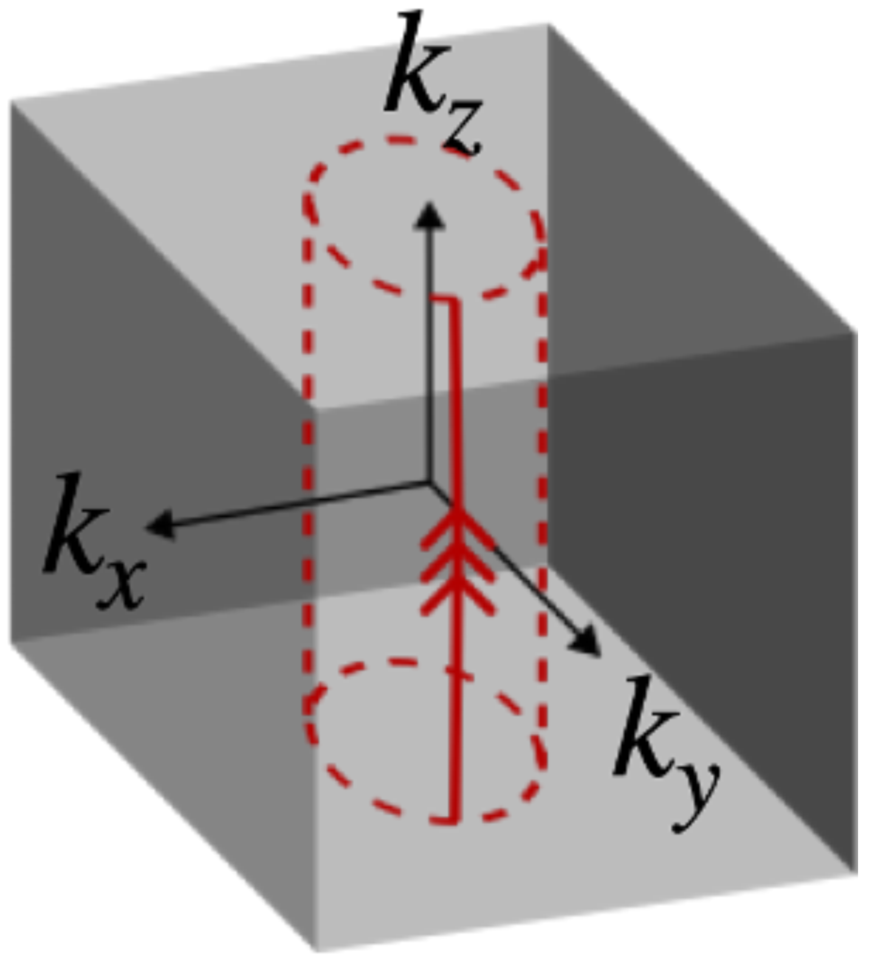
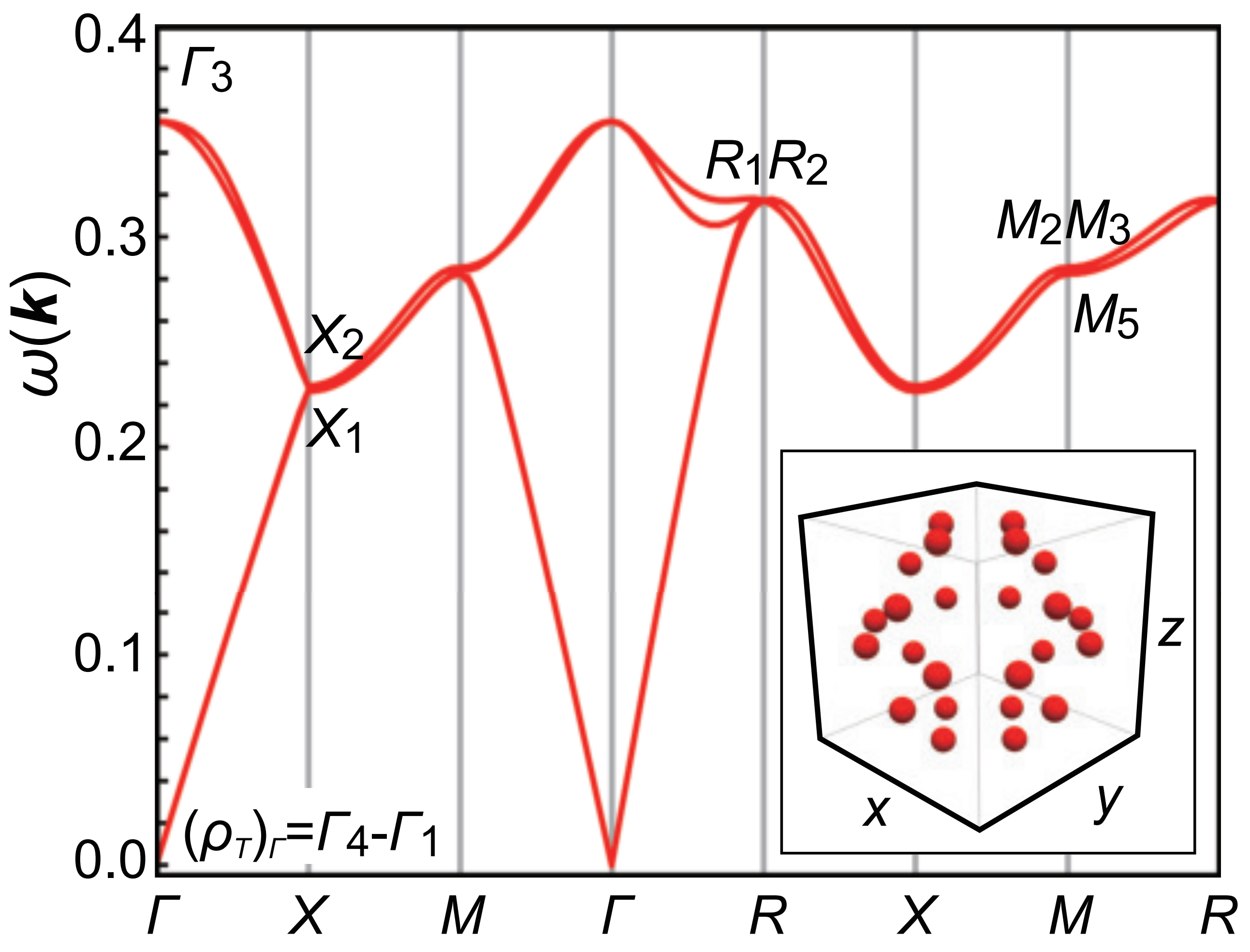
Computing Topological Invariants for 3D Photonic Crystals





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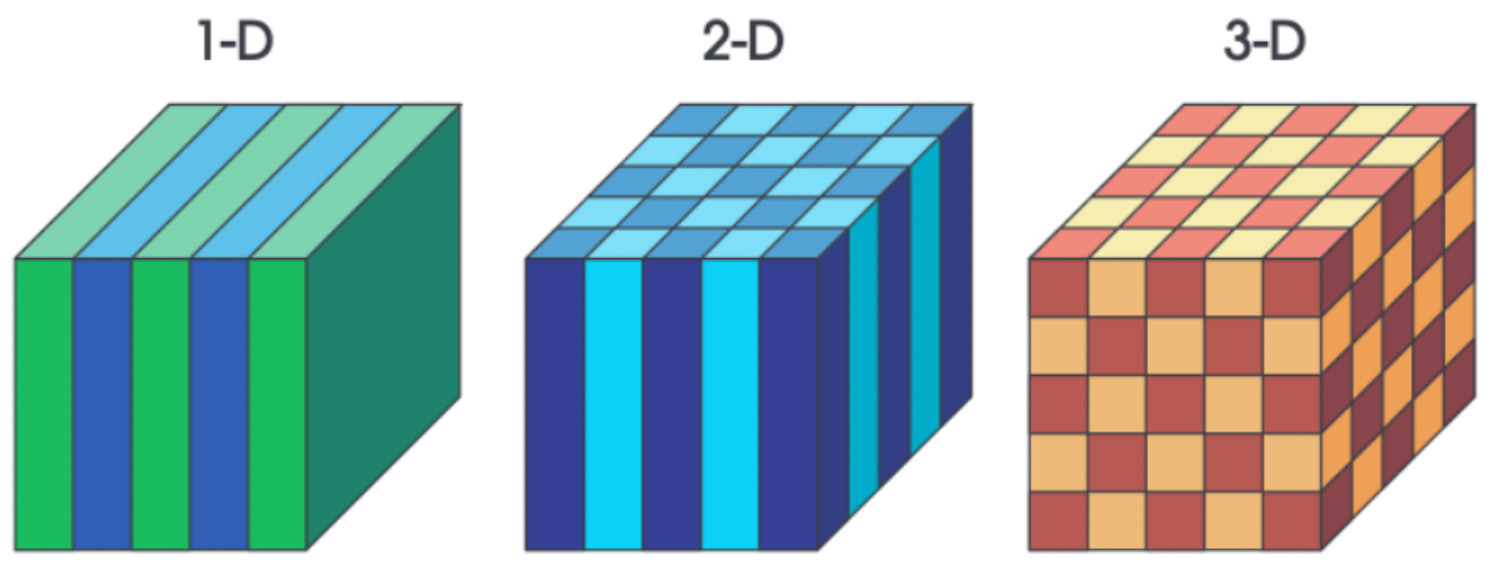
Motivation: Modeling Topological Photonic Crystals

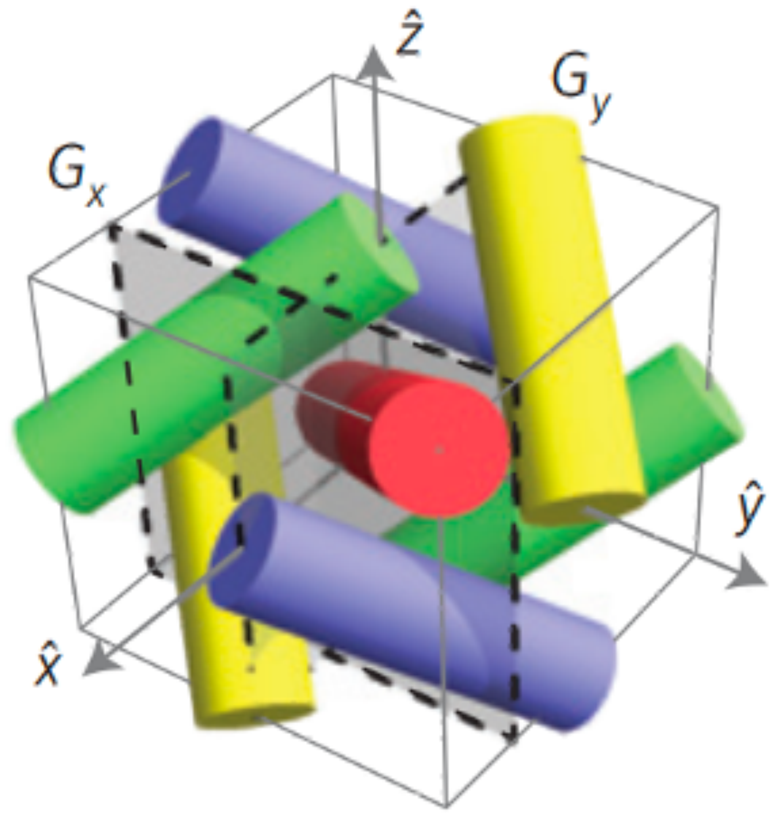


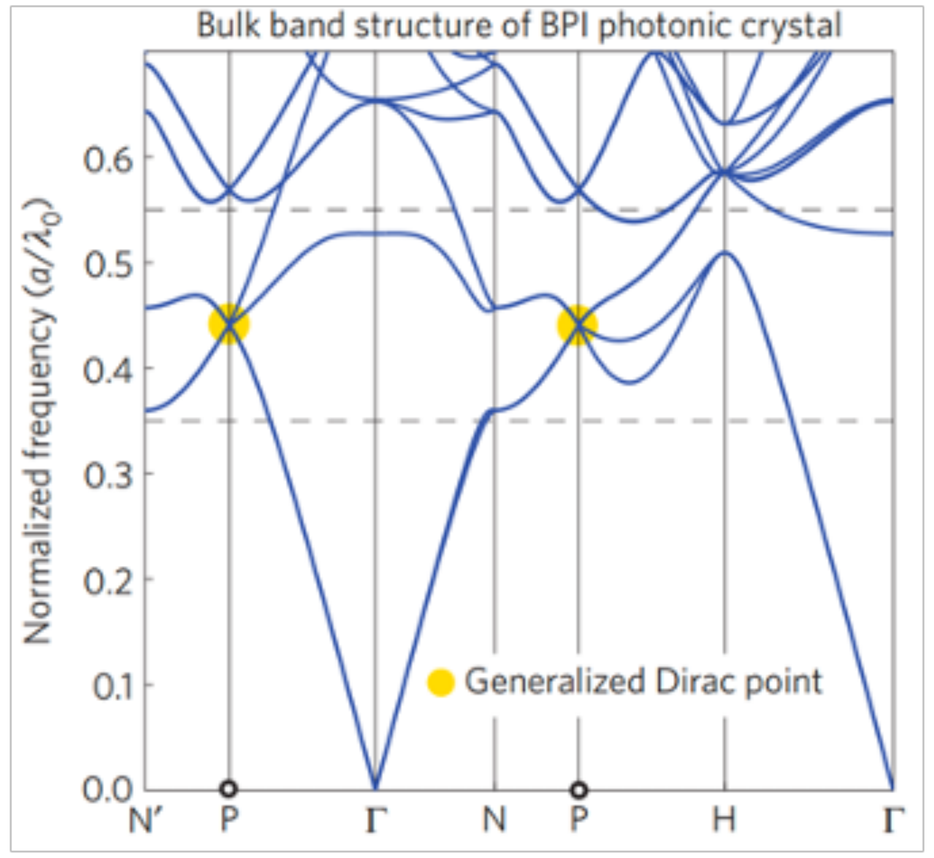
- **Periodic dielectric material** $\epsilon_{\mu\nu}(\mathbf{r}+\mathbf{a}) = \epsilon_{\mu\nu}(\mathbf{r})$
- **Maxwell Equations (source free) satisfy Bloch's Theorem**

$$\nabla \times (\nabla \times \mathbf{E}_{n\mathbf{k}}) = \omega_{n\mathbf{k}}^2 \mathbf{D}_{n\mathbf{k}}$$

$$\mathbf{D}_{n\mathbf{k}}(\mathbf{x}) = \varepsilon(\mathbf{x})\mathbf{E}_{n\mathbf{k}}(\mathbf{x})$$

- **Eigenfrequencies form bands**

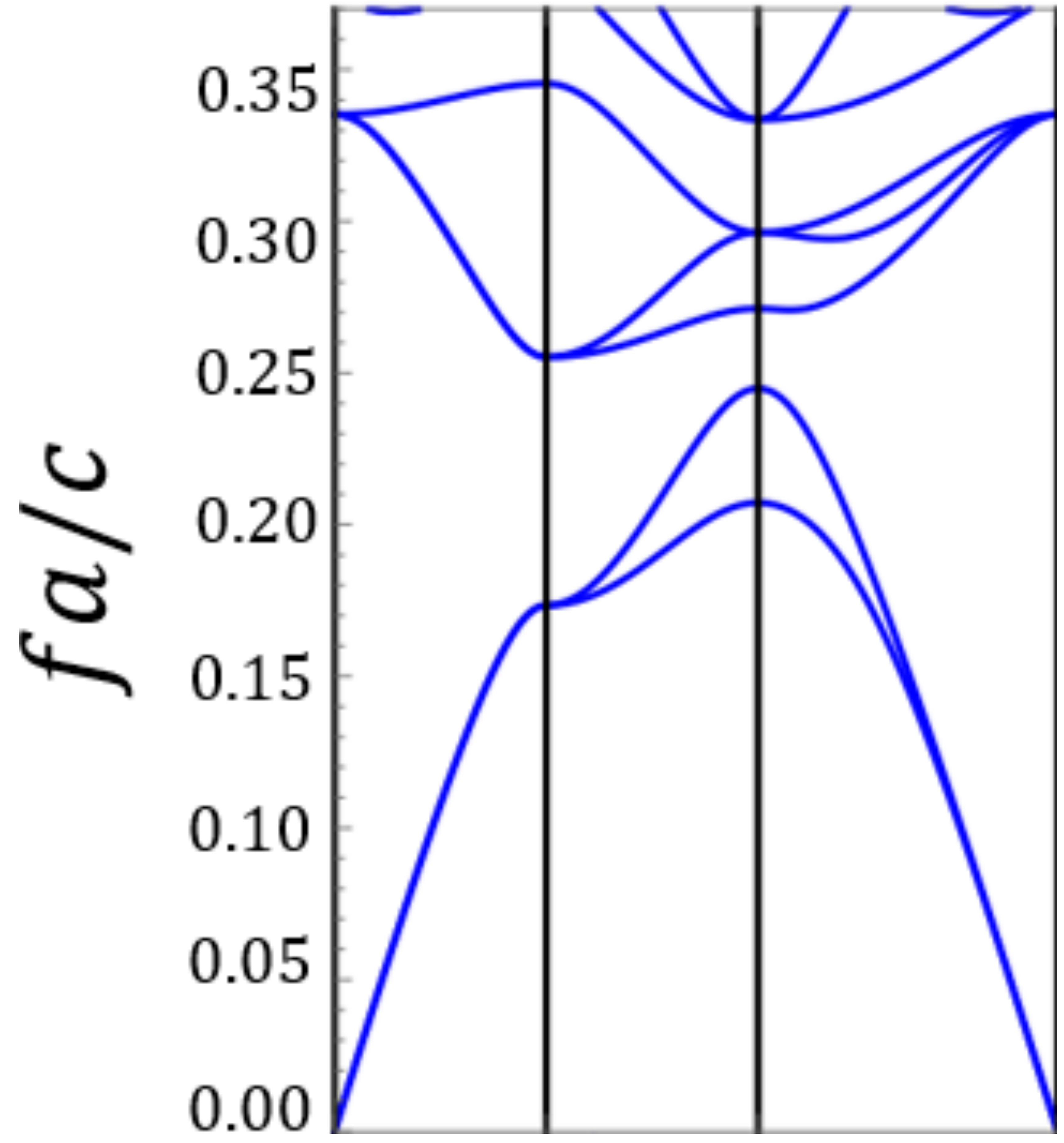




Lu et al, Nat Phys (2016)
Joannopoulos, John D., et al. Photonic Crystals: Molding the Flow of Light - Second Edition, Princeton University Press, 2008

Motivation: Modeling Topological Photonic Crystals

- **Gapped sets of bands can have nontrivial topology**
- **Need methods for computing topological invariants in the bulk**
- **Need models that capture topology without needing to solve Maxwell's equations for large finite systems**



Momentum Space: Wilson Loops and Symmetry Indicators

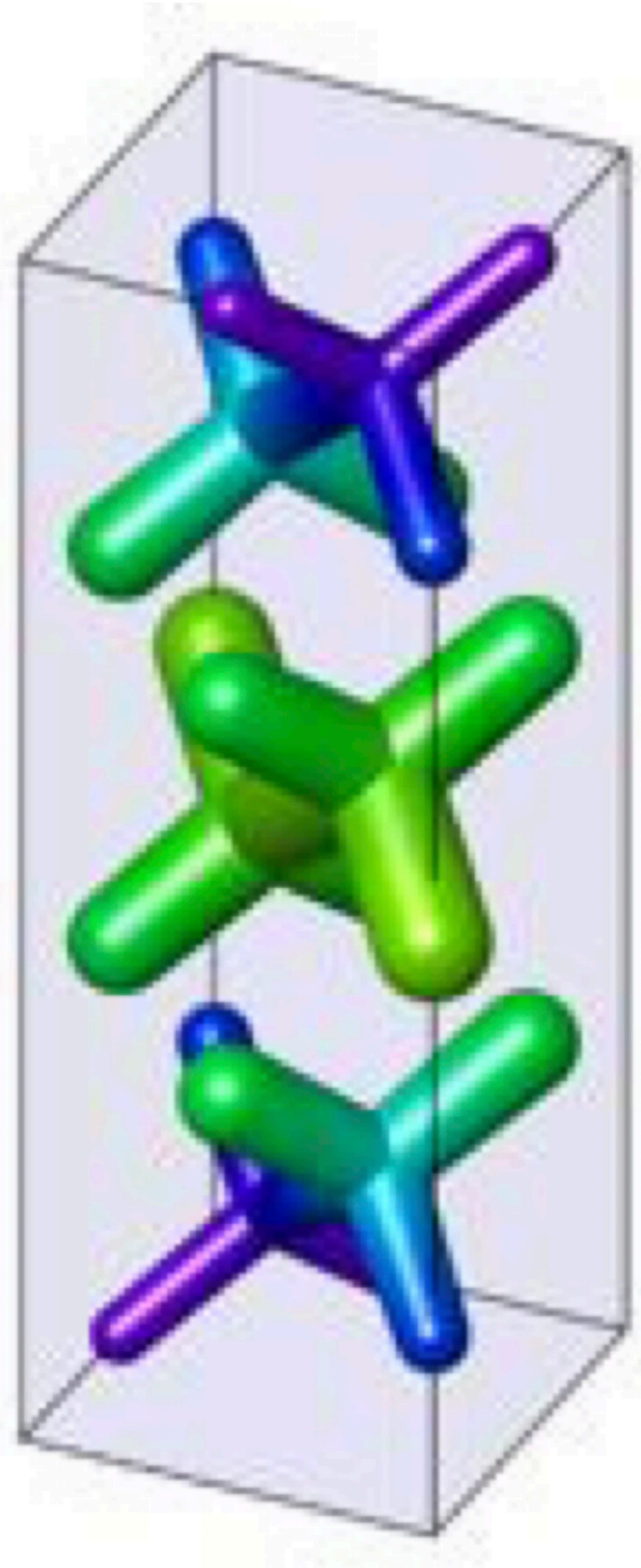
- **General magnetoelectric material:**

$$\begin{pmatrix}\mathbf{D}(t,\mathbf{r})\\ \mathbf{B}(t,\mathbf{r})\end{pmatrix} = \mathcal{K}(\mathbf{r})\begin{pmatrix}\mathbf{E}(t,\mathbf{r})\\ \mathbf{H}(t,\mathbf{r})\end{pmatrix}$$

$$\mathcal{K}(\mathbf{r}) = \begin{pmatrix}\varepsilon(\mathbf{r}) & \xi(\mathbf{r})\\ \xi^{\dagger}(\mathbf{r}) & \mu(\mathbf{r})\end{pmatrix}$$

$$\mathcal{M}(\mathbf{k})u_{n,\mathbf{k}}(\mathbf{r}) = \omega_n(\mathbf{k})u_{n,\mathbf{k}}(\mathbf{r}),$$

$$\mathcal{M}(\mathbf{k}) = \mathcal{K}^{-1}\mathrm{Rot}(\mathbf{k}) = \mathcal{K}^{-1}\begin{pmatrix}0 & +(i\nabla_{\mathbf{r}} - \mathbf{k})\times\\ -(i\nabla_{\mathbf{r}} - \mathbf{k})\times & 0\end{pmatrix}$$



Devescovi et al., Optical Materials Express 14, 2161-2177 (2024).

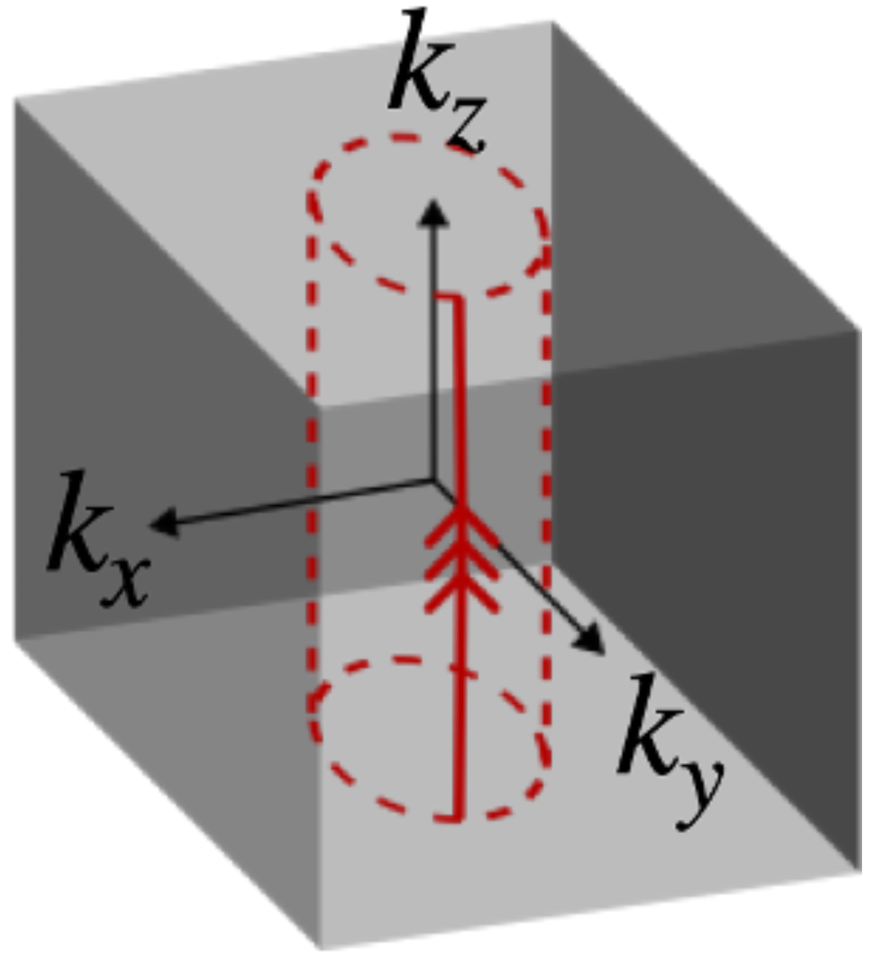
Momentum Space: Wilson Loops and Symmetry Indicators

- **Topological invariants can be computed from Berry phases**

$$\langle u|v\rangle_{\mathcal{K}} = \int_{\mathrm{UC}} d^3\mathbf{r}\, u^\dagger(\mathbf{r})\mathcal{K}(\mathbf{r})v(\mathbf{r}),$$

$$\mathcal{W}_{m,n}^{\mathbf{k}+\boldsymbol{\epsilon}\leftarrow\mathbf{k}}(\epsilon) = \langle u_{m,\mathbf{k}+\boldsymbol{\epsilon}}|u_{n,\mathbf{k}}\rangle_{\mathcal{K}},$$

$$\mathcal{W}_{m,n}(\ell) = \mathcal{W}_{m,p}^{\mathbf{k}_{(F)}\leftarrow\mathbf{k}_{(N)}}\mathcal{W}_{p,q}^{\mathbf{k}_{(N)}\leftarrow\mathbf{k}_{(N-1)}}\cdots\mathcal{W}_{r,s}^{\mathbf{k}_{(2)}\leftarrow\mathbf{k}_{(1)}}\mathcal{W}_{s,n}^{\mathbf{k}_{(1)}\leftarrow\mathbf{k}_{(0)}}$$

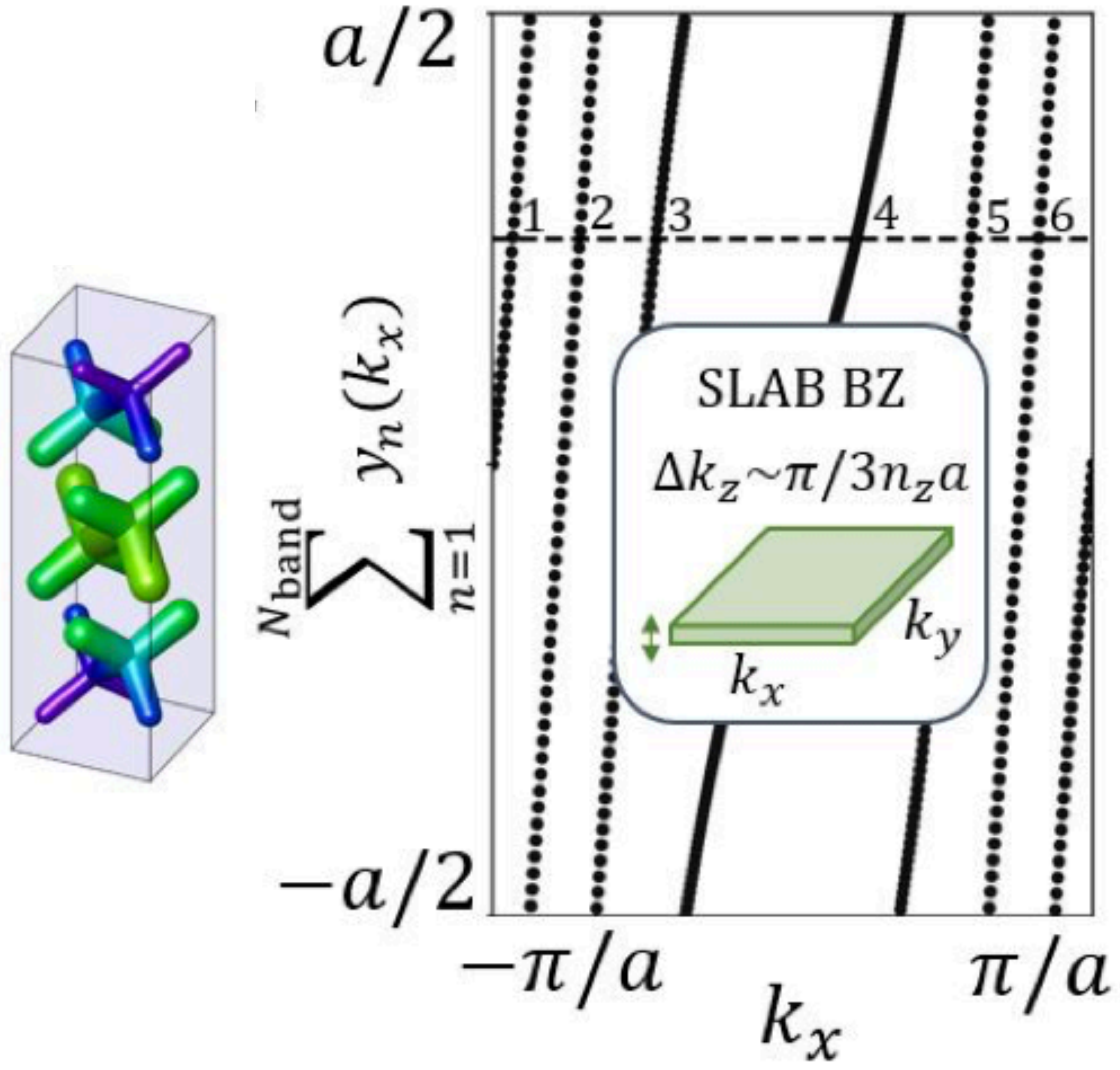


- **Eigenvalues $e^{iw_n(\mathbf{k}_\perp)}$ related to projected position eigenvalues** $\frac{w_n(\mathbf{k}_\perp)}{2\pi} = \frac{x_n(\mathbf{k}_\perp)}{a_x} \pmod 1,$

Devescovi et al., Optical Materials Express 14, 2161-2177 (2024).

Momentum Space: Wilson Loops and Symmetry Indicators

- **Nontrivial topology appears in the winding of $w_n(\mathbf{k}_\perp)$**



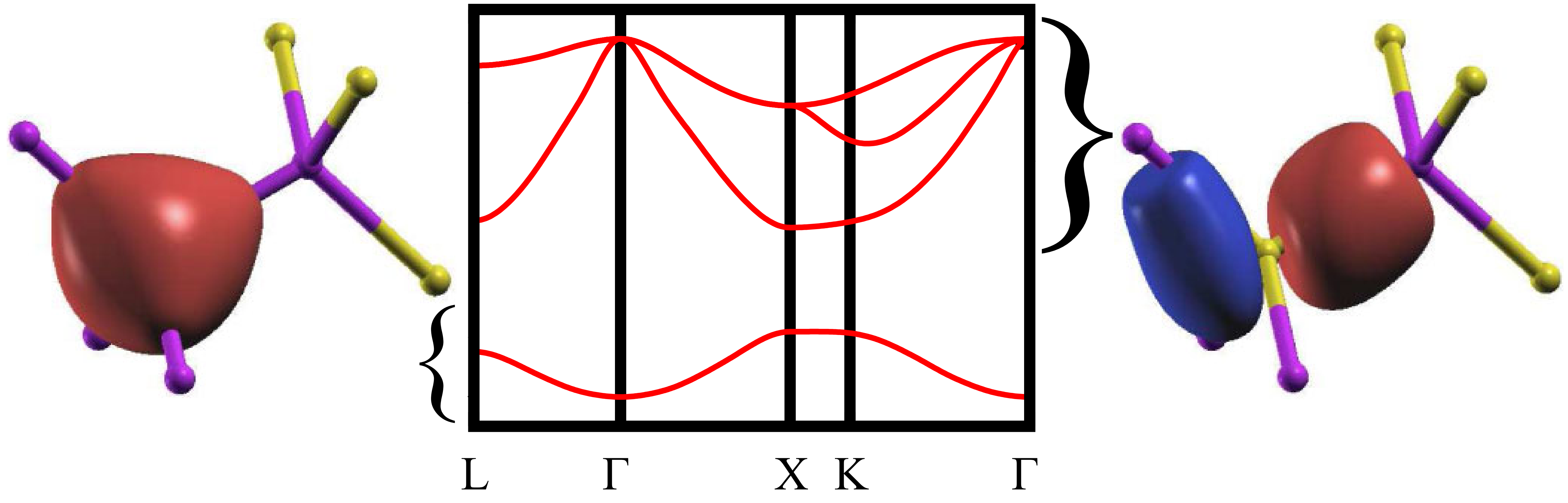
- **Bulk-boundary correspondence: Winding of $w_n(\mathbf{k}_\perp)$ related to edge spectral flow**

- $\dfrac{w_n(\mathbf{k}_\perp)}{2\pi} = \dfrac{x_n(\mathbf{k}_\perp)}{a_x} \pmod 1,$ **implies a deep connection between topology and localization**

Devescovi et al., Optical Materials Express 14, 2161-2177 (2024).

Momentum Space: Wilson Loops and Symmetry Indicators

- **Topological invariants are obstructions to finding exponentially localized Wannier functions**



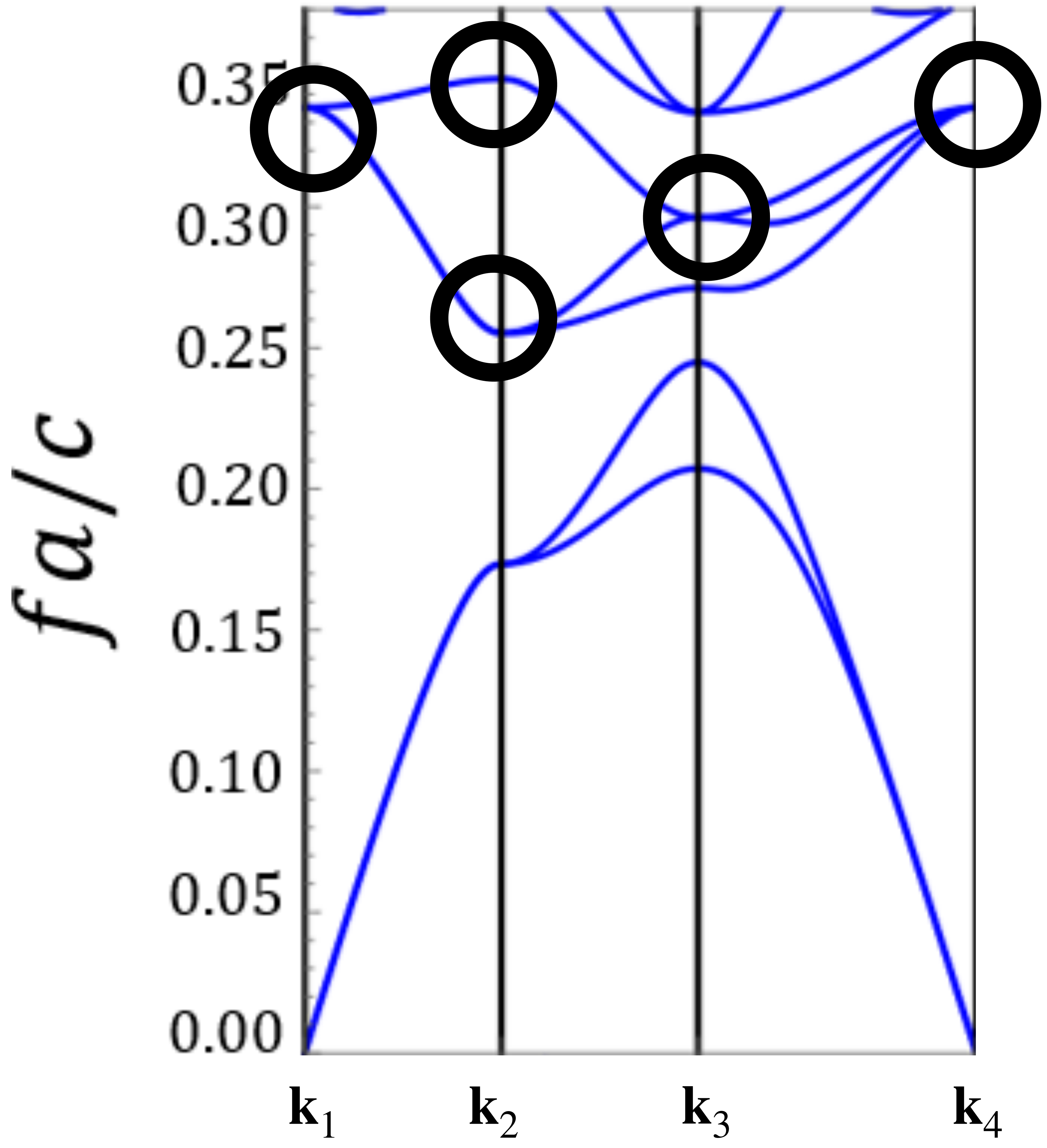
- **Wilson loop winding obstructs Fourier transform**

- **Space group symmetries let us define *symmetry indicators of band topology***

Marzari et al., Rev. Mod. Phys. 2012

Momentum Space: Wilson Loops and Symmetry Indicators

- **Eigenstates transform in irreps of the little group at each k**
- **Collect irreps at high symmetry points into a symmetry data vector v**
- **Topological Quantum Chemistry: for each space group, we can define a matrix Σ of symmetry indicators**
- $\Sigma\mathbf{v} \not\equiv 0$ **implies nontrivial topology**



Challenges

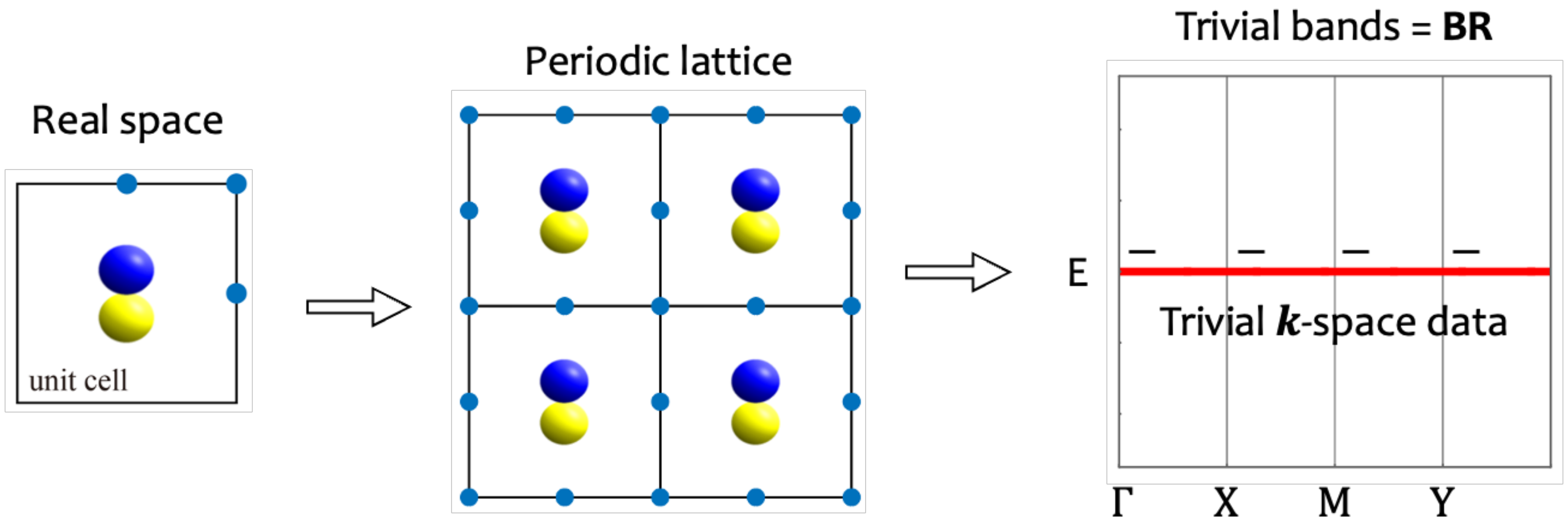
- General:
    - There exist topological invariants invisible to symmetry indicators
    - Symmetry indicators don't capture that different trivial phases may still be distinct
- Photonic specific: The lowest photonic bands are singular in 3D -> no well-defined symmetry data vector

All have the same solution - Stable Real Space Invariants

Wannier Functions and Band Representations

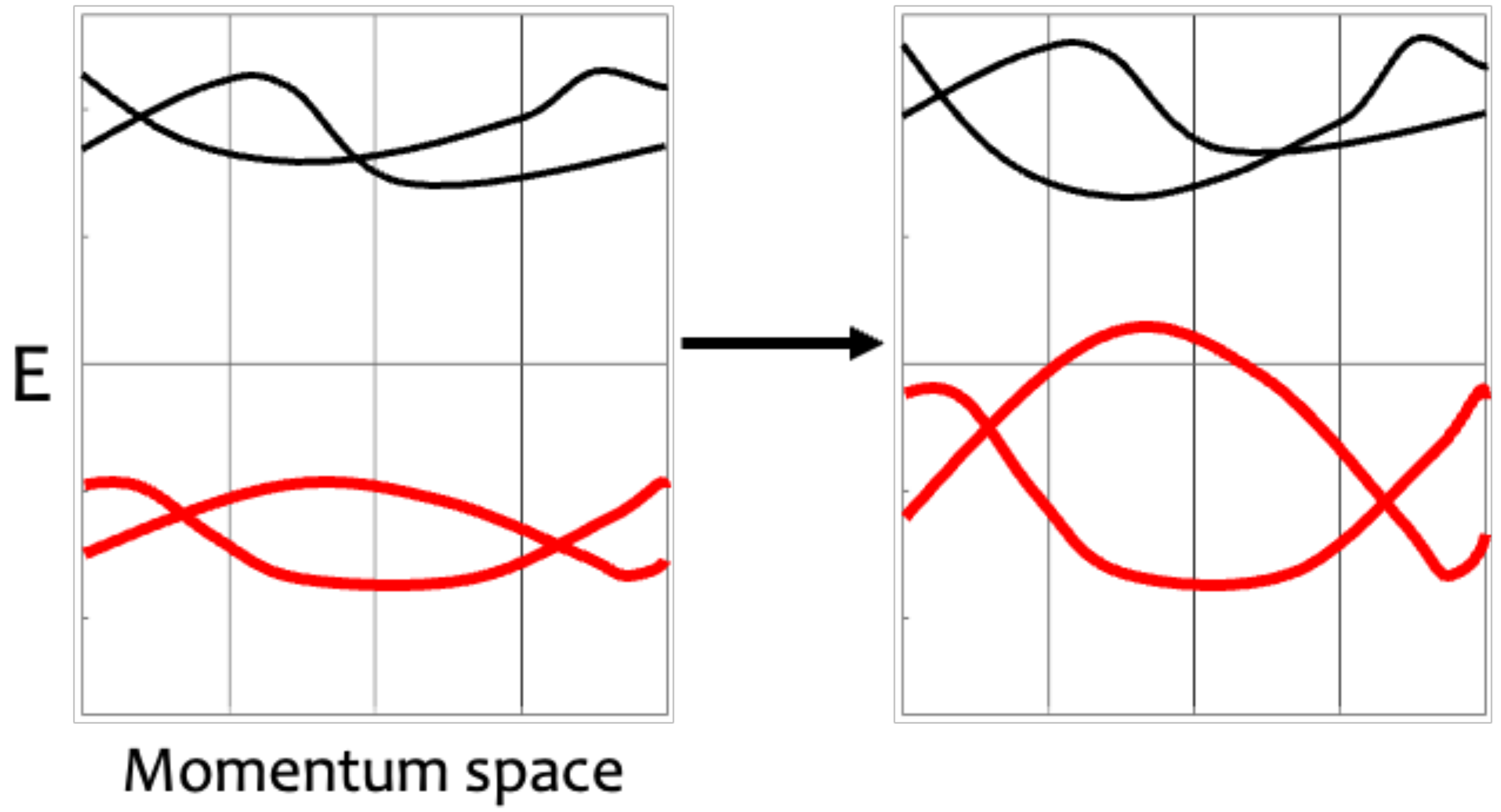
- **End goal will be finite-rank modeling of photonic bands via tight-binding models -> need to understand *where* symmetry data vectors come from**

- **Topologically trivial bands have symmetric, localized Wannier functions by definition, determine symmetry data vectors by *induction***



Wannier Functions and Band Representations

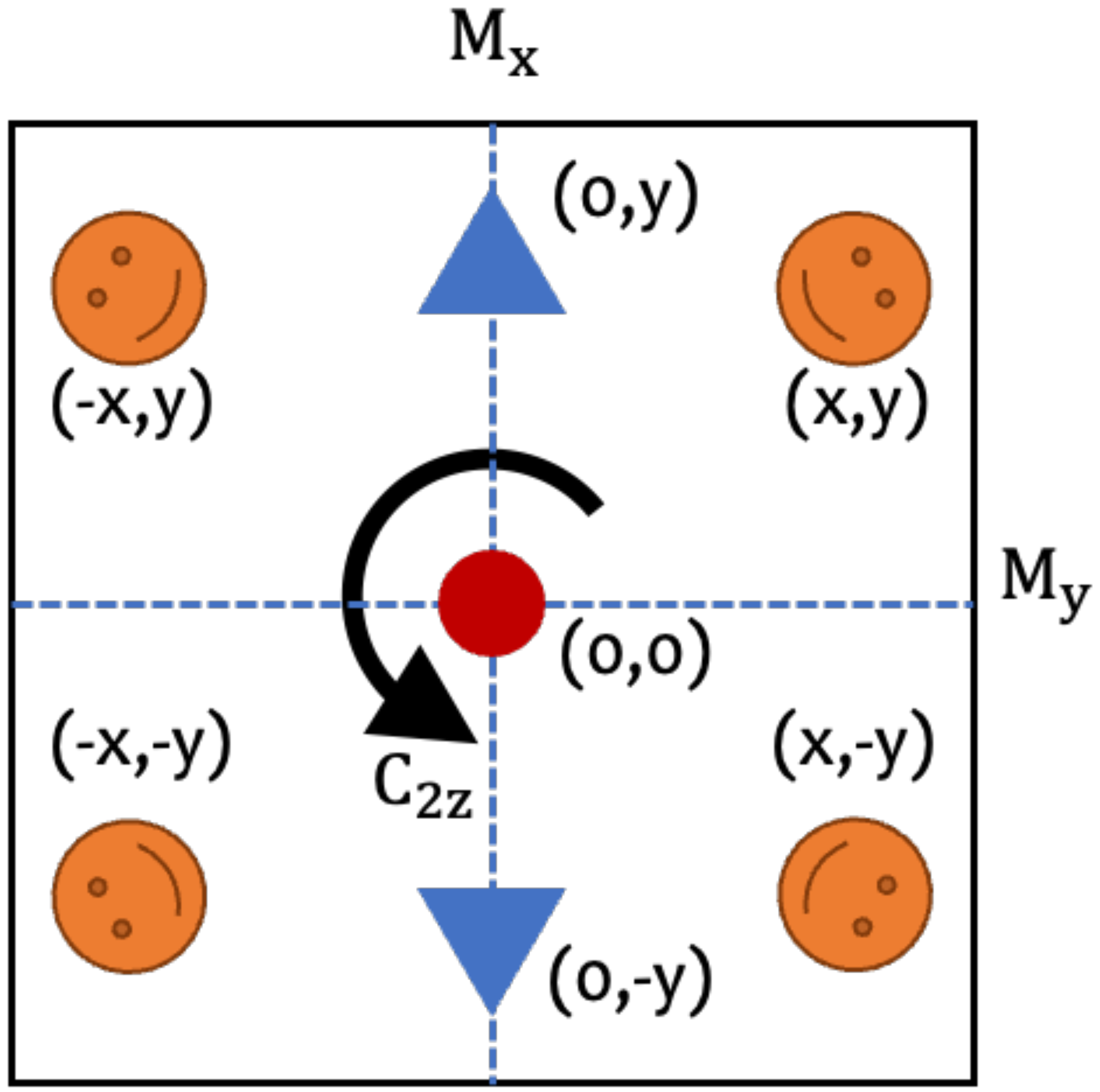
- **Band representations do not change under adiabatic deformation**



- **We can characterize trivial bands in terms of the symmetry properties and centers of the Wannier functions that induce them**

Wannier Functions and Band Representations

- **Classify trivial bands by specifying location and symmetry properties of Wannier functions (orbitals) in the unit cell**



Unit cell

Points in the unit cell fall into orbits called Wyckoff Positions (WP)

1. WP 1a(0,0)

At (0,0): PG mm2

	1	C_{2z}	M_x	M_y
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	-1	1
B_2	1	-1	1	-1

e.g. "A_1 orbital" or "irrep"

2. WP 2g{(0,y)+(0,-y)}

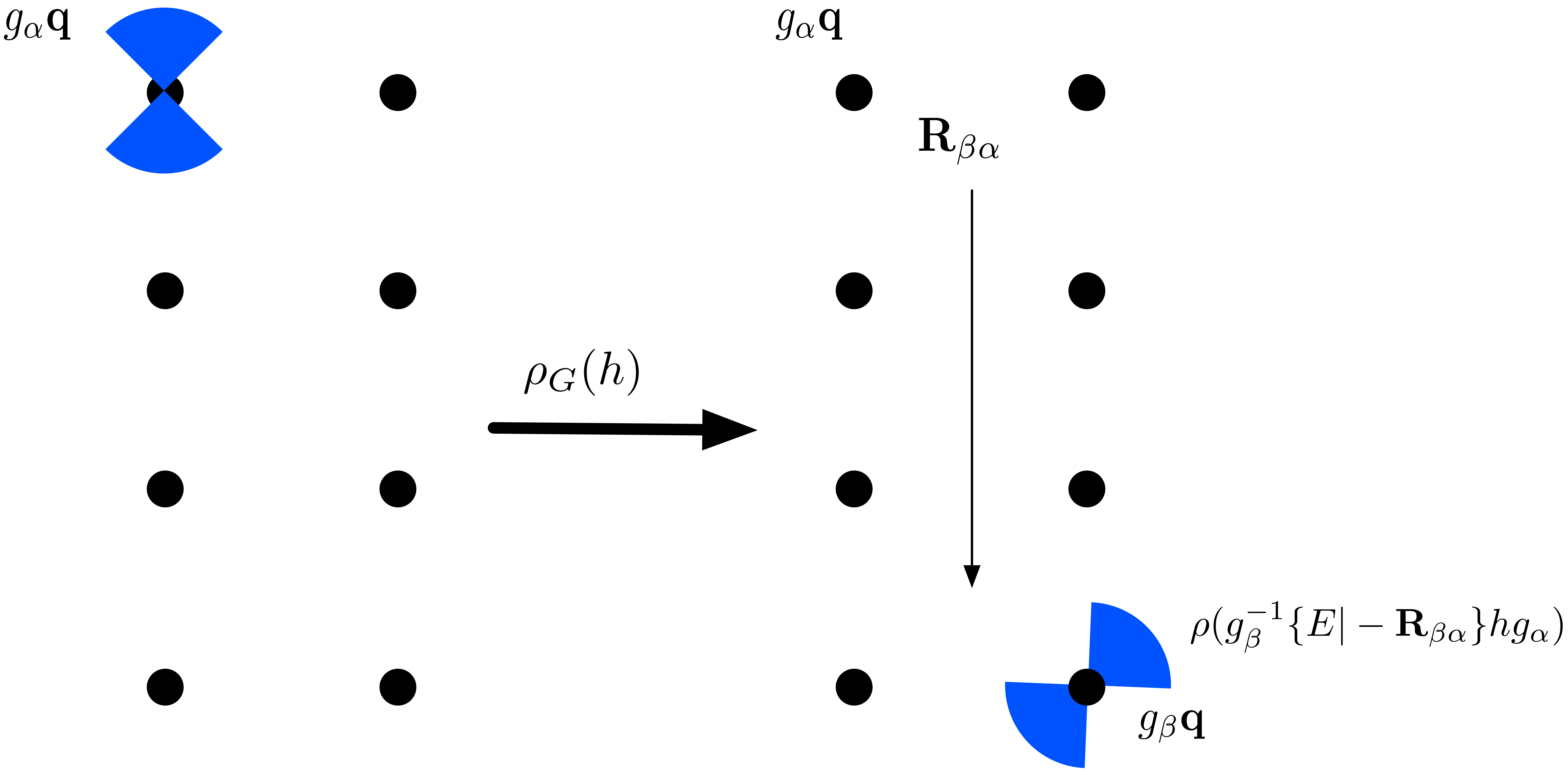
At (0,y): PG m
Same at (0,-y)

3. WP 4i{(x,y)+(-x,-y)+(-x,y)+(x,-y)}

At each position e.g. (x,y): trivial PG

Only trivial A orbital

Wannier Functions and Band Representations



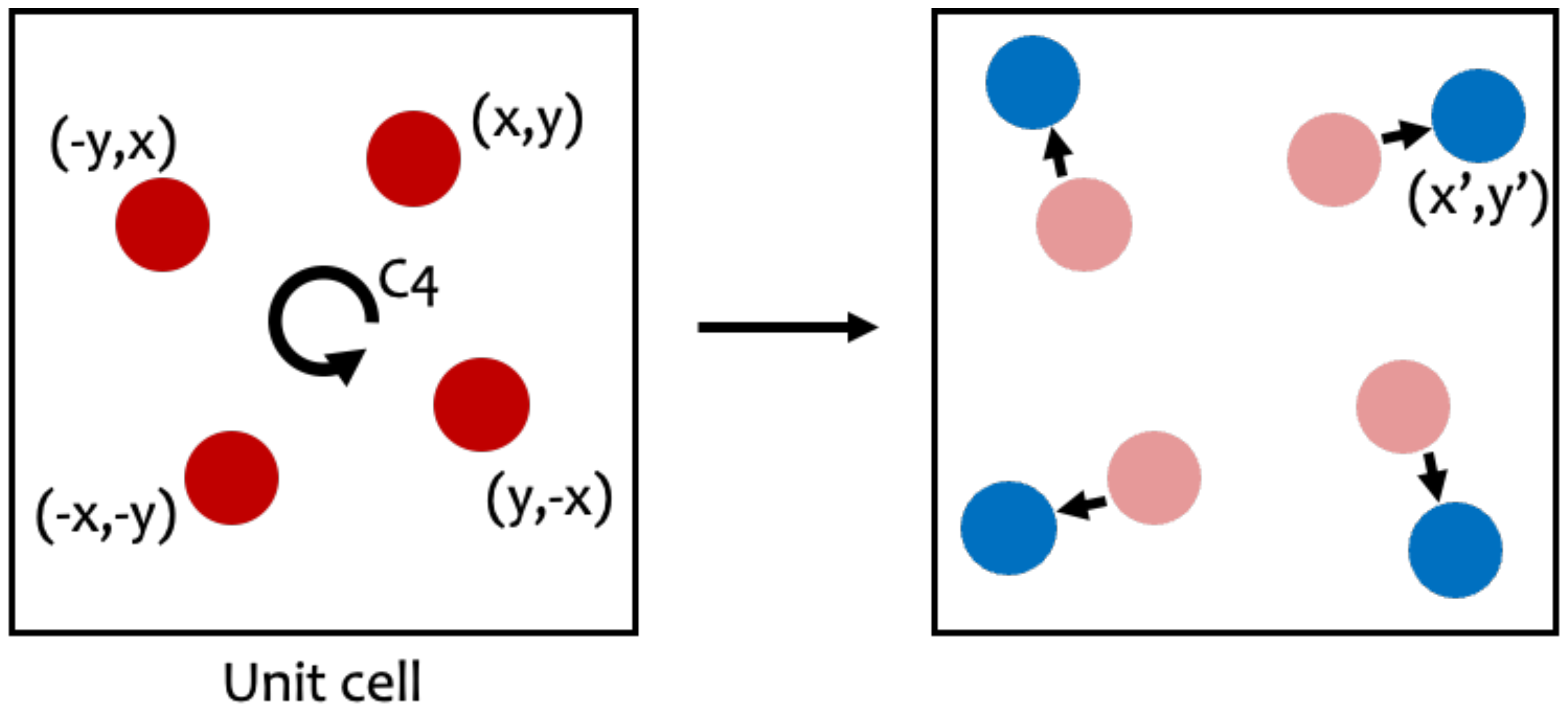
Wannier Functions and Topology

- **Trivial band Wannier functions characterized by orbital multiplicity vector**

$$\mathbf{m} = \left[\mathrm{m}\left(\mathrm{orb}_{1,\mathrm{WP}_1}\right), \mathrm{m}\left(\mathrm{orb}_{2,\mathrm{WP}_1}\right), \ldots, \mathrm{m}\left(\mathrm{orb}_{\mathrm{M},\mathrm{WP}_\mathrm{N}}\right)\right]^\mathrm{T}$$

- **Induction: $\mathrm{v} = B\mathbf{m}$. B is the *band representation matrix***

- **Overcomplete: m is not a topological invariant, can be changed by a choice of gauge. Corresponds to "moving" Wannier orbitals without breaking symmetry**



Wannier Functions and Topology

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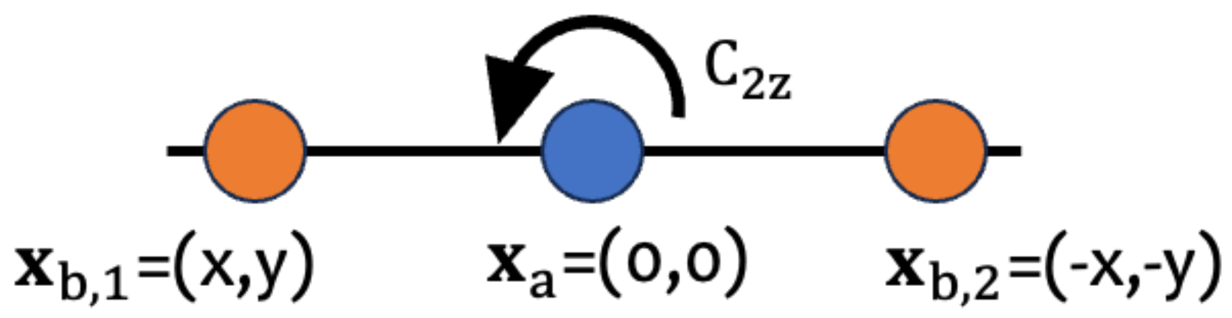
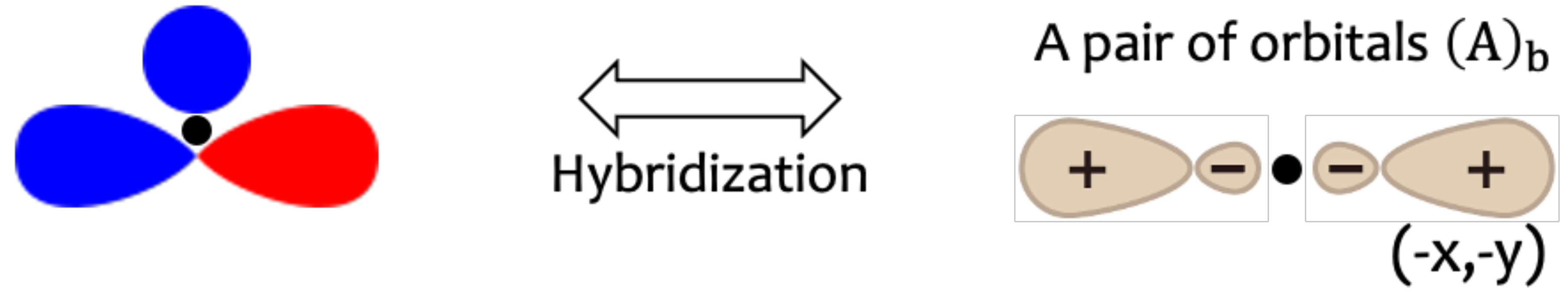
$$(\mathrm{A})_\mathrm{a} + (\mathrm{B})_\mathrm{a} \Leftrightarrow (\mathrm{A})_\mathrm{b}$$

$\mathrm{m}[(\mathrm{A})_\mathrm{a}] = 1, \mathrm{m}[(\mathrm{B})_\mathrm{a}] = 1$ $\qquad$ $\mathrm{m}[(\mathrm{A})_\mathrm{a}] = 0, \mathrm{m}[(\mathrm{B})_\mathrm{a}] = 0$





Adiabatic Processes

- **For every space group, we can consider the collection of these "adiabatic processes" into a matrix q**

- **Columns of q:** $(\mathrm{A})_\mathrm{a} + (\mathrm{B})_\mathrm{a} \Leftrightarrow (\mathrm{A})_\mathrm{b}$ **->** $(1,1,-1)$

- **Can use Smith Decomposition to derive a set of adiabatic invariants:**

$$q = L \cdot \Lambda \cdot R$$

$$\mathrm{diag}(\Lambda) = (\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_M, \ldots, \lambda_M, 0, \ldots, 0)$$

- **Since R and L are unimodular matrices, under any adiabatic process**

$$(L^{-1} \cdot \Delta m)_i = \begin{cases} 0 \mod \Lambda_{ii} & \text{for } i \leq \mathrm{rank}(q) \\ 0 & \text{for } i > \mathrm{rank}(q). \end{cases}$$

Stable Real Space Invariants - Construction

- This means that L^{-1} defines a set of invariants that do not change under any adiabatic process

$$\tilde{\Theta}^{(\lambda_1)} = (L^{-1})_{1,\ldots,n_{\lambda_1},*} \mod \lambda_1,$$
$$\tilde{\Theta}^{(\lambda_2)} = (L^{-1})_{n_{\lambda_1}+1,\ldots,n_{\lambda_1}+n_{\lambda_2},*} \mod \lambda_2, \quad \ldots$$
$$\tilde{\Theta}^{(\lambda_M)} = (L^{-1})_{n_{\lambda_1}+\cdots+n_{\lambda_{M-1}}+1,\ldots,N^{\rho}_{\mathrm{UC}}-n_0,*} \mod \lambda_M,$$
$$\tilde{\Theta}^{(0)} = (L^{-1})_{N^{\rho}_{\mathrm{UC}}-n_0+1,\ldots,N^{\rho}_{\mathrm{UC}},*},$$

- These are the stable real space invariants (stable RSIs). Tabulated for all space groups

Stable Real Space Invariants - Meaning

- **Theorem: Two sets of trivial bands $\mathbf{m}_1$ and $\mathbf{m}_1$ have the same stable RSIs, i.e.**

$$(L^{-1}\cdot(\mathbf{m}_1-\mathbf{m}_2))_i = \begin{cases} 0 \mod \Lambda_{ii} & \text{for } i \leq \text{rank}(q) \\ 0 & \text{for } i > \text{rank}(q). \end{cases}$$

if and only if there exists some auxiliary trivial bands $\mathbf{m}_{\text{aux}}$ such that $\mathbf{m}_1 + \mathbf{m}_{\text{aux}}$ and $\mathbf{m}_2 + \mathbf{m}_{\text{aux}}$ can be adiabatically deformed into each other -> stable equivalence

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Two phases AI1 and AI2 may not be deformable (topologically distinct)

(AI1 + Aux) and (AI2 + Aux) are equivalent for certain additional orbitals (trivial bands) Aux

AI1 $\mathbf{m}_1$ $(\boldsymbol{\theta}_{\mathrm{Z}},\boldsymbol{\theta}_{\mathrm{Z_n}})$ $\not\Leftrightarrow$ AI2 $\mathbf{m}_2$ $(\boldsymbol{\theta}_{\mathrm{Z}},\boldsymbol{\theta}_{\mathrm{Z_n}})$

AI1+Aux $\mathbf{m}_1 + \mathbf{m}_{\mathrm{aux}}$ $(\boldsymbol{\theta}_{\mathrm{Z}},\boldsymbol{\theta}_{\mathrm{Z_n}})+(\boldsymbol{\theta}_{\mathrm{Z,aux}},\boldsymbol{\theta}_{\mathrm{Z_n,aux}})$ $\Leftrightarrow$ AI2+Aux $\mathbf{m}_2 + \mathbf{m}_{\mathrm{aux}}$ $(\boldsymbol{\theta}_{\mathrm{Z}},\boldsymbol{\theta}_{\mathrm{Z_n}})+(\boldsymbol{\theta}_{\mathrm{Z,aux}},\boldsymbol{\theta}_{\mathrm{Z_n,aux}})$

*BR multiplicity $\mathbf{m}$ determines stable RSIs ($\boldsymbol{\theta}_{\mathrm{Z}}$, $\boldsymbol{\theta}_{\mathrm{Z_n}}$) [many-to-one]

Deformable with extra orbitals!

Stable Real Space Invariants - Meaning

- **Integer valued stable RSIs uniquely determine the symmetry data vector, and vice versa:** $\mathrm{v} = \mathscr{M}\theta_{\mathbb{Z}}$

- **mod n stable RSIs contain info beyond momentum space data**

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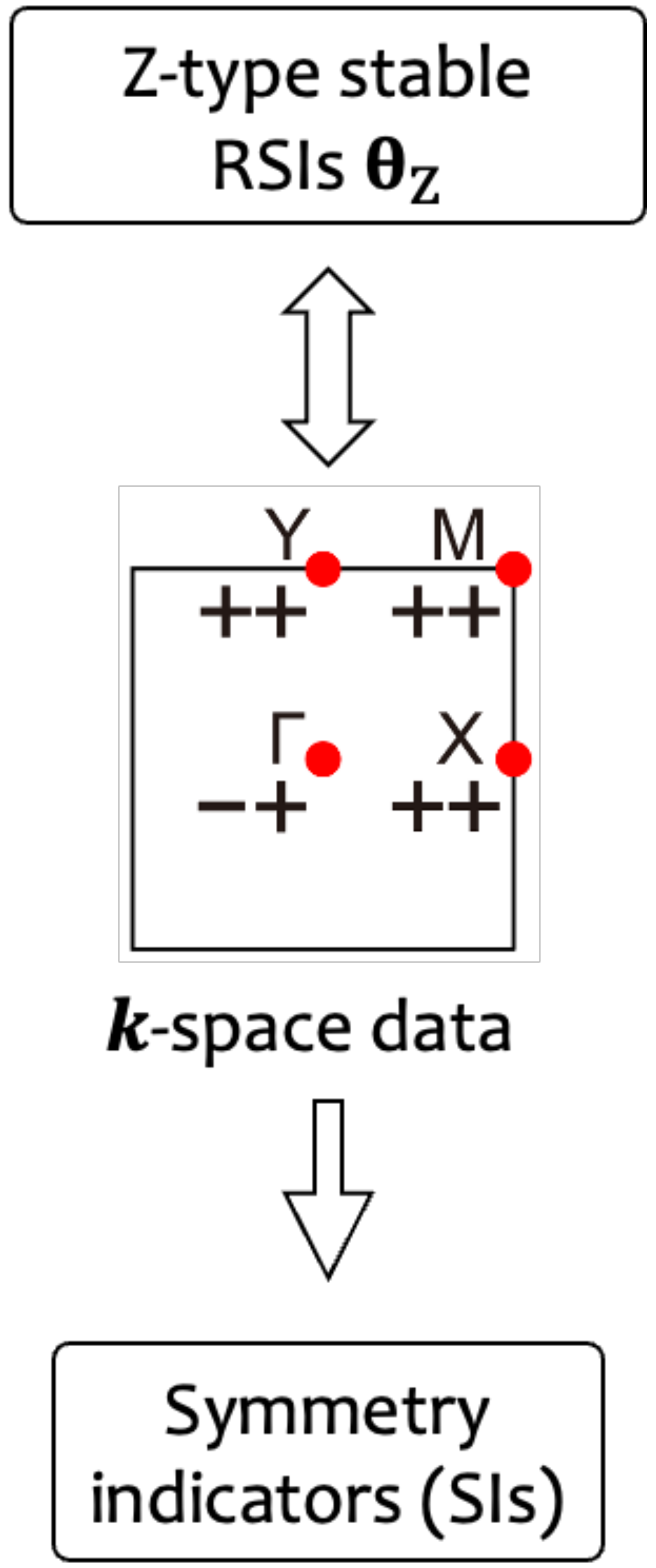


Atomic insulator (AI)

$\mathbf{m}_{\mathrm{AI}}$: well-defined BR/orbital

$\boldsymbol{k}$-space data : Well-defined

$(\boldsymbol{\theta}_Z, \boldsymbol{\theta}_{Z_n})$: Integer valued (Z or Zn)

Zero SI

Topological insulator (TI)

$\mathbf{m}_{\mathrm{TI}}$: NOT defined (by definition)

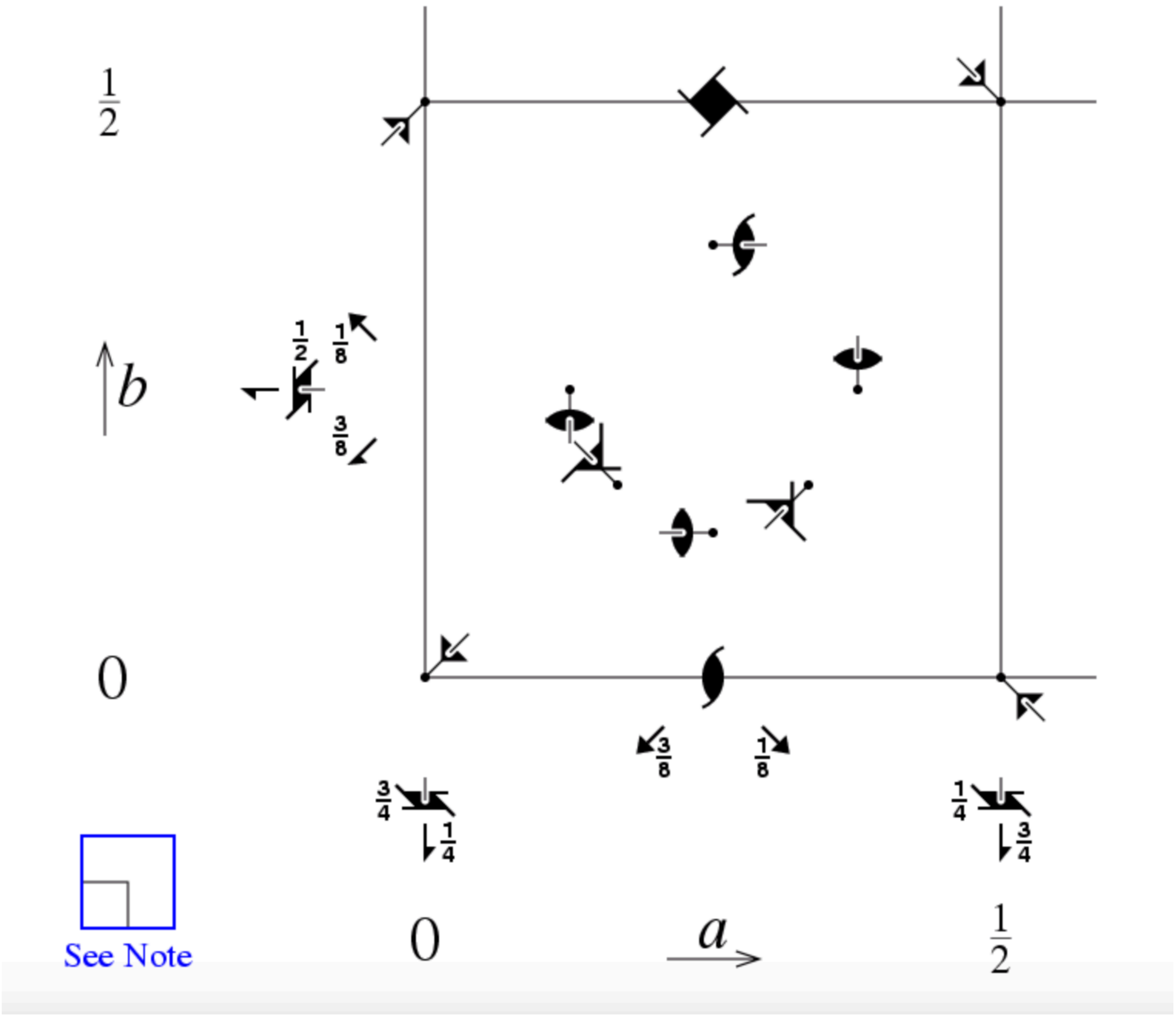
$\boldsymbol{k}$-space data : Still well-defined

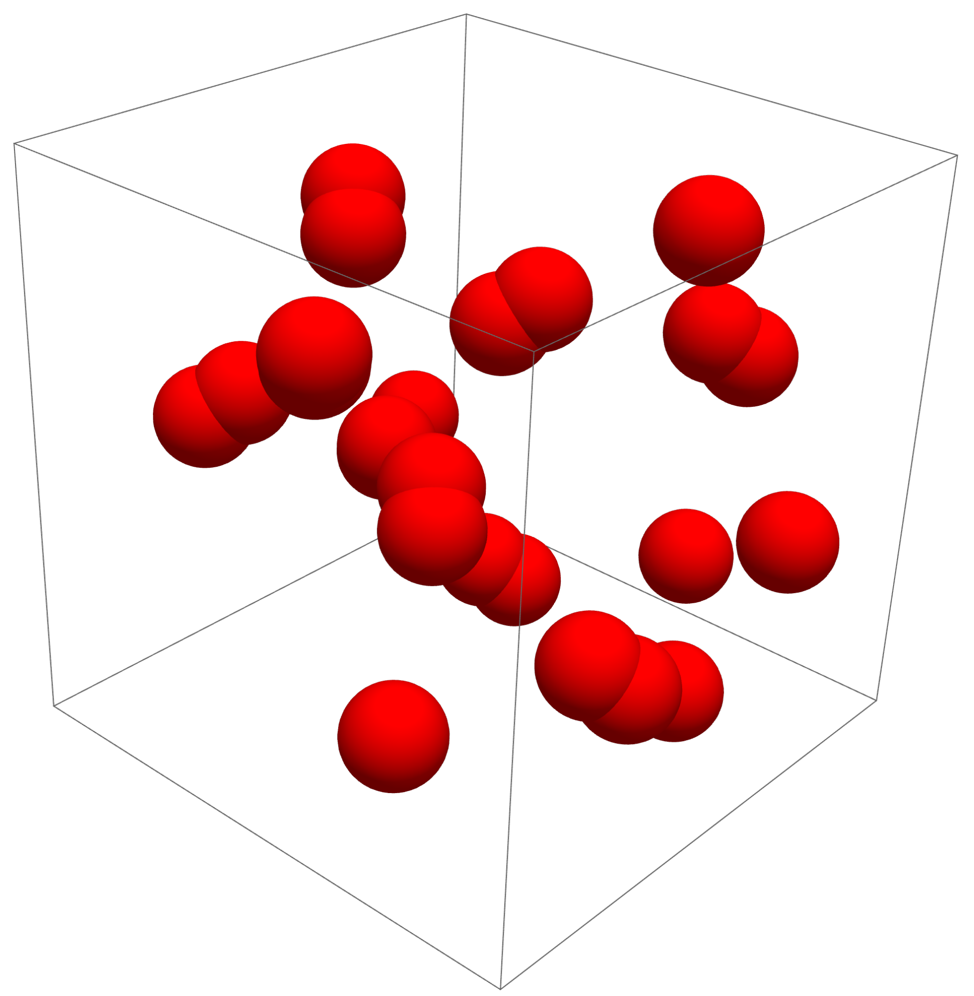
$\boldsymbol{\theta}_Z$: Determined from $\boldsymbol{k}$-space data

Fractional $\boldsymbol{\theta}_Z \Leftrightarrow$ Nonzero SI

Example: Space Group 212

- **Generators: Cubic 3fold rotation, fourfold screw rotation, twofold screw rotation**

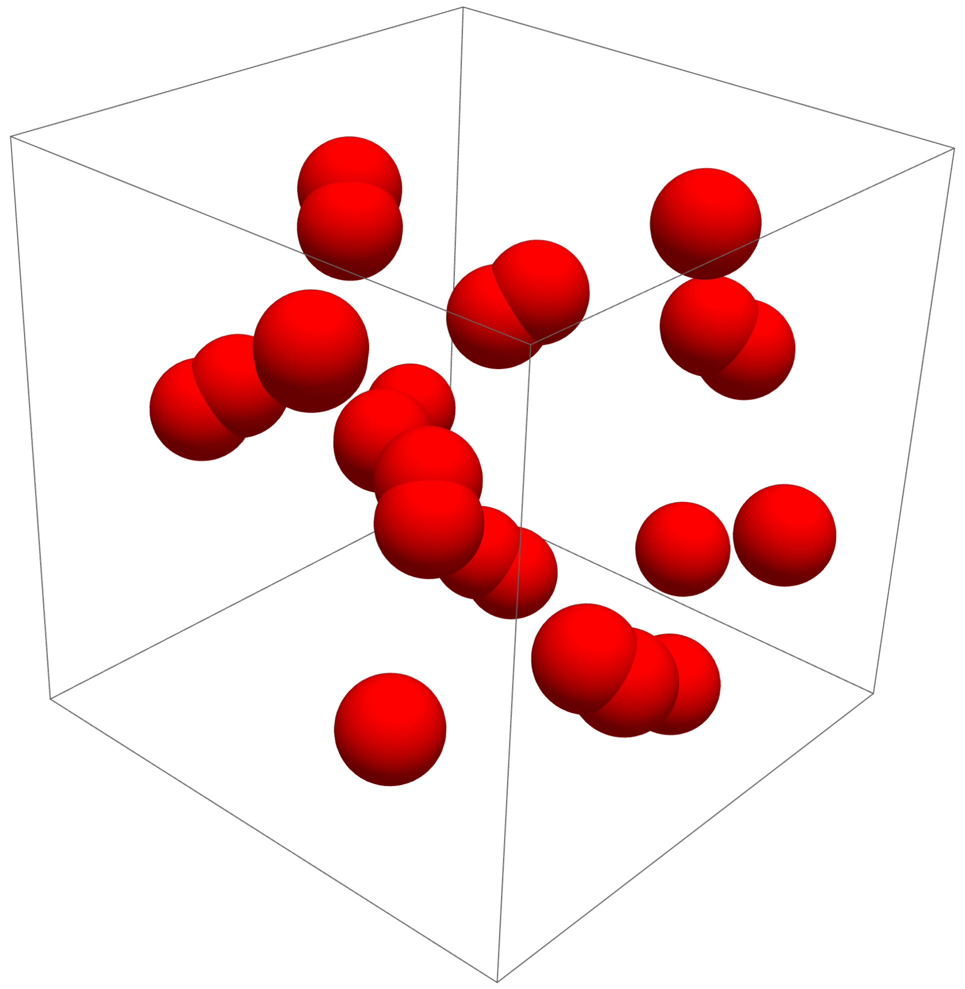




Example: Space Group 212

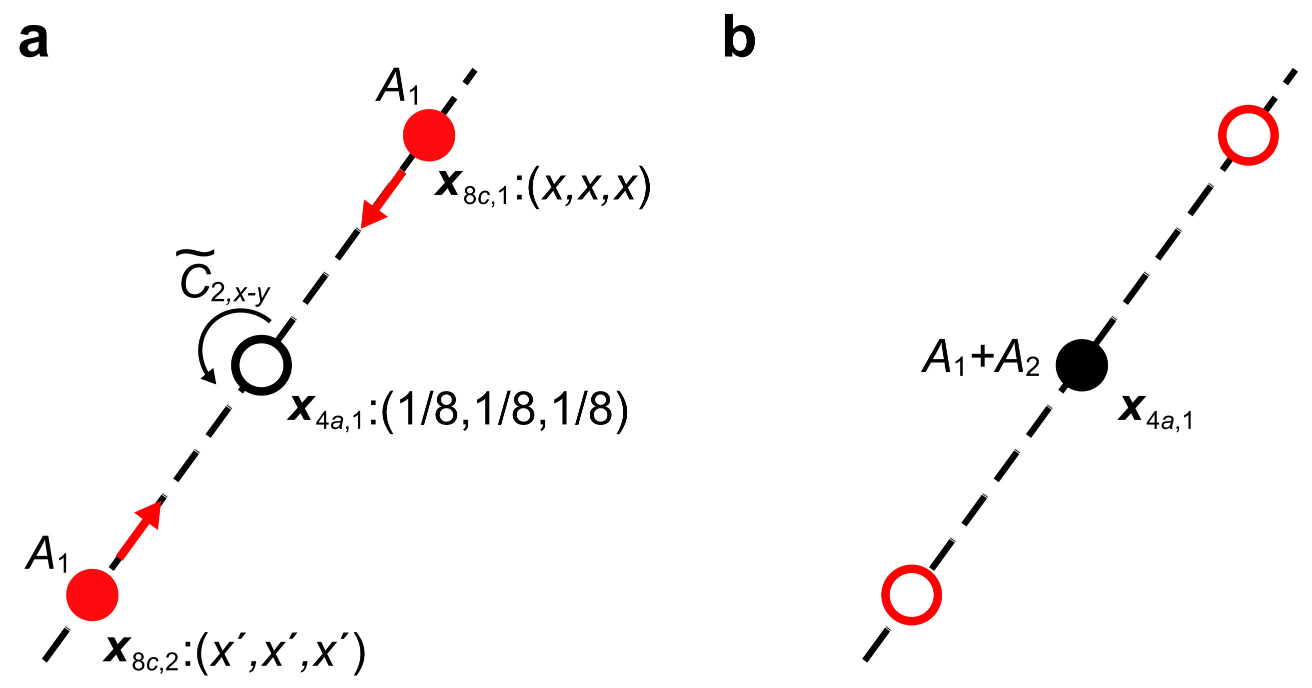
- **Generators: Cubic 3fold rotation, fourfold screw rotation, twofold screw rotation**

WP W	G_W	$(\rho)_W$
$4a(1/8,1/8,1/8)$	$C_{3,111}, \widetilde{C}_{2,x-y}$	$(A_1)_a, (A_2)_a, (E)_a$
$4b(5/8,5/8,5/8)$	$C_{3,111}, \widetilde{C}'_{2,x-y}$	$(A_1)_b, (A_2)_b, (E)_b$
$8c(x,x,x)$	$C_{3,111}$	$(A_1)_c, (^1E^2E)_c$
$12d(y,1/4-y,1/8)$	$\widetilde{C}_{2,x-y}$	$(A)_d, (B)_d$
$24e(x,y,z)$	$\{E\|\mathbf{0}\}$	$(A)_e$



- **Adiabatic processes**

$$\begin{aligned}
&(A_1+A_2)_a \Leftrightarrow (A_1)_c, \quad 2(E)_a \Leftrightarrow (^1E^2E)_c,\\
&(A_1+A_2)_b \Leftrightarrow (A_1)_c, \quad 2(E)_b \Leftrightarrow (^1E^2E)_c,\\
&(A_1+E)_a \Leftrightarrow (A)_d, \quad (A_2+E)_a \Leftrightarrow (B)_d,\\
&(A_1+E)_b \Leftrightarrow (A)_d, \quad (A_2+E)_b \Leftrightarrow (B)_d,\\
&(A+{}^1E^2E)_c \Leftrightarrow (A)_e, \quad (A+B)_d \Leftrightarrow (A)_e,
\end{aligned}$$



Example: Space Group 212

$$
q_{\text{adia}} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}
\begin{array}{r}
(A_1)_a \\
(A_2)_a \\
(E)_a \\
(A_1)_b \\
(A_2)_b \\
(E)_b \\
(A_1)_c \\
({}^1E\,{}^2E)_c \\
(A)_d \\
(B)_d \\
(A)_e
\end{array}
$$

Example: Space Group 212

$$
L = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 1 & -2 & -1 & -1 & 1 \\
-1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

$$
\operatorname{diag} \Lambda = (1, 1, 1, 1, 1, 1, 1, 2, 0, 0),
$$

$$
R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & -3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1
\end{pmatrix}.
$$

Example: Space Group 212

$$(L^{-1})_{8,*} \quad \mathrm{mod}\ 2 = (1,1,1,0,0,\ldots,0),$$

$$\begin{pmatrix}(L^{-1})_{9,*}\\(L^{-1})_{10,*}\\(L^{-1})_{11,*}\end{pmatrix} = \begin{pmatrix}1&0&0&1&0&0&1&0&1&0&1\\0&1&0&0&1&0&1&0&0&1&1\\0&0&1&0&0&1&0&2&1&1&2\end{pmatrix}.$$

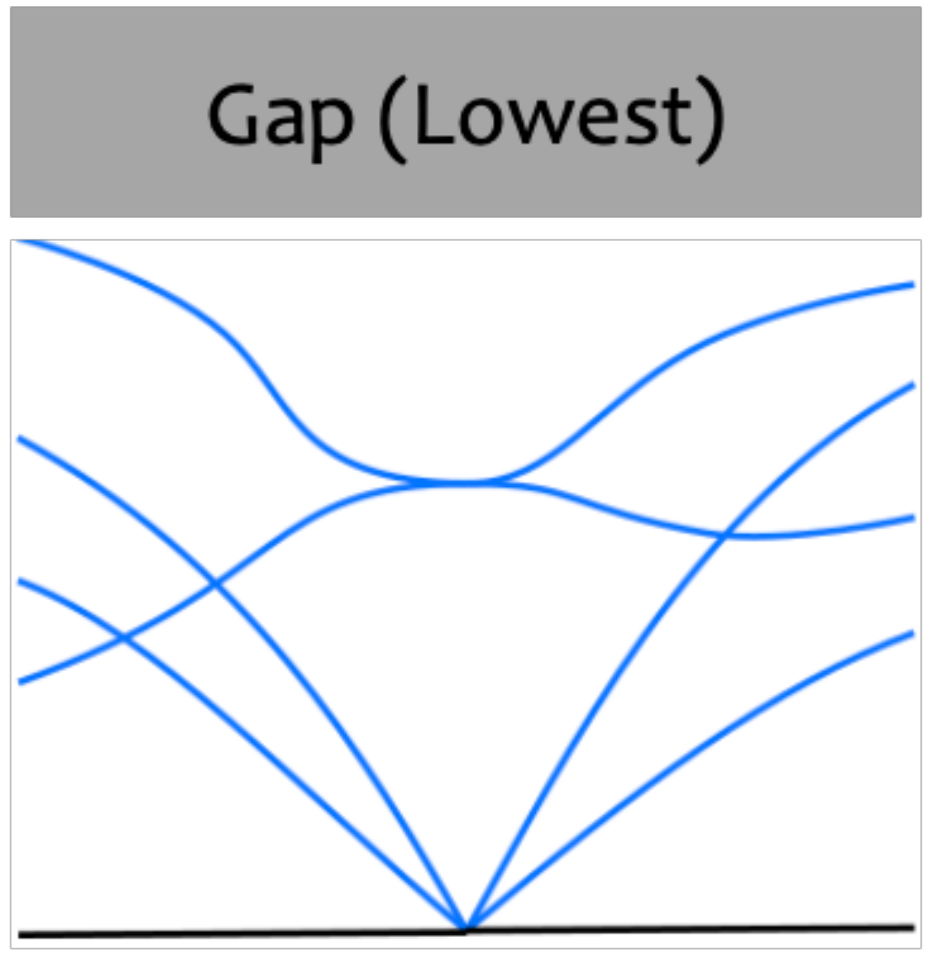
- **Three integer invariants, and one mod 2 invariant**

$$\begin{aligned}
\theta_1 =& m[(A_1)_a] + m[(A_1)_b] + m[(A_1)_c] \\
&+ m[(A)_d] + m[(A)_e], \\
\theta_2 =& m[(A_2)_a] + m[(A_2)_b] + m[(A_1)_c] \\
&+ m[(B)_d] + m[(A)_e], \\
\theta_3 =& m[(E)_a] + m[(E)_b] + 2m[({}^1E^2E)_c] \\
&+ m[(A)_d] + m[(B)_d] + 2m[(A)_e], \\
\theta_4^{(2)} =& m[(A_1)_a] + m[(A_2)_a] + m[(E)_a] \quad (\mathrm{mod}\ 2).
\end{aligned}$$

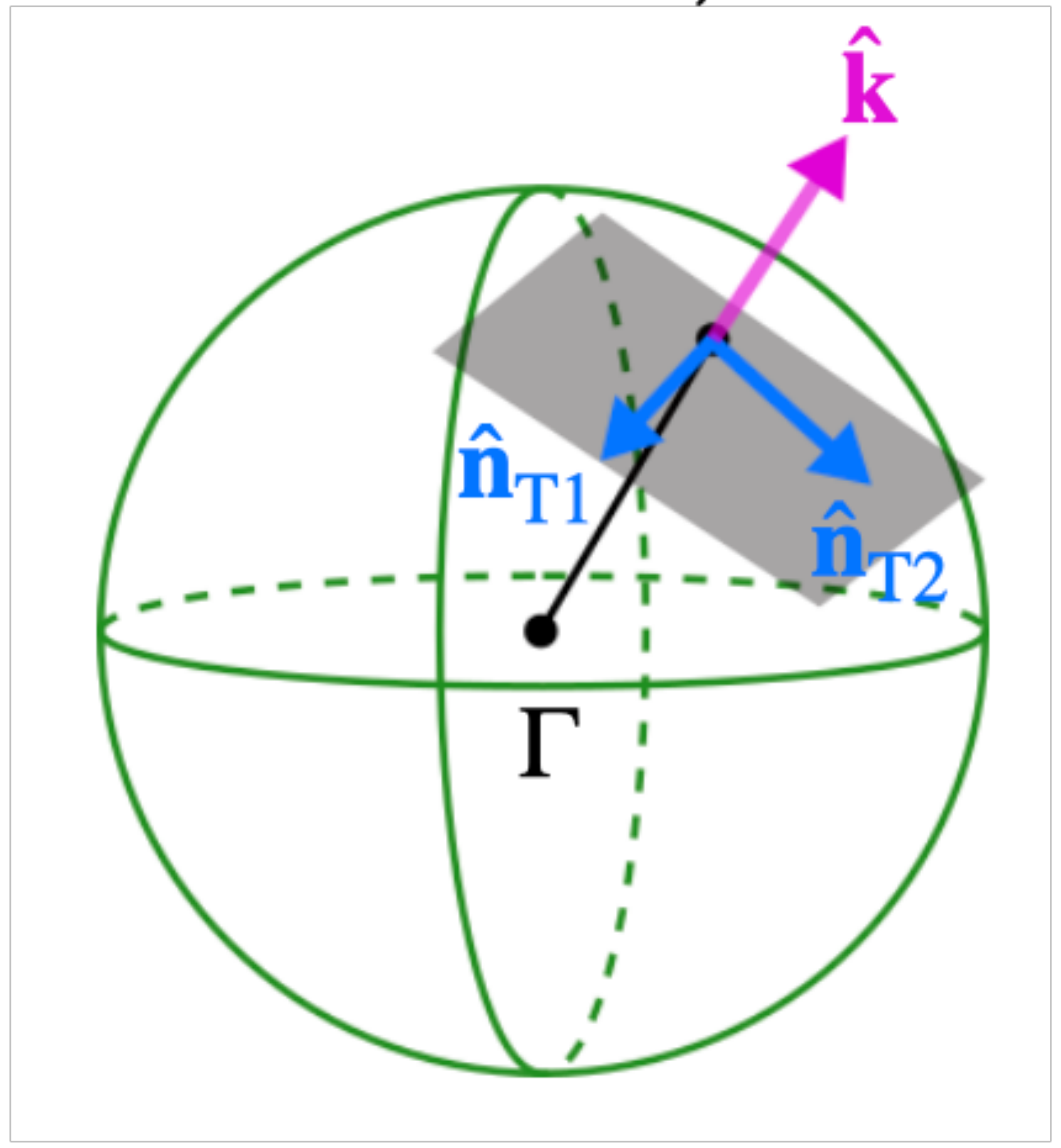
Photonics- The Lowest Bands

- **The lowest photonic bands are pathological**

Two **transverse (T)** modes w/ linear dispersion



Transverse **polarization vectors** $\mathbf{n}_{T1,T2}(\mathbf{k})$



$\mathbf{D}_{\mathbf{k}}^{T1}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{n}}_{T1}$ $\quad$ $\mathbf{D}_{\mathbf{k}}^{T2}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{n}}_{T2}$

2T modes

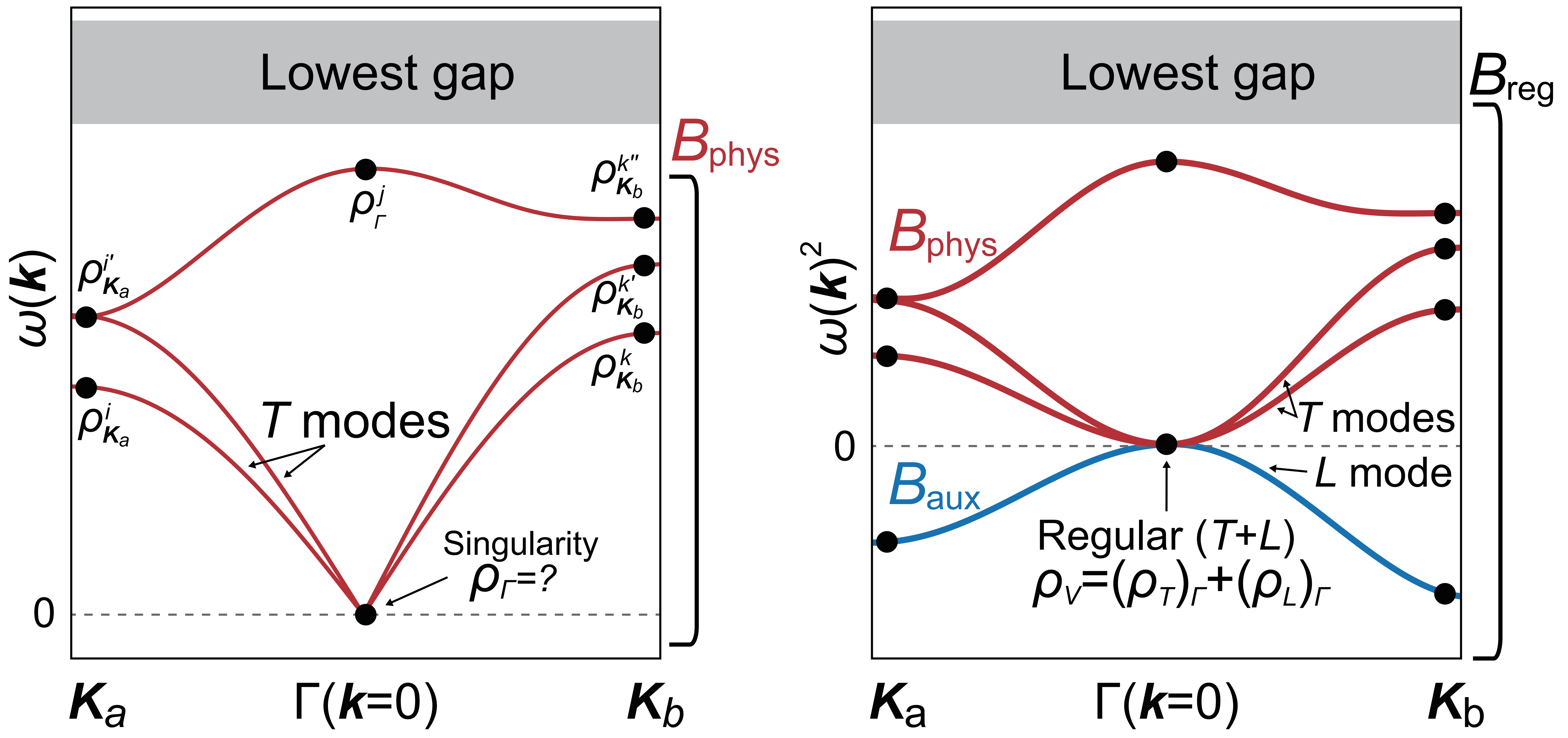
$\mathbf{D}_{\mathbf{k}}^{L}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{k}}$

1L mode

L mode forbidden by transversality constraint $\nabla \cdot \mathbf{D} = 0$ i.e. $\mathbf{k} \cdot \mathbf{D}_{n\mathbf{k}} = 0$

Photonics- The Lowest Bands

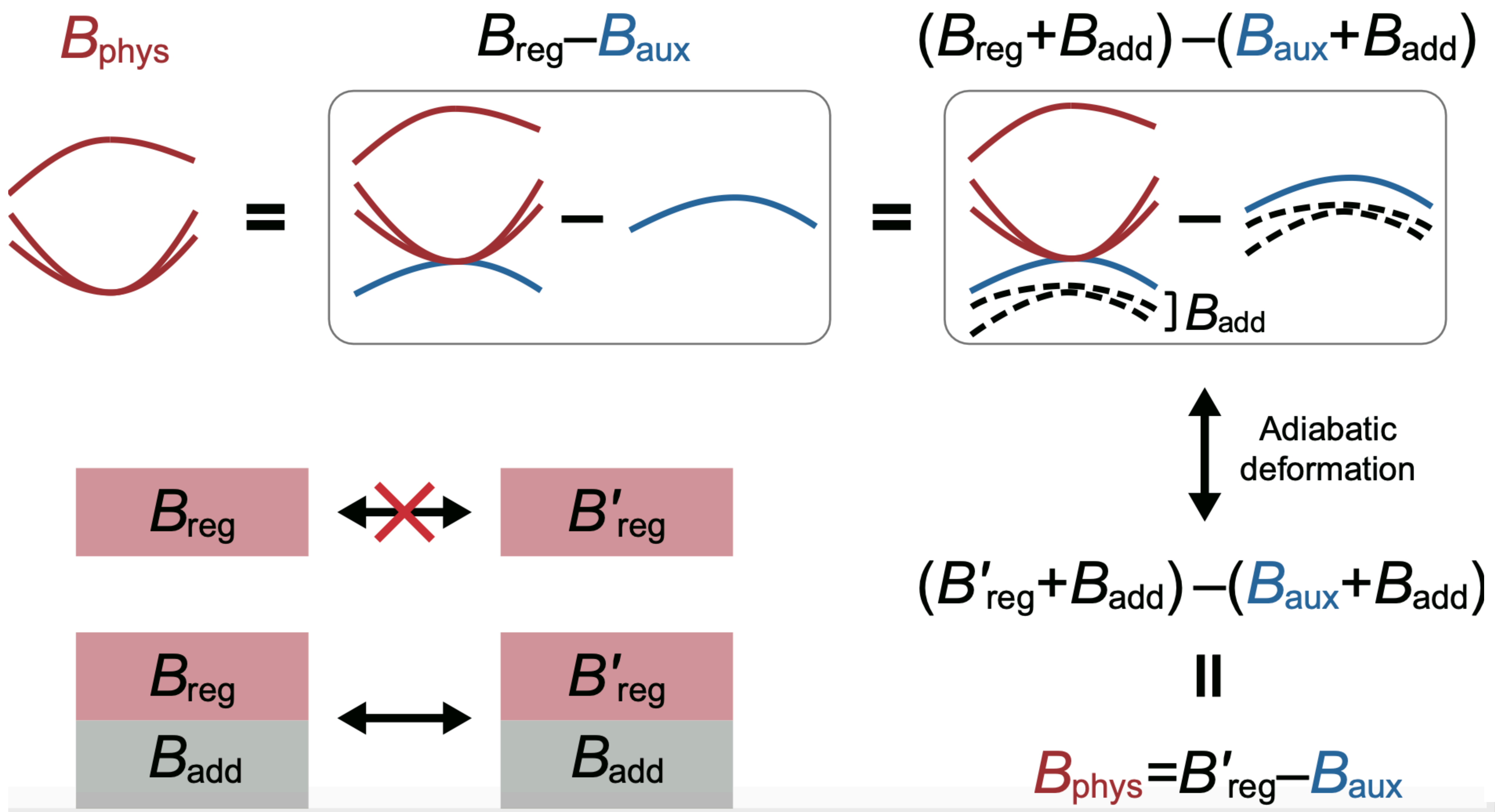
- **We can cure the pathology by adding back in ficticious longitudinal bands as a mathematical trick, as long as we can subtract them later**



Stable Equivalence and Photonic Bands

- Useful for modeling - can algorithmically search for the minimal auxiliary bands needed for a faithful tight binding model

- But what about classification? If we add even more auxiliary bands, nothing should change about the physics



Stable Equivalence and Photonic Bands

- **This means photonic bands are classified by stable equivalence classes of pairs** $(\mathscr{B}_{\text{reg}}, \mathscr{B}_{\text{aux}})$
- **These are classified by stable real space invariants!**
- **Transversality constraint: inequalities on the invariants that must be satisfied for any topologically trivial photonic band structure**

$$\mathcal{M}\theta_{\mathbb{Z}} \geq \mathbf{b}_{\text{ph}}$$

- **Can be solved via linear programming**

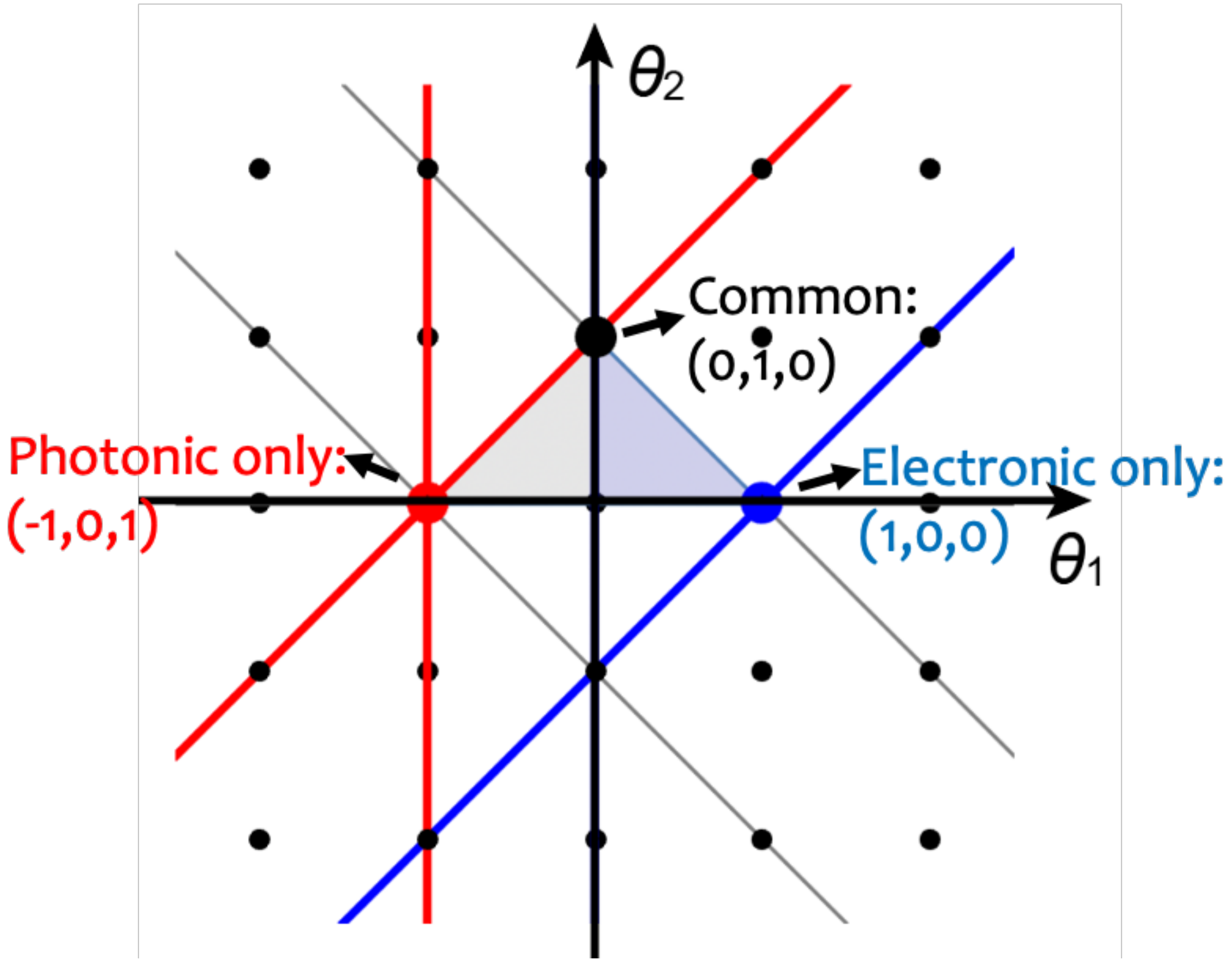
Stable Equivalence and Photonic Bands

- **Key point: This allows us to isolate "interesting" topology of the lowest band from the Euler characteristic of the polarization vector**

- $\mathbf{b}_{\mathrm{ph}} \neq \mathbf{b}_{\mathrm{electron}}$ **implies photonic systems can realize trivial band structures that cannot appear in condensed matter**

- **Allows us to model band structures in a way that captures topological properties of Wilson loops**

Example: Space Group 212



Photonic bands [$(\blacksquare)_T = \Gamma_4 - \Gamma_1$]

θ_Z=(0,1,0): $((\blacksquare)_T + \Gamma_1 + \Gamma_2, R_3, M_2M_3 + M_5, X_1 + X_2)$

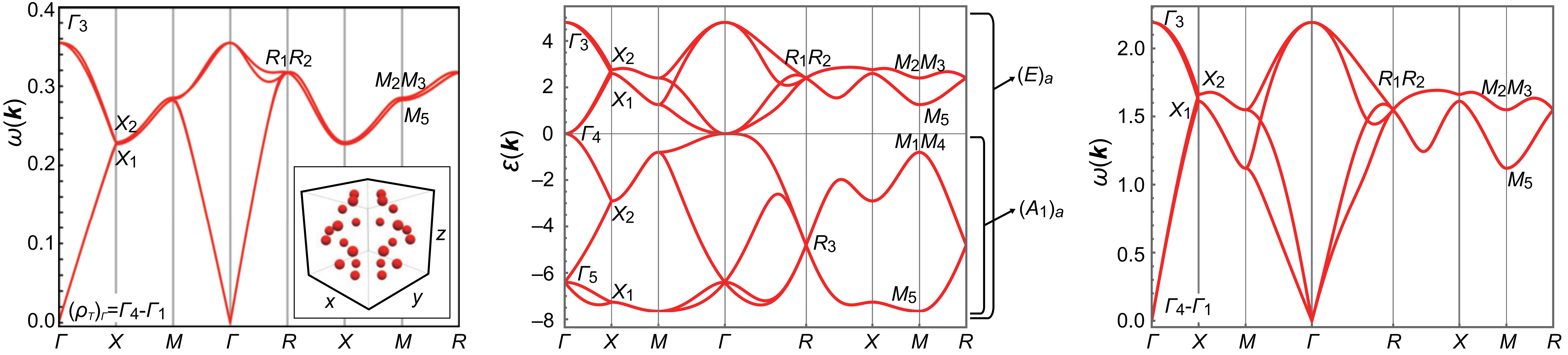
θ_Z=(-1,0,1): $((\blacksquare)_T + \Gamma_3, R_1R_2, M_2M_3 + M_5, X_1 + X_2),$

Electronic bands

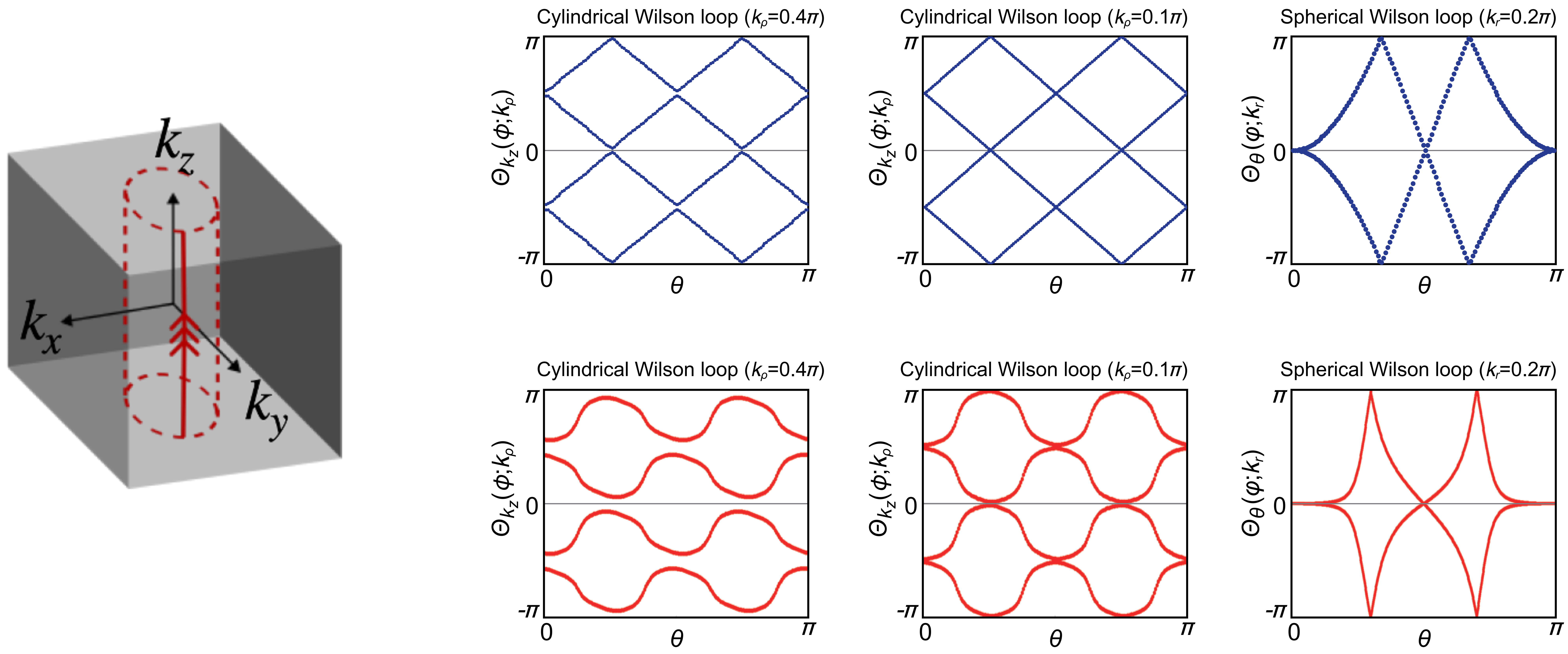
θ_Z=(0,1,0): $(\Gamma_2 + \Gamma_4, R_3, M_2M_3 + M_5, X_1 + X_2)$

θ_Z=(1,0,0): $(\Gamma_1 + \Gamma_5, R_3, M_1M_4 + M_5, X_1 + X_2)$

Example: Space Group 212



Example: Space Group 212



Conclusion

- **Stable real space invariants characterize band topology beyond symmetry indicators**

- **They also give the natural classification of topologically trivial photonic bands**

- **Complete the extension of the theory of topological quantum chemistry to photonics**

- **References:**

Devescovi et al., Optical Materials Express 14, 2161-2177 (2024).

Hwang et al., to appear 2025 (x2)