure is given by

$$\frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}\mathsf{P}^\nu}(\omega) = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\big(X_0(\omega)\big) \quad \mathsf{P}^\nu\text{-a.s. } \omega$$

- For each $t > 0$, $\pi_t^\mu \ll \pi_t^\nu$, $\mathsf{P}^\mu|_{\mathcal{Z}_t}$-a.s..

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