CONVEXIFICATION METHOD IN INVERSE PROBLEMS

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FOUR TOPICS:

1. Convexification for a Coefficient Inverse Problem (CIP) for the radiative transport equation
2. Convexification for the Retrospective Forward Problem of Mean Field Games
3. Convexification for a CIP of Mean Field Games
4. Convexification for the 3D problem of Travel Time Tomography (=Inverse Kinematic Problem of Seismic)

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Past results on numerical methods for the radiative transport equation are only for inverse source problems, which are linear:
1. H. Fujiwara, K. Sadiq, and A. Tamasan,
2. A. V. Smirnov, M. V. Klibanov, and L. H. Nguyen.
However, CIPs are nonlinear.

- I am unaware about past numerical methods for the above listed problems
- I am also unaware about past rigorously justified globally convergent numerical methods for problems of Mean Field Games.

A FEW WORDS ABOUT THE CONVEXIFICATION METHOD

- Currently the convexification is the single globally convergent numerical method for CIPs without an overdetermination.
- The vast majority of numerical methods for nonlinear Ill-Posed Problems, including CIPs, is based on the least squares minimization. This is a truly powerful method!
- Still, it has a drawback (so as everything in life): corresponding cost functionals are plagued by the phenomenon of local minima and ravines.

- The goal of the convexification concept is to improve the least square minimization via fixing this drawback.
- The convexification was introduced in 1997 by Klibanov in order to avoid that phenomenon. Active studies of the convexification have started in 2017.
- The convexification works for many nonlinear ill-posed problems for PDEs as well as for many CIPs.

HOW THE CONVEXIFICATION WORKS:

- The convexification constructs a weighted Tikhonov-like functional J, which is globally strongly convex on an appropriate convex bounded set $B\left(d\right)\subset H$, where $d>0$ is an arbitrary but fixed diameter of $B\left(d\right)$ and H is the Hilbert space of solutions of the corresponding PDE.

- The weight is the Carleman Weight Function for the corresponding PDE operator. CWF is used as the weight in the Carleman estimate for this operator.
- Convergence to the true solution of the gradient descent method of the minimization of J is established if starting from an arbitrary point of $B(d)$, provided that the noise in the data tends to zero. Rates of convergence are written explicitly.
- Since no restrictions are imposed on the diameter d, then this is global convergence.

Convexification for a Coefficient Inverse Problem for the Radiative Transport Equation

For $n \geq 1$, points in $\mathbb{R}^{n+1}$ are denoted below as $\mathbf{x} = (x_1, x_2, ..., x_n, y) \in \mathbb{R}^{n+1}$. Let numbers $A, a, b, d > 0$, where

$$1 < a < b. \tag{1}$$

Define the rectangular prism $\Omega \subset \mathbb{R}^{n+1}$ and parts $\partial_1\Omega, \partial_2\Omega, \partial_3\Omega$ of its boundary $\partial\Omega$ as:

$$\Omega = \{\mathbf{x} : -A < x_1, ..., x_n < A, a < y < b\}, \tag{2}$$

$$\partial_1\Omega = \{\mathbf{x} : -A < x_1, ..., x_n < A, y = a\}, \tag{3}$$

$$\partial_2\Omega = \{\mathbf{x} : -A < x_1, ..., x_n < A, y = b\}, \tag{4}$$

$$\partial_3\Omega = \{x_i = \pm A, y \in (a, b), i = 1, ..., n, \}. \tag{5}$$

Let Γ_d be the line where the external sources are,

$$\Gamma_d = \{\mathbf{x}_\alpha = (\alpha, 0, ..., 0) : \alpha \in [-d, d]\}. \tag{6}$$

Hence, Γ_d is a part of the x_1-axis. It follows from (1) and (2) that $\Gamma_d \cap \overline{\Omega} = \varnothing$.
Let the points of external sources $\mathbf{x}_\alpha$ run along Γ_d, $\mathbf{x}_\alpha \in \Gamma_d$. Let $\varepsilon > 0$ be a sufficiently small number. To avoid dealing with singularities, we model the $\delta(\mathbf{x})-$function as:

$$f(\mathbf{x}) = C_\varepsilon \begin{cases} \exp\left(\frac{|\mathbf{x}|^2}{\varepsilon^2 - |\mathbf{x}|^2}\right), |\mathbf{x}| < \varepsilon, \\ 0, |\mathbf{x}| \geq \varepsilon \end{cases}, \tag{7}$$

where the constant C_ε is chosen such that

$$C_\varepsilon \int_{|\mathbf{x}|<\varepsilon} \exp\left(\frac{|\mathbf{x}|^2}{\varepsilon^2 - |\mathbf{x}|^2}\right) d\mathbf{x} = 1. \tag{8}$$

Hence, the function
$f\left(\mathbf{x}-\mathbf{x}_{\alpha}\right)=f\left(x_{1}-\alpha, x_{2}, \ldots, x_{n}, y\right) \in C^{\infty}\left(\mathbb{R}^{n+1}\right)$ plays the role of the source function for the source $\mathbf{x}_{\alpha}$. We choose ε so small that

$$f\left(\mathbf{x}-\mathbf{x}_{\alpha}\right)=0, \forall \mathbf{x} \in \overline{\Omega}, \forall \mathbf{x}_{\alpha} \in \Gamma_{d}. \tag{9}$$

Let $A_{1}=\max(A, d)$. Introduce three domains $G \subset \mathbb{R}^{n+1}$ and $G_{a}^{+}, G_{a}^{-} \subset G$,

$$\begin{aligned} G &= \left\{\mathbf{x}:\left|x_{1}\right|, \ldots,\left|x_{n}\right|<A_{1}, y \in(0, b)\right\}, \; G_{a}^{+}=G \cap\{y>a\}, \\ G_{a}^{-} &= G \diagdown G_{a}^{+}. \end{aligned} \tag{10}$$

By (2), (6) and (10) $\Omega \subset G_{a}^{+}$. Everywhere below

$$(\mathbf{x}, \alpha) \in G \times(-d, d). \tag{11}$$

Let $u(\mathbf{x}, \alpha)$ denotes the steady-state radiance at the point $\mathbf{x}$ generated by the source function $f\left(\mathbf{x}-\mathbf{x}_{\alpha}\right)$.

The stationary RTE:

$$\nu(\mathbf{x},\alpha)\cdot\nabla_{\mathbf{x}}u(\mathbf{x},\alpha)+a\left(\mathbf{x}\right)u(\mathbf{x},\alpha)$$
$$=\mu_s(\mathbf{x})\int_{\Gamma_d}K(\mathbf{x},\alpha,\beta)u(\ \mathbf{x},\beta)d\beta+f\left(\mathbf{x}-\mathbf{x}_\alpha\right),\ \mathbf{x}\in G,\mathbf{x}_\alpha\in\Gamma_d. \tag{12}$$

The kernel $K(\mathbf{x},\alpha,\beta)$ is called the "phase function",

$$\begin{gathered}K(\mathbf{x},\alpha,\beta)\geq 0,\mathbf{x}\in\overline{G};\ \alpha,\beta\in[-d,d],\\ K(\mathbf{x},\alpha,\beta)\in C^1\left(\overline{G}\times[-d,d]^2\right).\end{gathered} \tag{13}$$

In equation (12),

$$a\left(\mathbf{x}\right)=\mu_a\left(\mathbf{x}\right)+\mu_s(\mathbf{x}), \tag{14}$$

where $\mu_a\left(\mathbf{x}\right)$ and $\mu_s(\mathbf{x})$ are the absorption and scattering coefficients respectively and $a\left(\mathbf{x}\right)$ is the emission coefficient.

We assume that

$$\mu_a(\mathbf{x}), \mu_s(\mathbf{x}) \geq 0, \mu_a(\mathbf{x}) = \mu_s(\mathbf{x}) = 0, \ \mathbf{x} \in \ G \setminus \Omega,$$
$$\mu_a(\mathbf{x}), \mu_s(\mathbf{x}) \in C^1\left(\overline{G}\right).$$

For two arbitrary points $\mathbf{x}, \mathbf{z} \in \mathbb{R}^{n+1}$ let $L(\mathbf{x}, \mathbf{z})$ be the line segment connecting these points and let ds be the element of the euclidean length on $L(\mathbf{x}, \mathbf{z})$. In (12) $\nu(\mathbf{x},\alpha)$ denotes the unit vector, which is parallel to $L(\mathbf{x}, \mathbf{x}_\alpha)$,

$$\nu(\mathbf{x},\alpha) = \frac{\mathbf{x} - \mathbf{x}_\alpha}{|\mathbf{x} - \mathbf{x}_\alpha|}. \tag{15}$$

Forward Problem. *Let (1)-(15) hold. Find the function* $u(\mathbf{x},\alpha) \in C^1\left(\overline{G} \times [-d,d]\right)$ *satisfying equation (12) and the initial condition*

$$u(\mathbf{x}_\alpha,\alpha) = 0 \text{ for } \mathbf{x}_\alpha \in \Gamma_d. \tag{16}$$

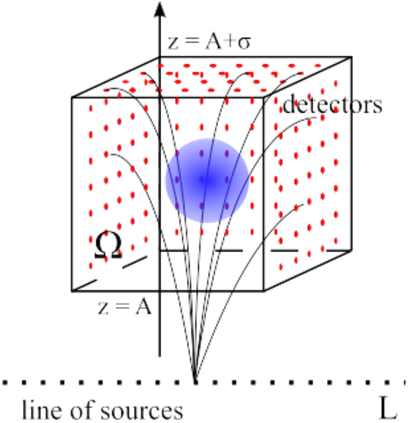

Figure 1: A schematic diagram of the source/detector configuration for CIP1.

Coefficient Inverse Problem 1 (CIP1). *Let (1)-(15) hold. Let the function* $u(\mathbf{x}, \alpha) \in C^1\left(\overline{G} \times [-d, d]\right)$ *be the solution of the Forward Problem. Assume that the coefficient* $a(\mathbf{x})$ *of equation (12) is unknown. Determine the function* $a(\mathbf{x})$, *assuming that the following function* $g(\mathbf{x}, \alpha)$ *is known:*

$$g(\mathbf{x}, \alpha) = u(\mathbf{x}, \alpha), \forall \mathbf{x} \in \partial\Omega \diagdown \partial_1\Omega, \forall \alpha \in (-d, d). \qquad (17)$$

Theorem 1 (existence and uniqueness of the forward problem). *Assume that (1), (2), (6)-(9) and (13)-(15) hold. Then there exists unique solution* $u(\mathbf{x},\alpha) \in C^1\left(\overline{G} \times [-d,d]\right)$ *of equation (12) with the initial condition (16). Furthermore, the following inequality holds:*

$$u(\mathbf{x},\alpha) \geq m > 0 \text{ for } (\mathbf{x},\alpha) \in \overline{G}_a^+ \times [-d,d], \tag{18}$$

$$m = \min_{\overline{G}_a^+ \times [-d,d]} \left[\exp\left(-\int_{L(\mathbf{x},\mathbf{x}_\alpha)} a(\mathbf{x}(s))\, ds \right) \cdot \left(\int_{L(\mathbf{x},\mathbf{x}_\alpha)} f(\mathbf{x}(s) - \mathbf{x}_\alpha)\, ds \right) \right],$$

where $L(\mathbf{x},\mathbf{x}_\alpha)$ *is the piece of the straight line connecting points* $\mathbf{x}$ *and* $\mathbf{x}_\alpha$.

THIS IS A NEW RESULT

Volterra Integral Equation of the second kind for $u(\mathbf{x}, \alpha)$ is used to prove this theorem and to solve the forward problem:

$$\begin{gathered}
u(\mathbf{x},\alpha) = u_0(\mathbf{x},\alpha) + \frac{1}{c(\mathbf{x},\alpha)} \int_{L(\mathbf{x},\mathbf{x}_\alpha)} c(\mathbf{x}(s), \alpha)\mu_s(\mathbf{x}(s)) \\
\left(\int_{\Gamma_d} K(\mathbf{x}(s), \alpha, \beta) u(\mathbf{x}(s), \beta) d\beta \right) ds, \\
u_0(\mathbf{x},\alpha) = \frac{1}{c(\mathbf{x},\alpha)} \int_{L(\mathbf{x},\mathbf{x}_\alpha)} f(\mathbf{x}(s) - \mathbf{x}_\alpha)\, ds, \\
c(\mathbf{x}, \alpha) = \exp\left(\int_{L(\mathbf{x},\mathbf{x}_\alpha)} a(\mathbf{x}(s)) ds \right).
\end{gathered} \tag{19}$$

Transformation

Change of variables:

$$w(\mathbf{x},\alpha) = \ln u(\mathbf{x},\alpha), \quad (\mathbf{x},\alpha) \in \overline{\Omega} \times [-d,d].$$

Then

$$\nu(\mathbf{x},\alpha) \cdot \nabla_{\mathbf{x}} w(\mathbf{x},\alpha) + a(\mathbf{x})$$
$$= e^{-w(\mathbf{x},\alpha)} \mu_s(\mathbf{x}) \int_{\Gamma_d} K(\mathbf{x},\alpha,\beta) e^{w(\mathbf{x},\beta)} d\beta, \mathbf{x} \in \Omega, \alpha \in (-d,d),$$
$$w(\mathbf{x},\alpha) \mid_{\partial\Omega} = \ln g_1(\mathbf{x},\alpha),$$
$$g_1(\mathbf{x},\alpha) = \begin{cases} g(\mathbf{x},\alpha), \mathbf{x} \in \partial\Omega \diagdown \partial_1\Omega, \alpha \in (-d,d), \\ u_0(\mathbf{x},\alpha), \mathbf{x} \in \partial_1\Omega, \alpha \in (-d,d). \end{cases}$$

Differentiating with respect to α and using $\partial_\alpha a(\mathbf{x}) \equiv 0$,

$$\begin{aligned}
&\nu(\mathbf{x},\alpha) \cdot \nabla_{\mathbf{x}} w_\alpha(\mathbf{x},\alpha) + \partial_\alpha \nu(\mathbf{x},\alpha) \cdot \nabla_{\mathbf{x}} w(\mathbf{x},\alpha) \\
&= \mu_s(\mathbf{x}) \frac{\partial}{\partial \alpha} \left[e^{-w(\mathbf{x},\alpha)} \int_{\Gamma_d} K(\mathbf{x},\alpha,\beta) e^{w(\mathbf{x},\beta)} d\beta \right], \qquad (20) \\
&\mathbf{x} \in \Omega, \alpha \in (-d,d).
\end{aligned}$$

An orthonormal basis in $L_2(-d, d)$ (Klibanov, 2017)

Consider the set of linearly independent functions, which is complete in $L^2(-d, d)$:

$$\{\alpha^s e^\alpha\}_{s=0}^\infty.$$

Applying the Gram-Schmidt orthonormalization procedure to this set, we obtain the orthonormal basis $\{\Psi_s\,(\alpha)\}_{s=0}^\infty$ in $L_2(-d, d)$.

$$\Psi_s(\alpha) = P_s(\alpha)e^\alpha, \forall s \geq 0,$$

where $P_s(\alpha)$ is a polynomial of the degree s.

$$a_{s,k} = \int_{-d}^{d} \Psi'_s\left(x\right)\Psi_k\left(x\right)dx,$$
$$\text{Matrix: } M_N = \left(a_{s,k}\right)_{(s,k)=(0,0)}^{(N-1,N-1)},$$
$$\det M_N = 1, \forall N$$

$$M_N = \left(a_{mn}\right)_{m,n=0}^{N-1} = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & * & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

Remark: The invertibility of analogs of M_N does not take place for classical orthonormal polynomials and trigonometric basis.

A coupled system of nonlinear integral differential equations

Represent

$$w(\mathbf{x},\alpha) = \sum_{s=0}^{N-1} w_s(\mathbf{x})\Psi_s(\alpha), \;\; w_\alpha(\mathbf{x},\alpha) = \sum_{s=0}^{N-1} w_s(\mathbf{x})\Psi_s'(\alpha). \qquad (21)$$

Therefore, the main problem now is: Find the vector function $W(\mathbf{x})$,

$$W(\mathbf{x}) = (w_0, ..., w_{N-1})^T(\mathbf{x}).$$

Substitute (21) in (20)

$$p_s(\mathbf{x}) = \int_{-d}^{d} \ln\left[g_1(\mathbf{x},\alpha)\right]\Psi_s(\alpha)\, d\alpha, s = 0, ..., N-1. \qquad (22)$$

Multiply sequentially resulting equation by functions $\Psi_k(\alpha), k = 0, ..., N-1$ and integrate with respect to $\alpha \in (-d, d)$. Use boundary conditions.

Boundary value problem:

$$\left(M_N + A_{n+1}(\mathbf{x})\right) W_y(\mathbf{x}) + \sum_{i=1}^{n} A_i(\mathbf{x}) W_{x_i}(\mathbf{x}) + F(W(\mathbf{x}), \mathbf{x}) = 0,$$
$$\mathbf{x} \in \Omega,$$
$$W(\mathbf{x}) \mid_{\partial\Omega} = P(\mathbf{x}), \tag{23}$$

where $A_{n+1} \in C_{N^2}\left(\overline{\Omega}\right)$ and $A_i \in C_{N^2}\left(\overline{\Omega}\right), i = 1, ..., n$ are $N \times N$ matrices, the $N-$D vector function

$$F(s, x) \in C^2\left(\mathbb{R}^{N+n+1}\right). \tag{24}$$

$F(W(\mathbf{x}), \mathbf{x})$ is nonlinear with respect to $W(\mathbf{x})$.
Recall that $a < y < b$. Then

$$\|A_i(\mathbf{x})\|_{C_{N^2}(\overline{\Omega})} \leq \frac{C}{a}, i = 1, ..., n+1. \tag{25}$$

Denote

$$D_N(\mathbf{x}) = (M_N + A_{n+1}(\mathbf{x})) = M_N\left(I + M_N^{-1}A_1(\mathbf{x})\right). \tag{26}$$

Since the matrix M_N is invertible, then it follows from (25) and (26) that there exists such a number $a_0 = a_0(A, d, M_N) > 1$ depending only on listed parameters that

$$\text{the matrix } D_N^{-1}(\mathbf{x}) \text{ exists for all } a \geq a_0 \text{ and for all } \mathbf{x} \in \overline{\Omega}. \tag{27}$$

Convexification Functional for Problem (23)

Let

$$k_n = \left[\frac{n}{2}\right] + 1.$$

Hence, by Sobolev embedding theorem $H^{k_n}\left(\Omega\right) \subset C^1\left(\overline{\Omega}\right)$. Let

$$H_0^1\left(\Omega\right) = \left\{u \in H^1\left(\Omega\right) : u \mid_{\partial\Omega} = 0\right\}.$$

Let $R > 0$ be an arbitrary number. Define the set $B\left(R\right)$ as

$$B\left(R\right) = \left\{W \in H_N^{k_n}\left(\Omega\right) : W\left(\mathbf{x}\right) \mid_{\partial\Omega} = P\left(\mathbf{x}\right), \left\|W\right\|_{H_N^1\left(\Omega\right)} < R\right\}. \tag{28}$$

Minimization Problem. *Minimize the following Tikhonov-like weighted functional on the set* $\overline{B(R)}$:

$$J_\lambda(W) = e^{-2\lambda b} \left\| \left(D_N W_y + \sum_{i=1}^{n} A_i W_{x_i} + F(W(\mathbf{x}), \mathbf{x}) \right) e^{\lambda y} \right\|_{L_N^2(\Omega)}^2 + \\ +\alpha \|W\|_{H^{k_n}(\Omega)}^2 . \tag{29}$$

Here, $e^{-2\lambda b}$ is the balancing multiplier since $\alpha \in (0,1)$.

Theorem 2 (Carleman estimate). *Assume that the number $a \geq a_0$, as in (27). Then there exists a sufficiently large number $\lambda_0 = \lambda_0 (d, N, a, b) \geq 1$ depending only on listed parameters such that the following Carleman estimate holds*

$$\left\| \left(D_N W_y + \sum_{i=1}^{n} A_i W_{x_i} \right) e^{\lambda y} \right\|_{L_N^2(\Omega)}^2 \geq C\lambda^2 \left\| W e^{\lambda y} \right\|_{L_N^2(\Omega)}^2 ,$$
$$\forall W \in H_{N,0}^1 (\Omega) , \forall \lambda \geq \lambda_0 .$$

THE CENTRAL RESULT: STRONG CONVEXITY Theorem 3 (strong convexity). *The following three assertions hold:*

1. *The functional $J_{\lambda}(W)$ in (29) has the Fréchet derivative $J_{\lambda}'(W) \in H_{N,0}^{1}(\Omega)$ at any point $W \in \overline{B(R)}$ and for any value of the parameter $\lambda \geq 0$. The Lipschitz condition holds*

$$\left\| J_{\lambda}'(W_1) - J_{\lambda}'(W_2) \right\|_{H_N^{k_n}(\Omega^h)} \leq C_1 \left\| W_1 - W_2 \right\|_{H_N^{k_n}(\Omega)},$$
$$\forall W_1^h, W_2^h \in \overline{B(R)}$$

for all $\lambda \geq 0$, where the number $C_1 > 0$.

Assume that the number $a \geq a_0$, *as in (27). Then:*
2. *There exists a sufficiently large number* λ_1

$$\lambda_1 = \lambda_1 (R, A, d, N, a, b) \geq \lambda_0 \geq 1 \tag{30}$$

depending only on listed parameters such that the functional $J_\lambda (W)$ *in (29) is strictly convex on the set* $\overline{B(R)}$, *i.e. the following inequality holds:*

$$\begin{gathered} J_\lambda (W_2) - J_\lambda (W_1) - J'_\lambda (W_1) (W_2 - W_1) \\ \geq C_2 e^{-2\lambda(b-a)} \| W_2 - W_1 \|^2_{H^1_N(\Omega)} + \alpha \| W_2 - W_1 \|^2_{H^{k_n}_N(\Omega)}, \\ \forall \lambda \geq \lambda_1, \ \forall W_1, W_2 \in \overline{B(R, P)}. \end{gathered} \tag{31}$$

3. For each $\lambda \geq \lambda_1$ there exists unique minimizer $W_{\min,\lambda} \in \overline{B(R)}$ of the functional $J_\lambda(W)$ on the set $\overline{B(R)}$. Furthermore, the following inequality holds:

$$J'_\lambda\left(W^h_{\min,\lambda}\right)\left(W^h - W^h_{\min,\lambda}\right) \geq 0,\ \forall W^h \in \overline{B(R, P^h)}. \tag{32}$$

Theorem 4 (uniqueness). *Assume that the number $a \geq a_0$, as in (27). Then there exists at most one pair of function $W \in H^1_N(\Omega)$ satisfying conditions (23).*

Estimating the accuracy of the minimizer $W^h_{\min,\lambda}$

Following the concept of Tikhonov for ill-posed problems, we assume the existence of the exact solution

$$W^* \in B^*(R) = \left\{ W \in H_N^{k_n}(\Omega) : W(\mathbf{x}) \mid_{\partial\Omega} = P^*(\mathbf{x}), \|W\|_{H_N^1(\Omega)} < R \right\} \tag{33}$$

of problem (23) with the exact, i.e. noiseless data P^*, i.e. for $\mathbf{x} \in \Omega$

$$D_N(\mathbf{x}) W_y^*(\mathbf{x}) + \sum_{i=1}^{n} A_i(\mathbf{x}) W_{x_i}^*(\mathbf{x}) + F(W^*(\mathbf{x}), \mathbf{x}) = 0, \tag{34}$$

$$W^*\left(\mathbf{x}^h\right) \mid_{\partial\Omega} = P^*(\mathbf{x}). \tag{35}$$

Suppose that there exists a vector function $S(\mathbf{x}) \in B(R)$. Let the vector function $S^* \in B^*(R)$. Let $\delta \in (0,1)$ be the noise level in the data. We assume that

$$\|S - S^*\|_{H_N^{k_n}(\Omega)} < \delta. \tag{36}$$

Theorem 5. *Assume that the number* $a \geq a_0$*, as in (27). Suppose that conditions (34)-(36) hold. Consider the number* λ_2*,*

$$\lambda_2 = \lambda_1 (2R, A, d, N, a, b), \tag{37}$$

where $\lambda_1 (R, A, d, N, a, b)$ *is the number of Theorem 3. Let* $W_{\min,\lambda_2}$ *be the minimizer of the functional* $J_\lambda (W)$ *on the set* $\overline{B(R)}$*, which was found in Theorem 3. Let* $\beta \in (0, R)$ *be a number. Suppose that*

$$W^* \in B^* (R - \beta) \tag{38}$$

and the noise level δ is so small that $C_2 \delta < \beta$. *Then the vector function* $W_{\min,\lambda_2}$ *belongs to the open set* $B(R)$ *and the following accuracy estimate holds:*

$$\| W_{\min,\lambda_2} - W^* \|_{H^1_N(\Omega)} \leq C_2 \delta. \tag{39}$$

The global convergence of the gradient descent method of the minimization of functional $J_\lambda(W)$ on the set $\overline{B(R)}$

Let the starting point of iterations $W_0 \in B(R/3)$ be an arbitrary point of this set. The sequence of this method is:

$$W_n = W_{n-1} - \gamma J'_{\lambda_2}(W_{n-1}), n = 1, 2, ..., \tag{40}$$

where $\gamma > 0$ is a small number and λ_2 is the same as in (37).

Theorem 6. *Let*

$$W^* \in B^*\left(\left(R-\alpha\right)/3\right) \text{ and } C_2\delta < \alpha/3. \tag{41}$$

Then there exists a sufficiently small number $\gamma > 0$ *and a number* $\theta = \theta\left(\gamma\right) \in \left(0,1\right)$ *such that in (40) all functions* $W_n \in B\left(R\right)$ *and the following convergence estimates hold:*

$$\left\|W_n - W_{\min,\lambda_2}\right\|_{H_N^1\left(\Omega^h\right)} \leq \theta^n \left\|W_0 - W_{\min,\lambda_2}\right\|_{H_N^{k_n}(\Omega)},$$

$$\left\|W_n - W^*\right\|_{H_N^{1,h}\left(\Omega^h\right)} \leq C_2\delta + \theta^n \left\|W_0^h - W_{\min,\lambda_2}^h\right\|_{H_N^{k_n}(\Omega)},$$

$$\left\|a_n - a^*\right\|_{L_2(\Omega)} \leq C_2\delta + \theta^n \left\|W_0 - W_{\min,\lambda_2}\right\|_{H_N^{k_n}(\Omega)}.$$

Numerical Examples

In our tests, we have perturbed the data with 3% and 5% of the random noise.

Choice of the number N:

Table 1: The $L_2(\Omega)$ −norms of functions $w_s(\mathbf{x})$, $s = 0, 1, ..., 11$ for the reference Test 1.

s	0	1	2	3	4	5
$\|w_s(\mathbf{x})\|_{L_2}$	5.7122	1.6383	0.1630	0.0118	0.0091	0.0077
s	6	7	8	9	10	11
$\|w_s(\mathbf{x})\|_{L_2}$	0.0067	0.0061	0.0055	0.0057	0.0058	0.0054

Thus, $N = 3$ is the optimal number.

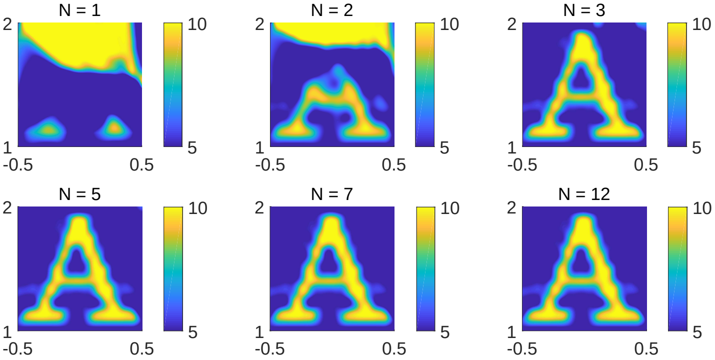

Figure 2: The choice of the optimal number N. Thus, $N = 3$ is the optimal number. The inclusion/background contrast here is $2:1$.

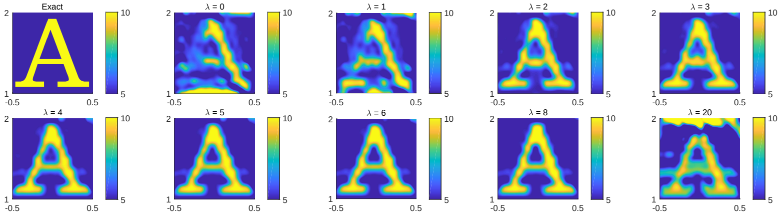

Figure 3: The choice of the optimal number λ in the Carleman Weight Function $e^{\lambda y}$. Here $N = 3$. Thus, $\lambda = 5$ is the optimal one. The inclusion/background contrast here is $2:1$.

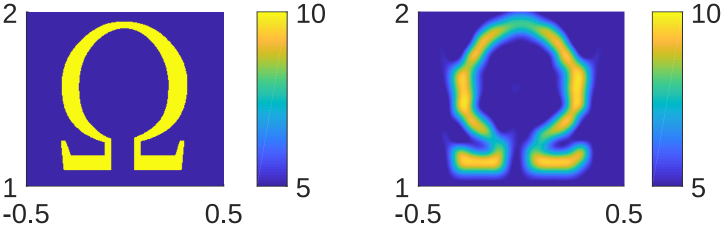

Figure 4: Numerical reconstruction of the $\Omega-$shaped abnormality. The inclusion/background contrast here is $2:1$.

MEAN FIELD GAMES: CARLEMAN ESTIMATES AND CONVEXIFICATION

The mean field games (MFG) theory examines the collective behavior of an infinite number of rational agents. This theory was initially introduced in 2006 in the seminal works of Lasry and Lions and Huang, Caines, and Malhamé.

In fact, the MFG theory can govern ALMOST ANY societal phenomenon via a system of two coupled nonlinear parabolic Partial Differential Equations!

This is the so-called MEAN FIELD GAMES SYSTEM.

PRICE TO PAY FOR THIS GREAT ADVANTAGE:

Those equations have two opposite directions of time. Therefore, the classical theory of parabolic equations is inapplicable.

The nonlinearity.

One of equations has an important term: an integral operator, which does not allow to “project” the theory of CIPs for one parabolic equation on CIPs for this system.

The **MOST CHALLENGING** difficulty is that the Laplace operator of the solution of the first equation is a part of the second equation.

Prior to eight works of Klibanov with coauthors in 2023-2024, there were no:

1. Uniqueness theorems for the forward problems for the MFGS without restrictive conditions.
2. No stability theorems for forward problems.
3. No rigorous numerical methods for forward problems.
4. No rigorous theoretical numerical investigations for inverse problems for the MFGS.
5. Introducing Carleman estimates in the MFG theory, Klibanov et. al. addressed the above points 1-4 in 2023-2024.
6. The ideology of the theory of Ill-Posed Problems was introduced in the MFG theory.

NOTATIONS:

$\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$, $Q_T = \Omega \times (0, T)$.

THE MEAN FIELD GAMES SYSTEM (MFGS)

$$\begin{gathered}
v_t(x,t) + \beta \Delta v(x,t) + k(x)(\nabla v(x,t))^2/2 + \\
+F\left(x, t, \int_\Omega K(x,y)\, m(y,t)\, dy, m(x,t)\right) = 0, \ \ (x,t) \in Q_T, \\
m_t(x,t) - \beta \Delta m(x,t) + \\
+\nabla \cdot (k(x) m(x,t) \nabla v(x,t)) = 0, \ \ (x,t) \in Q_T, \\
\beta = const. > 0.
\end{gathered} \tag{42}$$

The standard boundary conditions are either periodicity, or

$$\partial_n v \mid_{S_T} = \partial_n m \mid_{S_T} = 0. \tag{43}$$

The conventional initial and terminal conditions:

$$v(x,T) = v_T(x), m(x,0) = m_0(x). \tag{44}$$

Uniqueness theorems for problem (42)-(44) are known only under restrictive so-called "monotonicity" conditions of Lasry and Lions. $v(x,t)$ is the value function, $m(x,t)$ is the distribution of agents. $\varkappa(x)$ characterizes the reaction of the controlled object to an action applied at the point x.
The function F characterizes interaction between players.

$$\int\limits_{\Omega} K(x,y)\,m(y,t)\,dy$$

is the global interaction term. On the other hand, the term like $p(x,t)\,m(x,t)$ is the local interaction term.

RETROSPECTIVE PROBLEM. *Let the initial and terminal conditions (44) are supplemented by the following terminal condition:*

$$m(x, T) = m_T(x). \tag{45}$$

Find the pair of functions $(v(x,t), m(x,t))$ *satisfying conditions (42)-(45).*

$$H_0^2(Q_T) = \left\{ u \in H^2(Q_T) : \partial_n u \mid_{S_T} = 0 \right\}.$$

The Carleman Weight Function $\varphi_{\lambda,\nu}(t)$:

$$\varphi_{\lambda,\nu}(t) = e^{2\lambda(t+a)^{\nu}}, t \in (0, T); \lambda, \nu >> 1.$$

Theorem 7 (the first Carleman estimate). *There exists a number $C = C(T, a) > 0$ depending only on listed parameters such that the following Carleman estimate is valid:*

$$\int\limits_{Q_T} (u_t + \beta \Delta u)^2 \varphi^2_{\lambda,\nu} dxdt \geq \int\limits_{Q_T} \left(u_t^2/4 + \beta^2 (\Delta u)^2 \right) \varphi^2_{\lambda,\nu} dxdt +$$
$$+ C\lambda\nu\beta \int\limits_{Q_T} (\nabla u)^2 \varphi^2_{\lambda,\nu} dxdt + C\lambda^2\nu^2 \int\limits_{Q_T} u^2 \varphi^2_{\lambda,\nu} dxdt -$$
$$- e^{2\lambda(T+a)^\nu} \int\limits_{\Omega} \left[\beta (\nabla_x u)^2 + \lambda\nu (T+a)^{\nu-1} u^2 \right] (x, T)\, dt,$$
$$\forall \lambda > 0, \forall \nu > 2, \forall u \in H_0^2 (Q_T).$$

Theorem 8 (the second Carleman estimate)**.** *There exist a sufficiently large number* $\nu_0 = \nu_0(\beta, T, a) > 2$ *and a number* $C = C(T, a) > 0$ *depending only on listed parameters such that the following Carleman estimate holds:*

$$\begin{gathered}
\int\limits_{Q_T} (u_t - \beta \Delta u)^2 \varphi_{\lambda,\nu} dxdt \geq \\
\geq C\beta\sqrt{\nu} \int\limits_{Q_T} (\nabla u)^2 \varphi_{\lambda,\nu} dxdt + C\lambda\nu^2 \int\limits_{Q_T} u^2 \varphi_{\lambda,\nu} dxdt - \\
- C\lambda\nu (T + a)^{\nu-1} e^{2\lambda(T+a)^\nu} \int\limits_{\Omega} u^2(x, T)\, dx - \\
- Ce^{2\lambda a^\nu} \int\limits_{\Omega} \left[(\nabla u)^2 + \sqrt{\nu} u^2 \right](x, 0)\, dx, \\
\forall \lambda > 0, \forall \nu \geq \nu_0, \forall u \in H_0^2(Q_T).
\end{gathered}$$

Theorem 9 (Lipschitz stability estimate). *Let $M_1, M_2, M_3, M_4 > 0$ be certain numbers. Assume that in (42) the function $F = F(x, t, y, z) : \overline{Q}_T \times \mathbb{R}^2 \to \mathbb{R}$ be bounded in any bounded subset of the set $\overline{Q}_T \times \mathbb{R}^2$ and such that there exist derivatives $F_y, F_z \in C\left(\overline{Q}_T \times \mathbb{R}^2\right)$ satisfying*

$$\max\left(\sup_{Q_T \times \mathbb{R}^2} |F_y(x, t, y, z)|, \sup_{Q_T \times \mathbb{R}^2} |F_y(x, t, y, z)|\right) \leq M_1.$$

Let the function $K(x, y) \in L_\infty(\Omega \times \Omega)$, the function $\varkappa \in C^1\left(\overline{\Omega}\right)$ and

$$\sup_{\Omega \times \Omega} |K(x, y)|, \|k\|_{C^1(\overline{\Omega})} \leq M_2.$$

Consider the sets of functions $B_3(M_3), B_4(M_4)$ *defined as*

$$B_3(M_3) = \left\{ u \in H_0^2(Q_T) : \sup_{Q_T} |u|, \sup_{Q_T} |\nabla u|, \sup_{Q_T} |\Delta u| \le M_3 \right\},$$

$$B_4(M_4) = \left\{ u \in H_0^2(Q_T) : \sup_{Q_T} |u|, \sup_{Q_T} |\nabla u| \le M_4 \right\}.$$

Let

$$M = \max\left(M_1, M_2, M_3, M_4\right).$$

Assume that two pairs of functions

$$\left(v_1, m_1\right), \left(v_2, m_2\right) \in B_3\left(M_3\right) \times B_4\left(M_4\right)$$

satisfy equations (42), zero Neumann boundary conditions (43) as well as the following initial and terminal conditions:

$$v_1\left(x, T\right) = v_T^{(1)}\left(x\right),\ \ v_2\left(x, T\right) = v_T^{(2)}\left(x\right),\ \ x \in \Omega,$$

$$m_1\left(x, T\right) = m_T^{(1)}\left(x\right),\ \ m_2\left(x, T\right) = m_T^{(2)}\left(x\right),\ \ x \in \Omega,$$

$$m_1\left(x, 0\right) = m_0^{(1)}\left(x\right), m_2\left(x, 0\right) = m_0^{(2)}\left(x\right),\ \ x \in \Omega.$$

Then there exists a number $C_1 = C_1(\beta, M, T) > 0$ depending only on listed parameters such that the following two Lipschitz stability estimates are valid:

$$\|\partial_t v_1 - \partial_t v_2\|_{L_2(Q_T)} + \|\Delta v_1 - \Delta v_2\|_{L_2(Q_T)} + \|v_1 - v_2\|_{H^{1,0}(Q_T)} \leq$$
$$\leq C_1 \left(\left\| v_T^{(1)} - v_T^{(2)} \right\|_{H^1(\Omega)} + \left\| m_T^{(1)} - m_T^{(2)} \right\|_{L_2(\Omega)} \right) +$$
$$+ C_1 \left\| m_0^{(1)} - m_0^{(2)} \right\|_{H^1(\Omega)}, \tag{46}$$

$$\|m_1 - m_2\|_{H^{1,0}(Q_T)} \leq C_1 \left(\left\| v_T^{(1)} - v_T^{(2)} \right\|_{H^1(\Omega)} + \left\| m_T^{(1)} - m_T^{(2)} \right\|_{L_2(\Omega)} \right) +$$
$$+ C_1 \left\| m_0^{(1)} - m_0^{(2)} \right\|_{H^1(\Omega)}.$$

In particular, if the domain Ω is a rectangular prism, then estimate (46) is strengthened as

$$\|v_1 - v_2\|_{H^{2,1}(Q_T)} \leq C_1 \left(\left\| v_T^{(1)} - v_T^{(2)} \right\|_{H^1(\Omega)} + \left\| m_T^{(1)} - m_T^{(2)} \right\|_{L_2(\Omega)} \right) +$$
$$+ C_1 \left\| m_0^{(1)} - m_0^{(2)} \right\|_{H^1(\Omega)}.$$

Next, if

$$v_T^{(1)}(x) \equiv v_T^{(2)}(x), m_0^{(1)}(x) \equiv m_0^{(2)}(x), m_T^{(1)}(x) \equiv m_T^{(2)}(x), \ x \in \Omega,$$

then $v_1(x,t) \equiv v_2(x,t)$ *and* $m_1(x,t) \equiv m_2(x,t)$ *in* Q_T, *which means that problem (42)-(45) has at most one solution.*

CONVEXIFICATION NUMERICAL METHOD FOR THE RETROSPECTIVE PROBLEM

THIS IS THE FIRST NUMERICAL METHOD IN THE MFG THEORY WITH THE RIGOROUSLY ESTABLISHED GLOBAL **CONVERGENCE PROPERTY**

$$\begin{gathered} L_1(v,m) = \\ = v_t(x,t) + \beta\Delta v(x,t) + k(x)(\nabla v(x,t))^2/2 + \\ + \int\limits_{\Omega} K(x,y)\,m(y,t)\,dy + f(x,t)\,m(x,t) + F_1(x,t) = 0, \;\; (x,t)\in Q_T, \\ L_2(v,m) = \\ = m_t(x,t) - \beta\Delta m(x,t) + div(k(x)m(x,t)\nabla v(x,t)) + \\ + F_2(x,t) = 0, \;\; (x,t)\in Q_T, \end{gathered} \tag{47}$$

where $L_1(u,p)$ and $L_2(u,p)$ are two operators. Just as above, we add the zero Neumann boundary conditions

$$\partial_\nu v \mid_{S_T} = \partial_\nu m \mid_{S_T} = 0. \tag{48}$$

PROBLEM 2. Given:

$$\begin{gathered} v(x,T) = v_T(x), \;\; m(x,0) = m_0(x), \;\; x \in \Omega, \\ m(x,T) = m_T(x), \;\; x \in \Omega. \end{gathered} \tag{49}$$

Find the pair of functions (v,m).

$$k_n = [(n+1)/2] + 3.$$

By embedding theorem $H^{k_n}(Q_T) \subset C^2\left(\overline{Q}_T\right)$, and

$$\|g\|_{C^2\left(\overline{Q}_T\right)} \leq C_0 \|g\|_{H^{k_n}(Q_T)}, \; \forall g \in H^{k_n}(Q_T). \tag{50}$$

Let $R > 0$ be an arbitrary number. Consider the set $B(R)$,

$$B(R) = \left\{ \begin{array}{c} (v,m) \in H^{k_n}(Q_T) \times H^{k_n}(Q_T) : u, m \in H_0^2(Q_T), \\ v(x,T) = v_T(x), \ m(x,T) = m_T(x), \ m(x,0) = m_0(x), \\ \|v\|_{H^{k_n}(Q_T)}, \|m\|_{H^{k_n}(Q_T)} < R. \end{array} \right\} \tag{51}$$

CARLEMAN WEIGHT FUNCTION:

$$\varphi_{\lambda}(t) = e^{2(t+a)^{\lambda}}, t \in (0, T); \lambda >> 1. \tag{52}$$

Let $L_1(v, m)$ and $L_2(v, m)$ be two operators defined in (47). Consider four functionals

$$\begin{gathered} J_{1,\lambda}, J_{2,\lambda}, J_3, J : \overline{B(R)} \to \mathbb{R}, \\ J_{1,\lambda}(v, m) = \int\limits_{Q_T} (L_1(v, m))^2 \varphi_{\lambda} dxdt, \\ J_{2,\lambda}(v, m) = \int\limits_{Q_T} (L_2(v, m))^2 \varphi_{\lambda} dxdt, \\ J_3(v, m) = \gamma \left(\|v\|^2_{H^{k_n}(Q_T)} + \|m\|^2_{H^{k_n}(Q_T)} \right), \\ J_{\lambda,\gamma}(v, m) = J_{1,\lambda}(v, m) + \left(1/2 + C_1/\lambda^2\right) J_{2,\lambda}(v, m) + J_3(v, m). \end{gathered} \tag{53}$$

To solve our target Problem, we consider

Minimization Problem. *Minimize the functional $J_{\lambda,\gamma}(v,m)$ in (52) on the set $B(R)$ defined in (51).*

Below $[,]$ is the scalar product in the Hilbert space $H^{k_n}(Q_T)\times H^{k_n}(Q_T)$. Define the subspace $\widetilde{H}$ of this space as

$$\widetilde{H}=\left\{\begin{array}{c} (h,q)\in H^{k_n}(Q_T)\times H^{k_n}(Q_T): h,q\in H_0^2(Q_T), \\ h(x,T)=\ q(x,T)=q(x,0)=0, \\ \|(h,q)\|_{\widetilde{H}}^2=\|h\|_{H^{k_n}(Q_T)}^2+\|q\|_{H^{k_n}(Q_T)}^2. \end{array}\right\}$$

Theorem 10 (brief formulation).
1. The functional $J_{\lambda,\gamma}$ *has the Fréchet derivative* $J'_{\lambda,\gamma}(v,m) \in \widetilde{H}$ *at every point* $(v,m) \in \overline{B(R)}$*. The Fréchet derivative* $J'_{\lambda,\gamma}(v,m)$ *is Lipschitz continuos on* $\overline{B(R)}$*, i.e. the following inequality holds:*

$$\begin{gathered} \left\| J_{\lambda,\gamma}(v_1,m_1) - J_{\lambda,\gamma}(v_2,m_2) \right\|_{H^{k_n}(Q_T)\times H^{k_n}(Q_T)} \leq \\ \leq D \left\| (v_1,m_1) - (v_2,m_2) \right\|_{H^{k_n}(Q_T)\times H^{k_n}(Q_T)}, \\ \forall (v_1,m_1),(v_2,m_2) \in \overline{B(R)}, \end{gathered}$$

where the number $D = D(\lambda,\gamma,\Omega,T,M,R) > 0$ *depends only on listed parameters.*

2. There exists a sufficiently large number $\overline{\lambda} = \overline{\lambda}(\Omega, T, M, R) > 1$ such that for all $\lambda \geq \overline{\lambda}$ the functional $J_{\lambda,\gamma}$ is strongly convex on the set $\overline{B(R)}$, i.e. there exists a number $C_1 = C_1(\Omega, T, M, R) > 0$ such that the following inequality holds:

$$\begin{gathered}
J_{\lambda,\gamma}(v_1, m_1) - J_{\lambda,\gamma}(v, m) - \left[J'_{\lambda,\gamma}(v, m), (v_1 - v, m_1 - m) \right] \geq \\
\geq C_1 e^{2a^{\lambda}} \left(\|\Delta v_1 - \Delta v\|^2_{L_2(Q_T)} + \|v_1 - v\|^2_{H^{1,1}(Q_T)} + \|m_1 - m\|^2_{H^{1,0}(Q_T)} \right) \\
+\gamma \left(\|v_1 - v\|^2_{H^{k_n}(Q_T)} + \|m_1 - m\|_{H^{k_n}(Q_T)} \right), \\
\forall (v, m), (v_1, m_1) \in \overline{B(R)}, \ \forall \gamma > 0, \ \forall \lambda \geq \overline{\lambda}.
\end{gathered} \tag{54}$$

In particular, numbers C_1 and λ are also involved in the term $(1 + C_1/\lambda^2)J_{2,\lambda}(v, m)$ in (53). Both numbers $\overline{\lambda}$ and C_1 depend only on listed parameters.

3. For every $\lambda \geq \overline{\lambda}$ and for every $\gamma > 0$ there exists unique minimizer $(v_{\min,\lambda,\gamma},\ m_{\min,\lambda,\gamma}) \in \overline{B(R)}$ of the functional $J_{\lambda,\alpha}(u,p)$ on the set $\overline{B(R)}$ and the following inequality holds:

$$\left[J'_{\lambda,\gamma}\left(v_{\min,\lambda,\gamma}, m_{\min,\lambda,\gamma}\right), \left(v_{\min,\lambda,\gamma} - v, m_{\min,\lambda,\gamma} - m\right)\right] \leq 0,$$

$\forall\,(v,m) \in \overline{B(R)}$.(55)

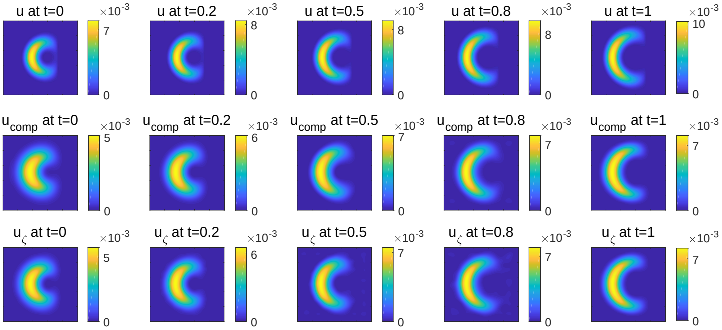

Figure 5: An example of the solution of the retrospective problem of Mean Field Games via the convexification method. This is the evolution of the function $v(x,t)$. The optimal $\lambda = 2$.

OTHER FORWARD PROBLEMS BEING ADDRESSED:

PROBLEM 3. Known:

$$v\left(x,T\right),v\left(x,0\right),m\left(x,0\right).$$

Lipschitz stability estimate.

PROBLEM 4. Known:

$$\text{either } \left(v\left(x,T\right),m\left(x,T\right)\right) \text{ or } \left(v\left(x,0\right),m\left(x,0\right)\right).$$

Hölder stability estimates.

PROBLEM 5. Known lateral Cauchy data:

$$\begin{gathered}
v\mid_{S_T}, m\mid_{S_T}, \partial_n v\mid_{S_T}, \partial_n m\mid_{S_T}, \\
\text{either } K(x,y) = \delta(y_1 - x_1) K_1(x,y), \\
\text{or } K(x,y) = H(y_1 - x_1) K_2(x,y), \\
K_1, K_2 \in L_\infty(\Omega \times \Omega), \\
H(z) \text{ is the Heaviside function.}
\end{gathered} \tag{56}$$

Hölder stability estimate. The Carleman Weight Function is an unusual one:

$$\psi_\lambda(x,t) = \exp\left[2\lambda\left(x_1^2 - c^2(t - T/2)^2\right)\right], \lambda >> 1. \tag{57}$$

COEFFICIENT INVERSE PROBLEMS OF MEAN FIELD GAMES

$$
\begin{gathered}
v_t(x,t) + \Delta v(x,t) - k(x)(\nabla v(x,t))^2/2 - \\
- \int_\Omega K\,(x,y)\, m\,(y,t)\, dy - s\,(x,t)\, m\,(x,t) = 0, \ \ (x,t) \in Q_T, \\
m_t(x,t) - \Delta m(x,t) - div(k(x) m(x,t) \nabla u(x,t)) = 0, \ \ (x,t) \in Q_T.
\end{gathered}
\tag{58}
$$

Let $x = (x_1, x_2, ..., x_n)$ denotes points in $\mathbb{R}^n$ and let $\overline{x} = (x_2, ..., x_n)$. To simplify the presentation, we assume that our domain of interest $\Omega \subset \mathbb{R}^n$ is a rectangular prism. Let $a, b, B_i > 0, i = 1, ..., n$ be some numbers and $a < b$.

$$\begin{gathered}
\Omega = \left\{ x : a < x_1 < b, -B_i < x_i < B_i, i = 2, ..., n \right\}, \\
\Omega_1 = \left\{ \overline{x} : -B_i < x_i < B_i, i = 2, ..., n \right\}, \\
\Gamma_1^+ = \left\{ x \in \partial\Omega : x_1 = b \right\}, \Gamma_1^- = \left\{ x \in \partial\Omega : x_1 = a \right\}, \Gamma_{1,T}^\pm = \Gamma_1^\pm \times (0, T), \\
\Gamma_i^\pm = \left\{ x \in \partial\Omega : x_i = \pm B_i \right\}, \ \Gamma_{iT}^\pm = \Gamma_i^\pm \times (0, T), \ i = 2, ..., n.
\end{gathered} \tag{59}$$

Coefficient Inverse Problem 2 (CIP2). *Assume that functions $v, m \in C^4\left(\overline{Q}_T\right)$ satisfy equations (58). Let*

$$\begin{gathered}
u\left(x, T/2\right) = u_0\left(x\right),\ m\left(x, T/2\right) = m_0\left(x\right),\ x \in \Omega, \\
u\mid_{S_T} = g_0\left(x,t\right),\ \partial_n u\mid_{S_T} = g_1\left(x,t\right), \\
m\mid_{S_T} = p_0\left(x,t\right),\ \partial_n m\mid_{S_T} = p_1\left(x,t\right).
\end{gathered} \tag{60}$$

Determine the coefficient $k\left(x\right) \in C^1\left(\overline{\Omega}\right)$.

Coefficient Inverse Problem 2 (CIP2). Assume that conditions of CIP1 are satisfied. Determine the coefficient $k\left(x\right) \in L_\infty\left(\Omega\right)$ in

$$k\left(x\right) \int\limits_{\Omega} K\left(x,y\right) m\left(y,t\right) dy.$$

TWO MAIN DIFFICULTIES COMPARED WITH THE CASE OF A SINGLE PARABOLIC PDE:

1. The presence of the integral operator

$$\int\limits_{\Omega} K\left(x,y\right) m\left(y,t\right) dy$$

   does not allow to "project" methods for CIPs for one parabolic PDE on the case of MFGS (*58*).

2. In CIP1, the unknown coefficient $k(x)$ is involved with its first derivatives, whereas still only a single measurement is taken.

A NEW CARLEMAN ESTIMATE FOR THE VOLTERRA-LIKE INTEGRAL

Theorem 11. *Let the number* α *be such that*

$$\alpha \in \left(0, \frac{1}{3}\right), \tag{61}$$

Let $d > 0$ *be a number and let the number* α *be as in (61), where* n_1, n_2 *are two odd numbers. Then the following Carleman estimate of the Volterra integral holds for all functions* $f \in L_2(-d, d)$ *and all* $\lambda > 0$:

$$\int_{-d}^{d} e^{-2\lambda|t|^{1+\alpha}} \left(\int_{0}^{t} f(\tau)\, d\tau\right)^2 dt \leq \frac{1}{\lambda^{3/2}} \cdot \frac{d^{(1-3\alpha)/2}}{\sqrt{2}\,(1+\alpha)^{3/2}} \int_{-d}^{d} f^2 e^{-2\lambda|t|^{1+\alpha}} dt.$$

- The new element here is the multiplier $1/\lambda^{3/2}$. In the conventional case $1+\alpha=2$ and the multiplier is $1/\lambda >> 1/\lambda^{3/2}$ for sufficiently large λ.

Let $N_2, N_3 > 0$ be two numbers,

$$S_1(N_2) = \left\{ v \in C^4\left(\overline{Q}_T\right) : \|v\|_{C^4\left(\overline{Q}_T\right)} \leq N_2 \right\},$$
$$S_2(N_3) = \left\{ k \in C^1\left(\overline{\Omega}\right) : \|k\|_{C^1\left(\overline{\Omega}\right)} \leq N_3 \right\},$$
$$N = \max(N_1, N_2, N_3).$$

Suppose that we have two triples

$$(u_i, m_i, k_i) \in S_1^2(N_2) \times S_2(N_3), i = 1, 2.$$

Let

$$
\begin{gathered}
u_i\left(x, T/2\right) = u_{0,i}\left(x\right), \; m_i\left(x, T/2\right) = m_{0,i}\left(x\right), \; x \in \Omega, \; i = 1, 2, \\
u_i \mid_{S_T} = g_{0,i}\left(x, t\right), \; \partial_n u_i \mid_{S_T} = g_{1,i}\left(x, t\right), \; i = 1, 2, \\
m_i \mid_{S_T} = p_0\left(x, t\right), \; \partial_n m_i \mid_{S_T} = p_{1,i}\left(x, t\right), \; i = 1, 2.
\end{gathered}
$$

Denote

$$\widetilde{u} = u_1 - u_2,\ \widetilde{m} = m_1 - m_2,\ \widetilde{k} = k_1 - k_2,$$
$$\widetilde{u}_0 = u_{0,1} - u_{0,2},\ \widetilde{m}_0 = m_{0,1} - m_{0,2},$$
$$\widetilde{g}_0 = g_{0,1} - g_{0,2},\ \widetilde{g}_1 = g_{1,1} - g_{1,2},\ \widetilde{p}_0 = p_{0,1} - p_{0,2},\ \widetilde{p}_1 = p_{1,1} - p_{1,2}.$$

Let the number $\varepsilon \in (0, T/2)$. Denote

$$Q_{\varepsilon,T} = \Omega \times (\varepsilon, T - \varepsilon).$$

Theorem 12 (Hölder stability estimate)**.** Consider CIP1. *Assume that the kernel $K(x,y)$ has either of forms (56). Also, let*

$$\frac{1}{2}\left|\nabla u_{0,1}(x)\right|^2 \geq c,\ x \in \overline{\Omega},$$

where $c > 0$ is a number. Let $\delta > 0$ be a sufficiently small number. Assume that

$$\begin{gathered}\left\|\widetilde{u}_0\right\|_{H^1(\Omega)}, \left\|\widetilde{m}_0\right\|_{H^1(\Omega)} \leq \delta, \\ \left\|\partial_t^s \widetilde{g}_0\right\|_{H^{2,1}(S_T^-)}, \left\|\partial_t^s \widetilde{p}_0\right\|_{H^{2,1}(S_T)} \leq \delta,\ s = 0,1,2, \\ \left\|\partial_t^s \widetilde{g}_1\right\|_{H^{1,0}(S_T)}, \left\|\partial_t^s \widetilde{p}_1\right\|_{H^{1,0}(S_T)} \leq \delta,\ s = 0,1,2.\end{gathered}$$

Let $\rho \in (0,1)$ be an arbitrary number. Then for every ε satisfying

$$\frac{T}{2}(1-\sqrt{\rho}) < \varepsilon < \frac{T}{2}.$$

there exists a sufficiently small number

$$\delta_0 = \delta_0 (N, \varepsilon, \Omega, T, c, \rho) \in (0,1),$$

and a number $C = C(N, \varepsilon, \Omega, T, c) > 0$, both numbers depending only on listed parameters, such that the following Hölder stability estimate holds:

$$\|\partial_t^s \widetilde{u}\|_{H^{2,1}(Q_{\varepsilon,T})}, \|\partial_t^s \widetilde{m}\|_{H^{2,1}(Q_{\varepsilon,T})}, \left\|\widetilde{k}\right\|_{L_2(\Omega)} \leq C\delta^{1-\rho}, \ \forall \delta \in (0, \delta_0), \ s = 0$$

Also, CIP2 has at most one solution $(u, m, k) \in S_1^2(N_2) \times S_2(N_3)$.

THE GLOBALLY CONVERGENT CONVEXIFICATION METHOD FOR CIP2

1. The same Carleman Weight Function is

$$\varphi_\lambda(x_1,t)=\exp\left[2\lambda\left(x_1^2-(t-T/2)^{1+\alpha}\right)\right].$$

   used to construct a globally strongly convex cost functional.

2. $\lambda=3$ is an optimal one.
3. This is the FIRST numerical method for a CIP for the MFGS with the rigorously proven GLOBAL convergence.
4. We have recovered the coefficient $k(x)$ in

$$K(x,y)=k(x)\,\delta(y_1-x_1)\,K_1(x,y).$$

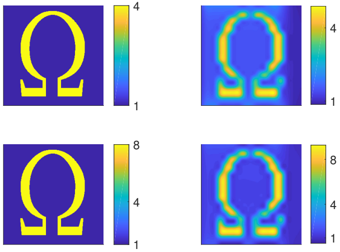

Figure 6: An example of the solution of CIP2 for Mean Field Games by the convexification method. The optimal $\lambda = 3$. Inclusion/background contrasts are 4:1 and 8:1, which are considered as high contrasts.

SUMMARY

1. Convexification is a versatile method, which is applicable to a broad class of Coefficient Inverse Problems with non-overdetermined data.
2. This is the only method at the time being, which has the global convergence property.

TRAVEL TIME TOMOGRAPHY PROBLEM IN 3-D

(=Inverse Kinematic Problem of Seismic)

This is the single MOST CHALLENGING INVERSE PROBLEM I am aware of.

Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Let $c(\mathbf{x})$ be the speed of waves propagation, $c(\mathbf{x}) = 1/n(\mathbf{x})$, where $n(\mathbf{x})$ is the refractive index. The function $n(\mathbf{x})$ generates the Riemannian metric [**?**, Chapter 3]

$$d\tau = n(\mathbf{x}) \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \tag{62}$$

Let $\mathbf{x}_0$ be a source of waves and $\mathbf{x}$ be an observation point.

Let $\Gamma(\mathbf{x}, \mathbf{x}_0)$ be the geodesic line generated by metric (62) and connecting points $\mathbf{x}$ and $\mathbf{x}_0$. The first arrival time $\tau(\mathbf{x}, \mathbf{x}_0)$ is

$$\tau(\mathbf{x}, \mathbf{x}_0) = \int\limits_{\Gamma(\mathbf{x}, \mathbf{x}_0)} n(\mathbf{y}(s))\, ds.$$

Eikonal Equation:

$$(\nabla_{\mathbf{x}} \tau)^2 = n^2(\mathbf{x}),$$
$$\tau(\mathbf{x}, \mathbf{x}_0) = O(|\mathbf{x} - \mathbf{x}_0|) \text{ as } \mathbf{x} \to \mathbf{x}_0.$$

Travel Time Tomography Problem (=Inverse Kinematic Problem of Seismic)
Assume that the function $\tau(\mathbf{x}, \mathbf{x}_0)$ known for all source positions running along the axis $\{x = 0, y = 0\}$ of the cylinder of Figure 7. Assume also that the function $n(\mathbf{x}) = 1$ for $\sqrt{x^2 + y^2} < \varepsilon$.

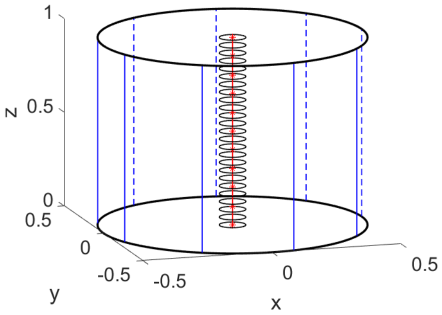

Figure 7: Schematic diagram of the source/detectors configuration.

Find the function $n(\mathbf{x})$ inside that cylinder, assuming that the function

$$\tau(\mathbf{x}, \mathbf{x}_0)$$

is known for all $\mathbf{x}$ on the entire surface of that cylinder for all $\mathbf{x}_0$ running along the axis of that cylinder.
Convexification works for this problem!

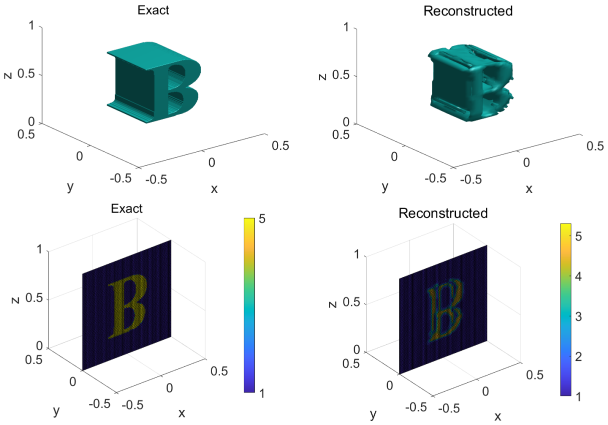

Figure 8: A result of the convexification.

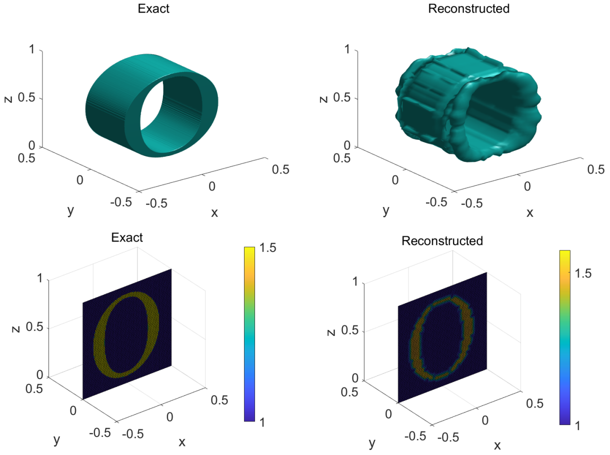

Figure 9: A result of the convexification.