Bounds and Optimal Performance in Linear Electromagnetic Systems

Owen Miller

Yale Applied Physics
& Energy Sciences Institute

millergroup.yale.edu/{people,publications,talks}

Shai Gertler

Andrew Shim

Hanwen Zhang

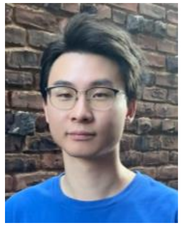
Lang Zhang

Wenjin Xue

Zeyu Kuang

Colin Christie

Zhaowei Dai



Hao Li

Bounds and optimal performance in linear EM

- Channel counting in wave volumes, w/ David Miller @Stanford

  Tunneling escape of waves, D. A. B. Miller, Z. Kuang, and O. D. Miller, *Nature Photonics* (Dec. 2024)

- Causality in nonreciprocal LTI systems

  In preparation, C. Christie & O. D. Miller

- Linear programming for high-power lasers, w/ Hui Cao & Doug Stone @Yale

  Optimal input excitations for suppressing nonlinear instabilities in multimode fibers, K. Wisal, C.-W. Chen, Z. Kuang, O. D. Miller, H. Cao, and A. D. Stone, *Optica* (Dec. 2024)

Communication Channels

D. A. B. Miller et al. 2000+

- Consider transmit and receive volumes:

- How many independent channels can one use to communicate on?

- Consider a basis of transmit and receive waves, excited with coefficient vectors c_{send} and c_{receive}. The matrix relating them is known:

$$c_{\text{receive}} = T c_{\text{send}}$$

- The number of independent degrees of freedom is captured in singular value decomposition (SVD) of T

D. A. B. Miller, Appl. Opt. 39, 1681 (2000); D. A. B. Miller, Adv. in Opt. and Phot. 11, 679 (2019)

Communication Channels

$$c_{\text{receive}} = T c_{\text{send}}$$

$$T = U \Sigma V^{\dagger}$$

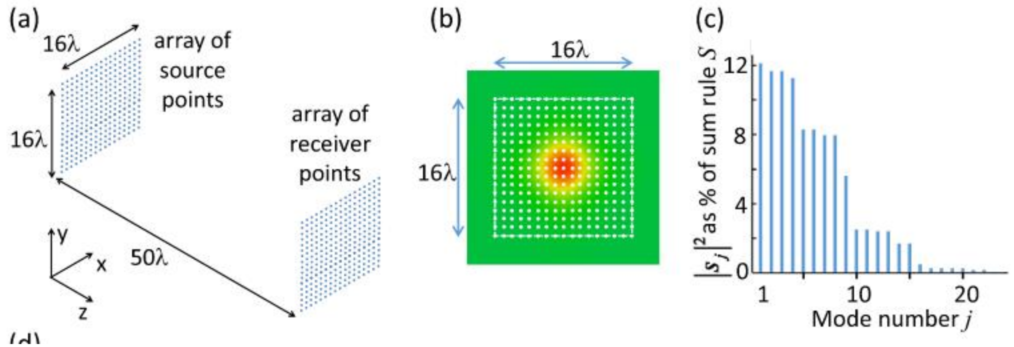
basis for received waves

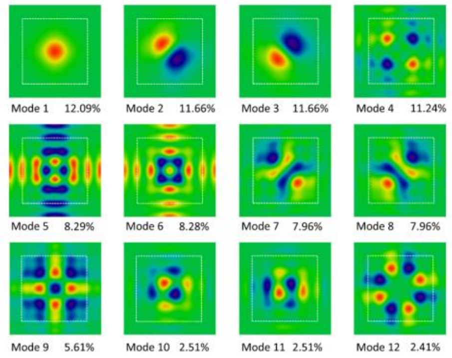
basis for transmitted waves

connection strengths between transmit eigenvectors and receive eigenvectors!

Can Σ have arbitrarily many nontrivial singular values? No!

Sum rule: $\mathrm{Tr}\left(T^{\dagger}T\right) = \int_{V_R}\int_{V_T} \|G(x_T, x_R)\|^2 dx_T dx_R = \mathrm{Tr}\left(\Sigma^{\dagger}\Sigma\right)$





D. A. B. Miller, Appl. Opt. 39, 1681 (2000); D. A. B. Miller, Adv. in Opt. and Phot. 11, 679 (2019)

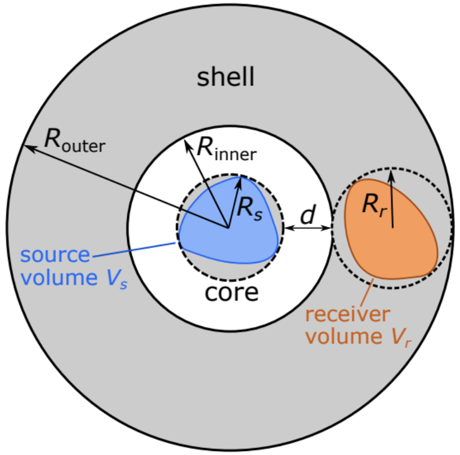
Bounds on the Coupling Strengths of Communication Channels and Their Information Capacities

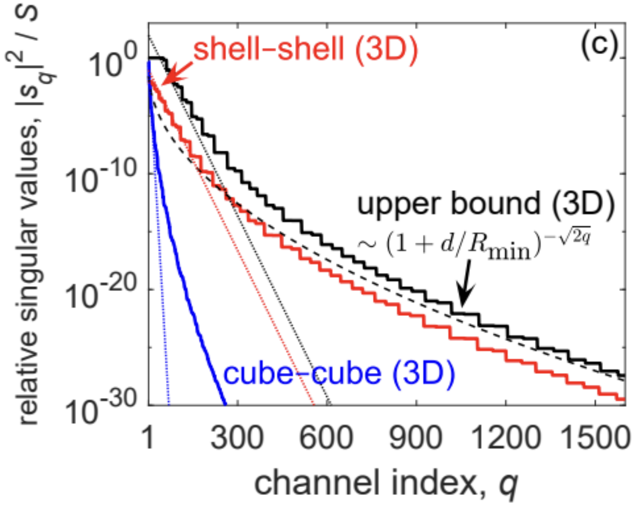
Zeyu Kuang,[1] David A. B. Miller,[2] and Owen D. Miller[1]
[1] Department of Applied Physics and Energy Sciences Institute, Yale University, New Haven, Connecticut 06511, USA
[2] Ginzton Laboratory, Stanford University, 348 Via Pueblo Mall, Stanford, California 94305-4088, USA
(Dated: May 12, 2022)

[near publication in *IEEE TAP*]

The singular values of the T matrix obey a monotonicity theorem

Semi-analytical bounds to communication strengths follow using spherical / shell bounding volumes:





$$\frac{|s_q|^2}{S} \leq \frac{2\pi^2 d_{\max}^2}{k^4 V_s V_r (1 + d/R_{\min})^{\sqrt{2q}+1}},$$

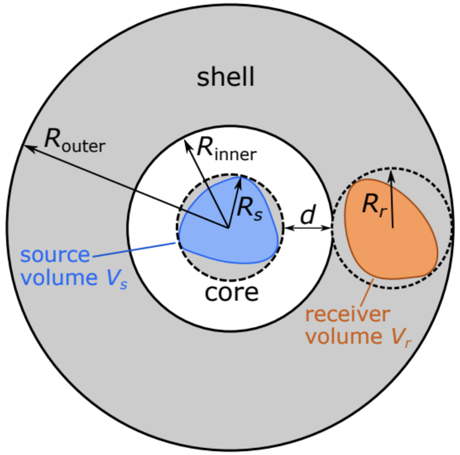
Bounds on the Coupling Strengths of Communication Channels and Their Information Capacities

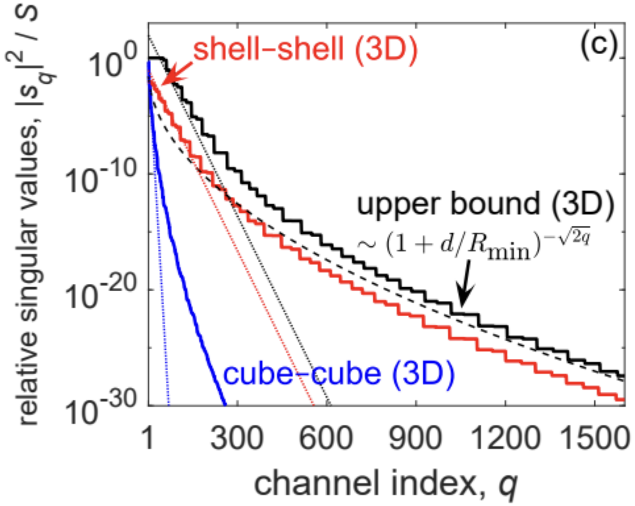
Zeyu Kuang,[1] David A. B. Miller,[2] and Owen D. Miller[1]
[1] *Department of Applied Physics and Energy Sciences Institute, Yale University, New Haven, Connecticut 06511, USA*
[2] *Ginzton Laboratory, Stanford University, 348 Via Pueblo Mall, Stanford, California 94305-4088, USA*
(Dated: May 12, 2022)

[near publication in *IEEE TAP*]

The singular values of the T matrix obey a monotonicity theorem

Semi-analytical bounds to communication strengths follow using spherical / shell bounding volumes:



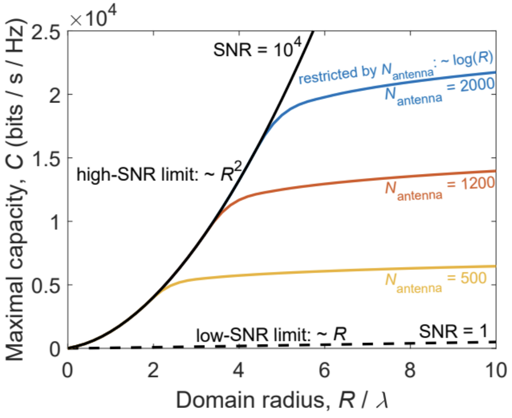


$$\frac{|s_q|^2}{S} \le \frac{2\pi^2 d_{\text{max}}^2}{k^4 V_s V_r (1+d/R_{\text{min}})^{\sqrt{2q}+1}},$$

Capacity bounds

$$C = \sum_{i=q}^{N} \log_2\left(1 + \frac{P_q |s_q|^2}{P_{\text{noise}}}\right) \quad \text{bits/s/Hz},$$

$$C \le 2\log_2(\text{SNR}) k^2 R^2.$$



Bounds on the Coupling Strengths of Communication Channels and Their Information Capacities

Zeyu Kuang,[1] David A. B. Miller,[2] and Owen D. Miller[1]

[1] *Department of Applied Physics and Energy Sciences Institute, Yale University, New Haven, Connecticut 06511, USA*

[2] *Ginzton Laboratory, Stanford University, 348 Via Pueblo Mall, Stanford, California 94305-4088, USA*

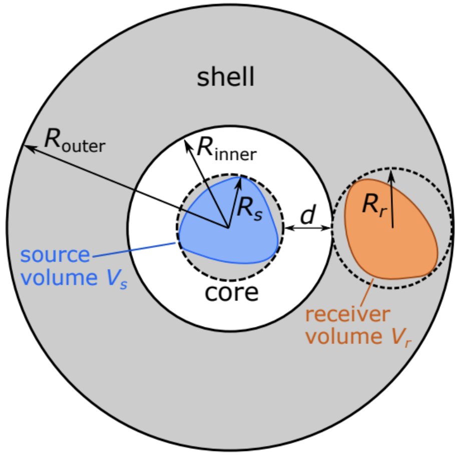
(Dated: May 12, 2022)

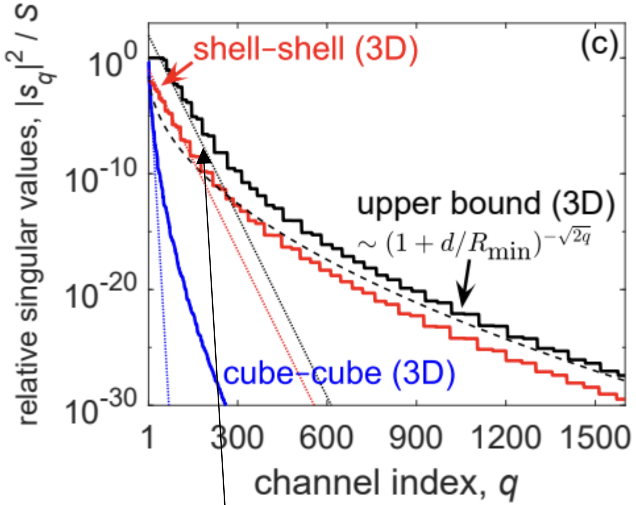
[near publication in *IEEE TAP*]

The singular values of the T matrix obey a monotonicity theorem

Semi-analytical bounds to communication strengths follow using spherical / shell bounding volumes:

$$\frac{|s_q|^2}{S} \leq \frac{2\pi^2 d_{\max}^2}{k^4 V_s V_r (1 + d/R_{\min})^{\sqrt{2q}+1}},$$





Channel strength scaling

Channel strength falls off quasi-exponentially after threshold.
Why?

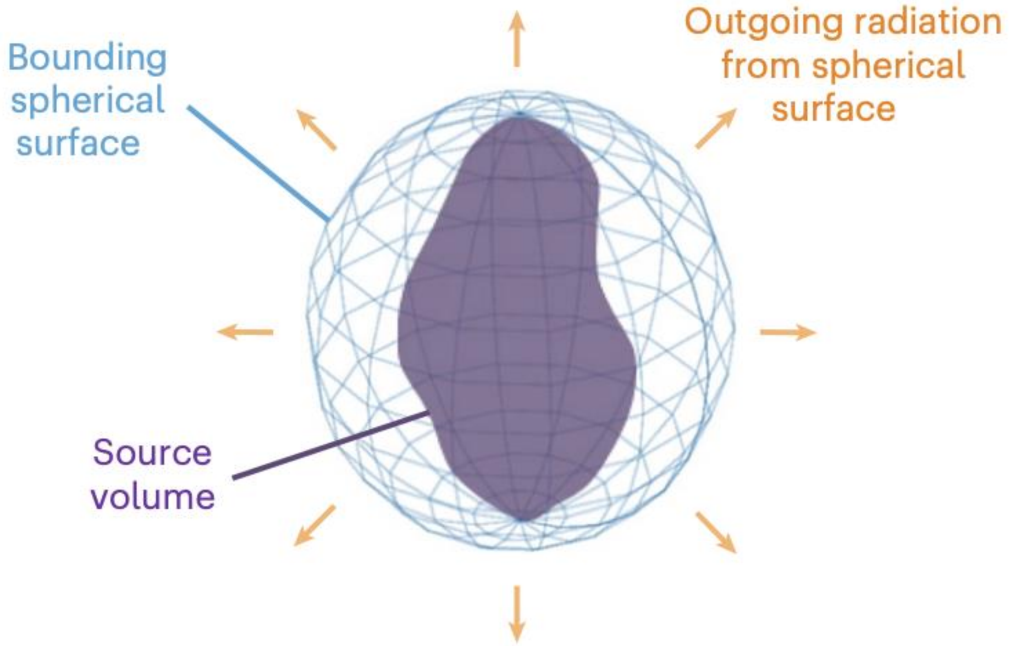
Waves from arbitrary volumes

There are a finite number of well-coupled channels from any volume
(before exponential fall-off)

Classical “diffraction” explanation suffices in paraxial scenarios

- Waves “miss” receiver
- But what about enclosed volumes?

We find an upper limit to the number of channels by surrounding the volume with a spherical bounding surface, and counting the maximum number of well-coupled waves from this surface



Tunneling in the 1D Schrodinger equation

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

$$\Downarrow$$

$$\frac{d^2}{dx^2}\psi(x)=[V(x)-E]\psi(x)$$

$V(x) > E$: exponential growth/decay (e.g., tunneling)
$V(x) < E$: oscillations

Scalar wave "tunneling"

$$\nabla^2 U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$$

$\Downarrow$ plane-wave solutions

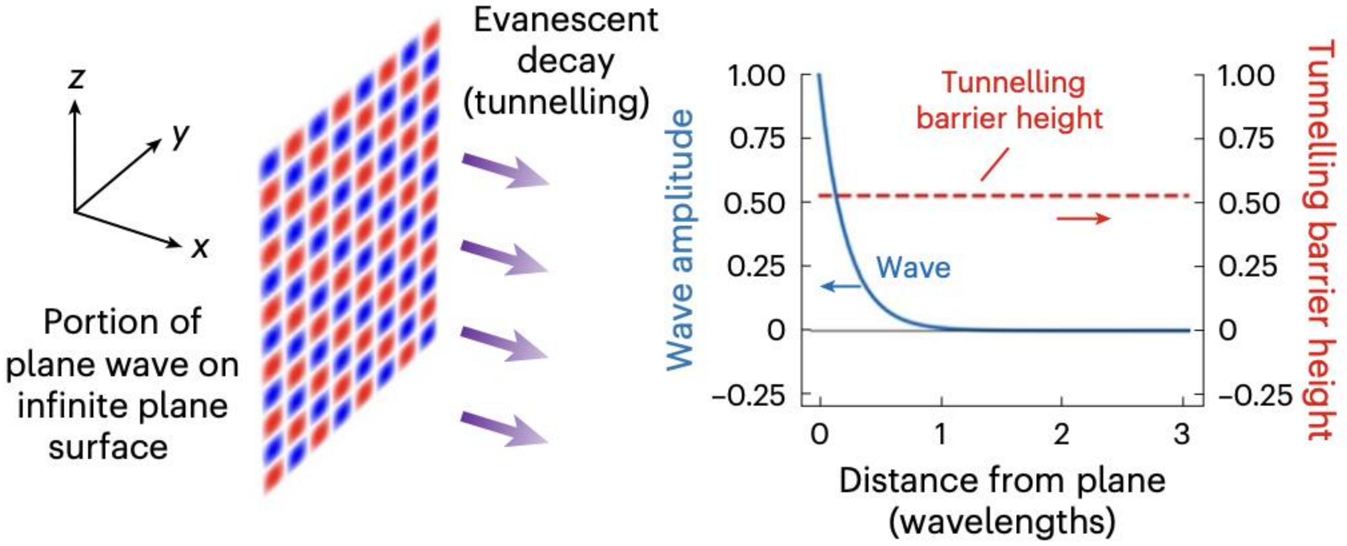
$$\frac{d^2}{d\rho^2} U(x) = [V - E] U(x)$$

$\rho = k_x x$

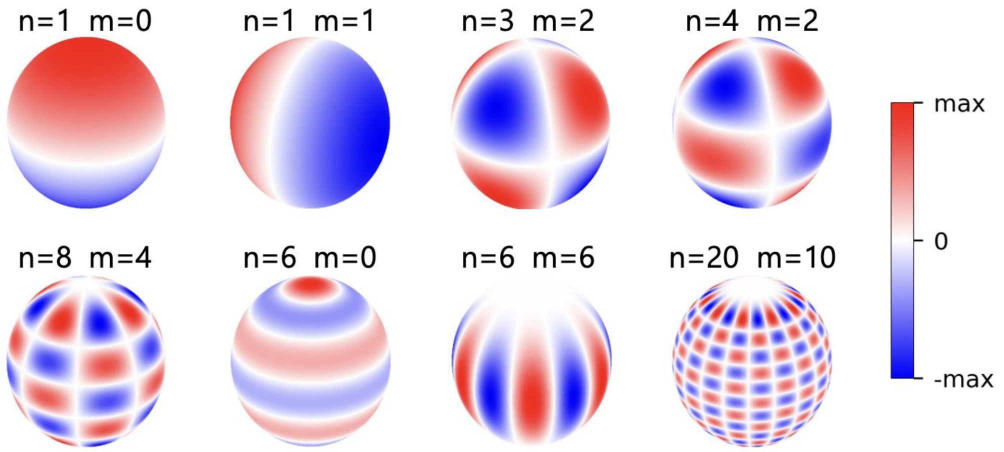
$V = (k_y^2 + k_z^2)/k^2$

$E = 1$ "eigen-energy"

$V - E$ tunneling "barrier height"



Scalar spherical waves



$$\nabla^2 U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$$

$$\Downarrow$$

$$U_{nm}(\boldsymbol{r}) = z_n(kr) Y_{nm}(\theta, \phi)$$

Y_{nm}: spherical harmonics

n nodal circles,

$|m|$ of which cut through poles

$z_n(kr)$ are the spherical Bessel functions,

which satisfy

$$\rho^2 \frac{d^2 z_n(\rho)}{d\rho^2} + 2\rho \frac{dz_n(\rho)}{d\rho} + [\rho^2 - n(n+1)] z_n(\rho) = 0 \qquad \rho = kr = \frac{2\pi r}{\lambda}$$

The Riccati-Bessel "Schrodinger" equation

We can recast the spherical Bessel functions, removing 1/radius asymptotic behavior. Such functions are known as the Riccati-Bessel functions ζ_n, which satisfy the Riccati-Bessel differential equation

$$\rho^2 \frac{d^2\zeta_n}{d\rho^2} + [\rho^2 - n(n+1)]\zeta_n = 0$$

Which we can rearrange to the form

$$-\frac{d^2\zeta_n}{d\rho^2} + \frac{n(n+1)}{\rho^2}\zeta_n = \zeta_n$$

This is a Schrodinger equation!
With effective radial "potential" $V(\rho) = \dfrac{n(n+1)}{\rho^2}$ and "eigen-energy" $E = 1$

Tunneling escape and escape radius

If the "potential energy" exceeds the "total energy," i.e., if

$$\frac{n(n+1)}{\rho^2} > 1 \qquad \text{or, equivalently,} \qquad n(n+1) > \rho^2$$

Then the wave will be tunneling rather than propagating.

So, for each n, there is an "escape radius"

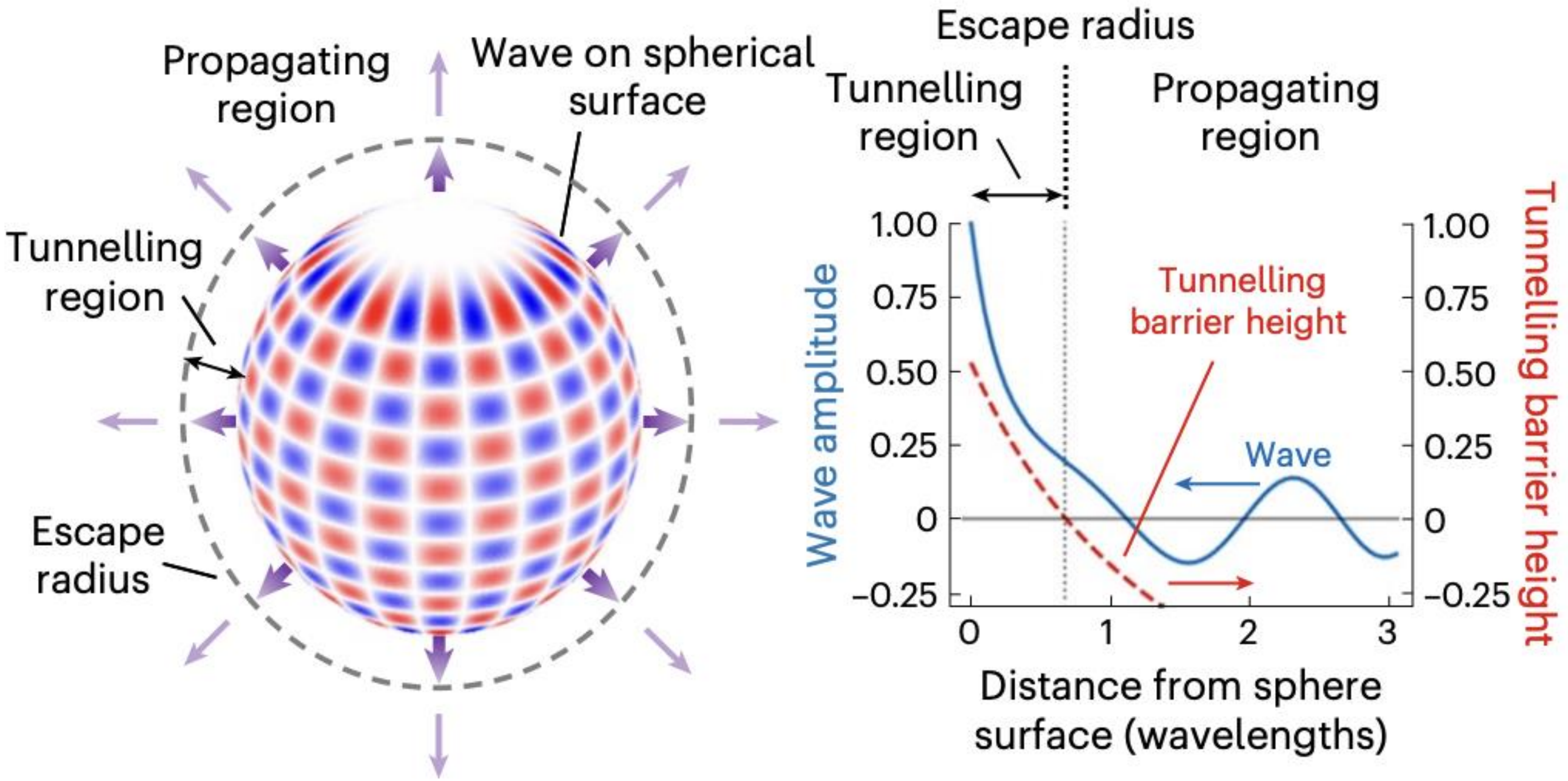
$$\boxed{\rho_{\mathrm{esc},n} = \sqrt{n(n+1)}}$$

Or, equivalently, in dimensioned form

$$\boxed{r_{\mathrm{esc},n} = \frac{\sqrt{n(n+1)}}{k} = \frac{\lambda_0}{2\pi}\sqrt{n(n+1)}}$$

Spherical escaping waves

The barrier height *falls* with distance for spherical waves (!)



$n = 22$
$m = 12$
$r_0 = 2.9\lambda$
$r_{\text{esc}} \approx 3.58\lambda$

$$r_{\text{esc},n} = \frac{\lambda_0}{2\pi}\sqrt{n(n+1)} \longrightarrow$$

At a given radius, there is a finite number of channels that do not have to tunnel. All others must tunnel.

Channel counting

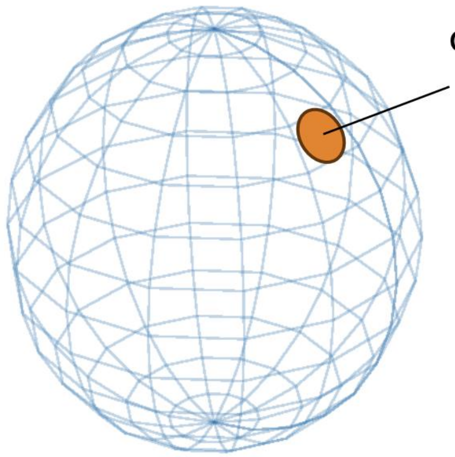
For a radius r_0, the largest value n_p of n that satisfies $\sqrt{n(n+1)} < kr_0$ is

$$n_p(r_0) = \text{floor}\left[\sqrt{1/4 + N_{SH}}\,\right]$$

where N_{SH} is a "spherical heuristic number"

$$N_{SH} = (kr_0)^2 = \frac{4\pi r_0^2}{\lambda_0^2/\pi} = \frac{A_S}{\lambda_0^2/\pi}$$

A_S = area of spherical surface



Then the total number of well-coupled (scalar) waves is

$$\boxed{N_{ps} = \left(n_p + 1\right)^2}$$

The diffraction limit of a volume

We can now construct a precise definition of the “diffraction limit” of a volume:

> For a wave interacting with a volume
>
> the wave passes the diffraction limit
>
> if any spherical component of the wave must tunnel to enter or leave the bounding spherical surface enclosing the volume

More consequences & conclusions

Electromagnetic waves have 2 polarizations and no $n = 0$ channel

- Proper counting yields per-polarization well-coupled “channel” count of

$$N_{ps} = n_p(n_p + 2)$$

- For r_0 less than approx. 0.225λ, there are no well-coupled channels (consistent with Wheeler-Chu limit)
- Scalar waves always have at least 1 well-coupled channel

Based on onset of spherical wave tunneling, this approach gives clear intuition that

- Explains how many waves can easily get in or out of a volume…
- … and why the fall-off is so abrupt past this number
- Gives a rigorous and precise diffraction limit definition
- Can also derive previous heuristic results

Tunneling escape of waves, D. A. B. Miller, Z. Kuang, and O. D. Miller, *Nature Photonics* (Dec. 2024)

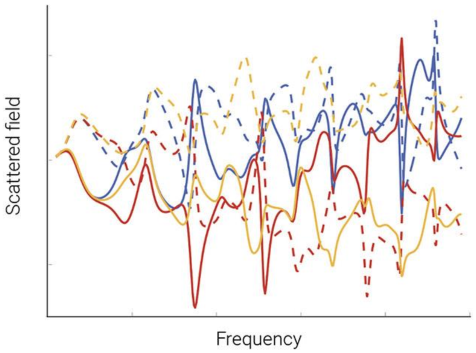
Extending causality-based bounds in electromagnetism

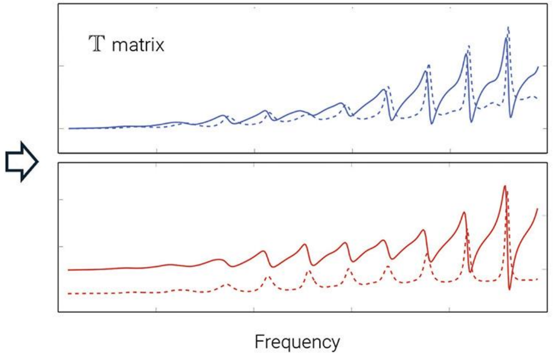
Pioneering works: Bode-Fano, Purcell, Gustafsson, Rozanov

2018: Near-field power-bandwidth limits *(Phys Rev X)*

2021: Refractive index bounds *(Advanced Materials)*

2023: EM scatterers as matrix-valued oscillators (*Nature Communications*)

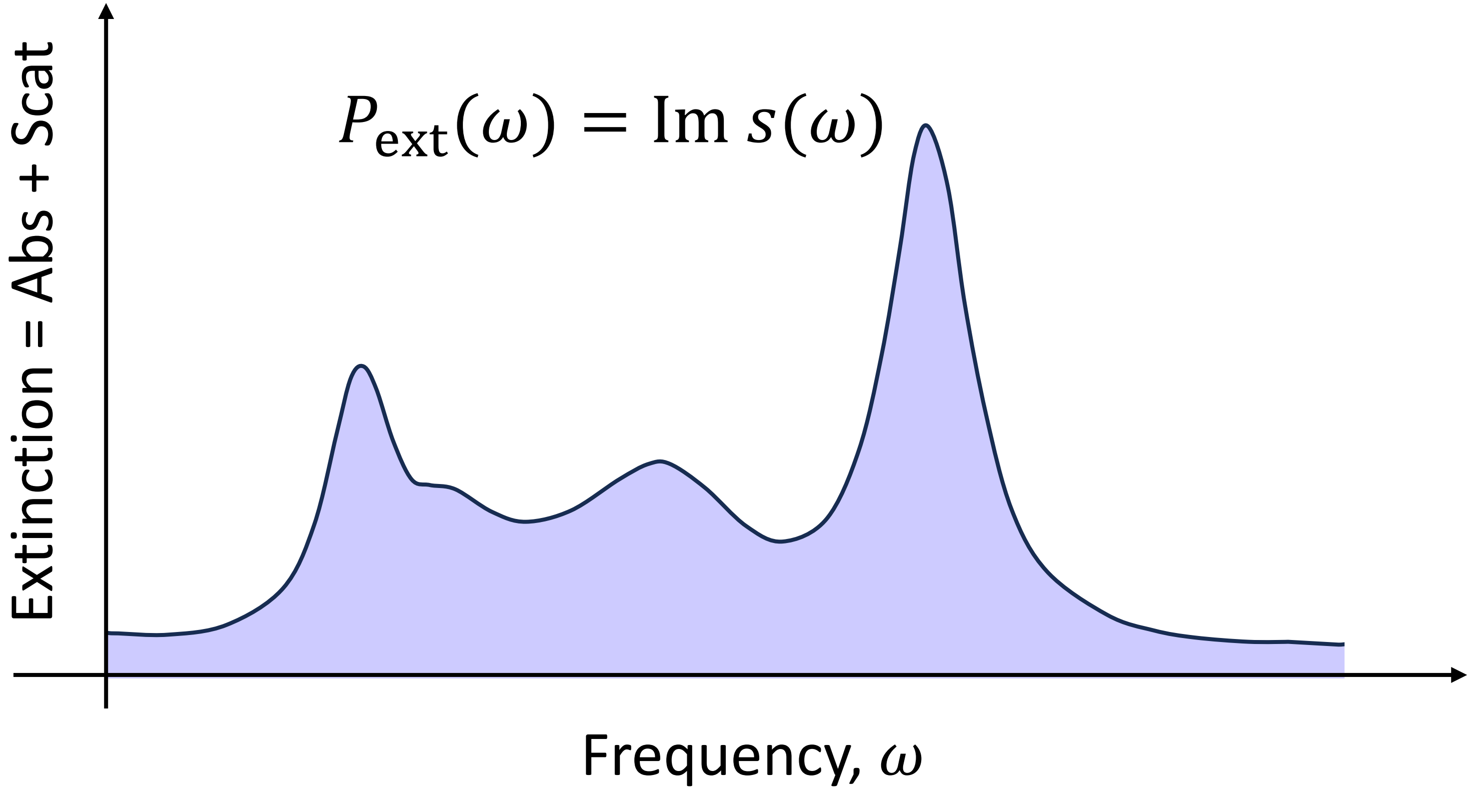




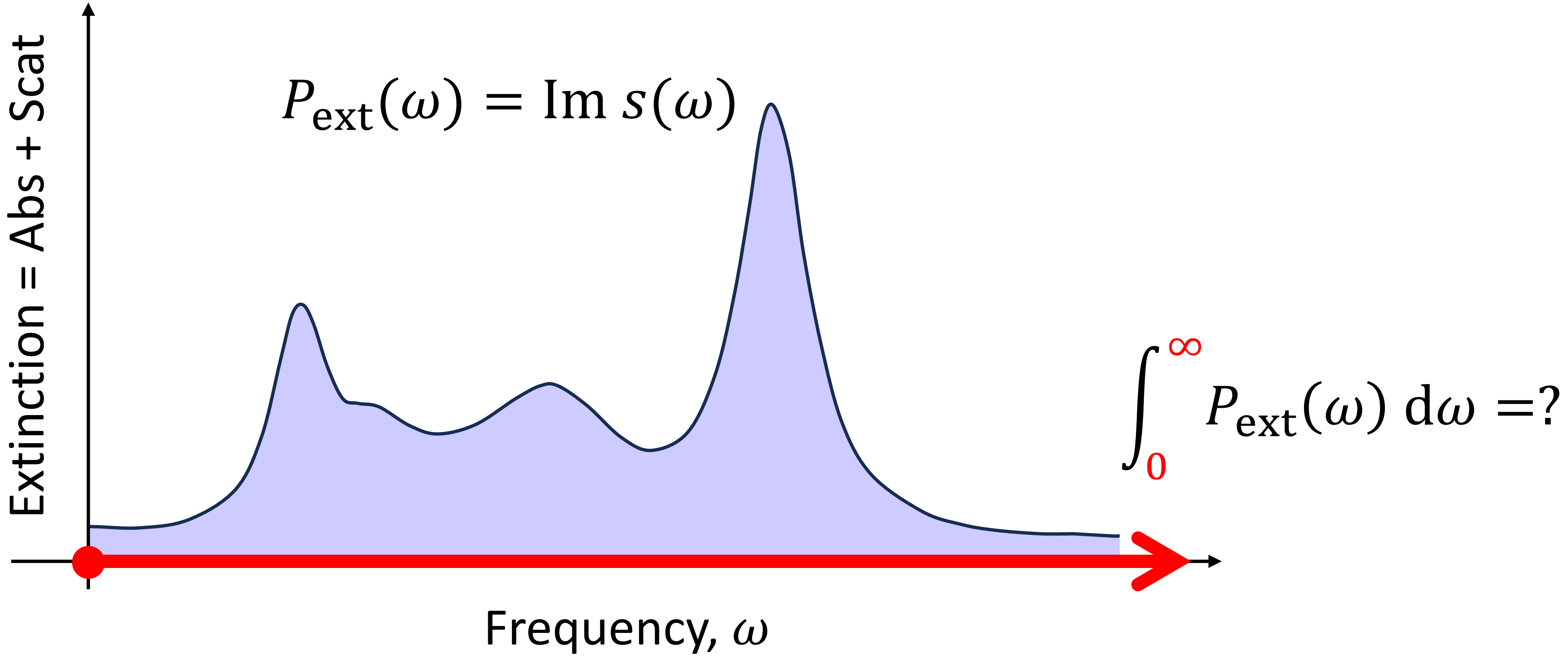
2024 *Optics & Photonics News* Research Highlight

Today: Nonreciprocal, linear time-invariant (LTI) systems

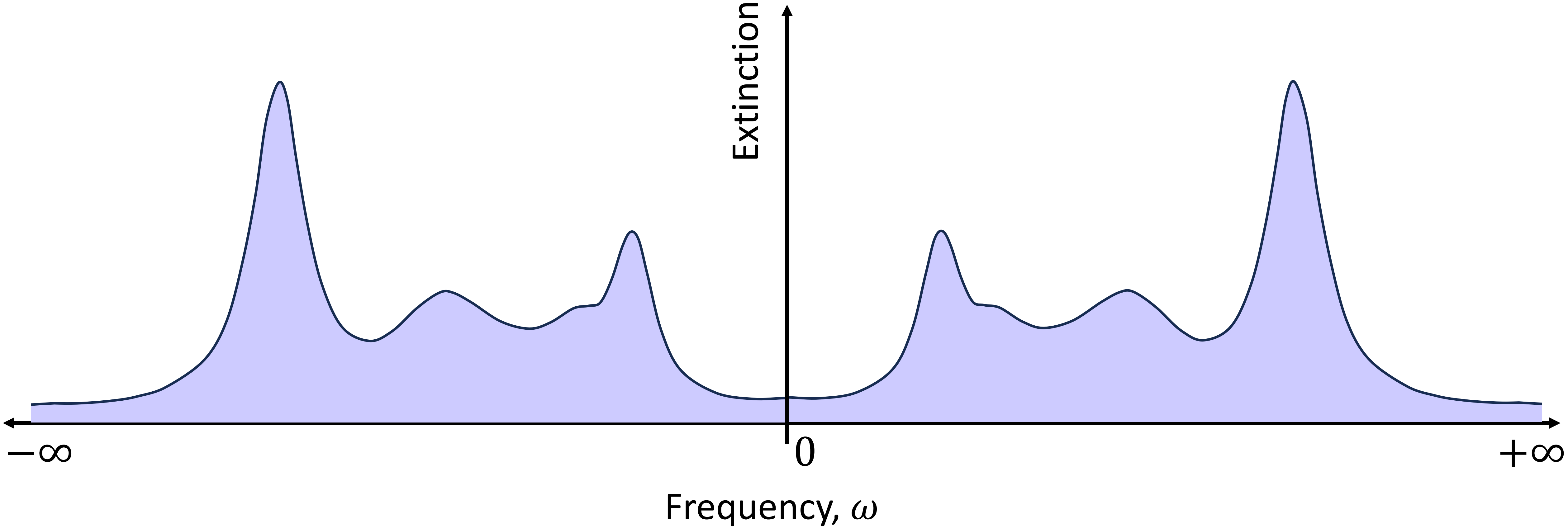
Causality-based sum rules and bounds



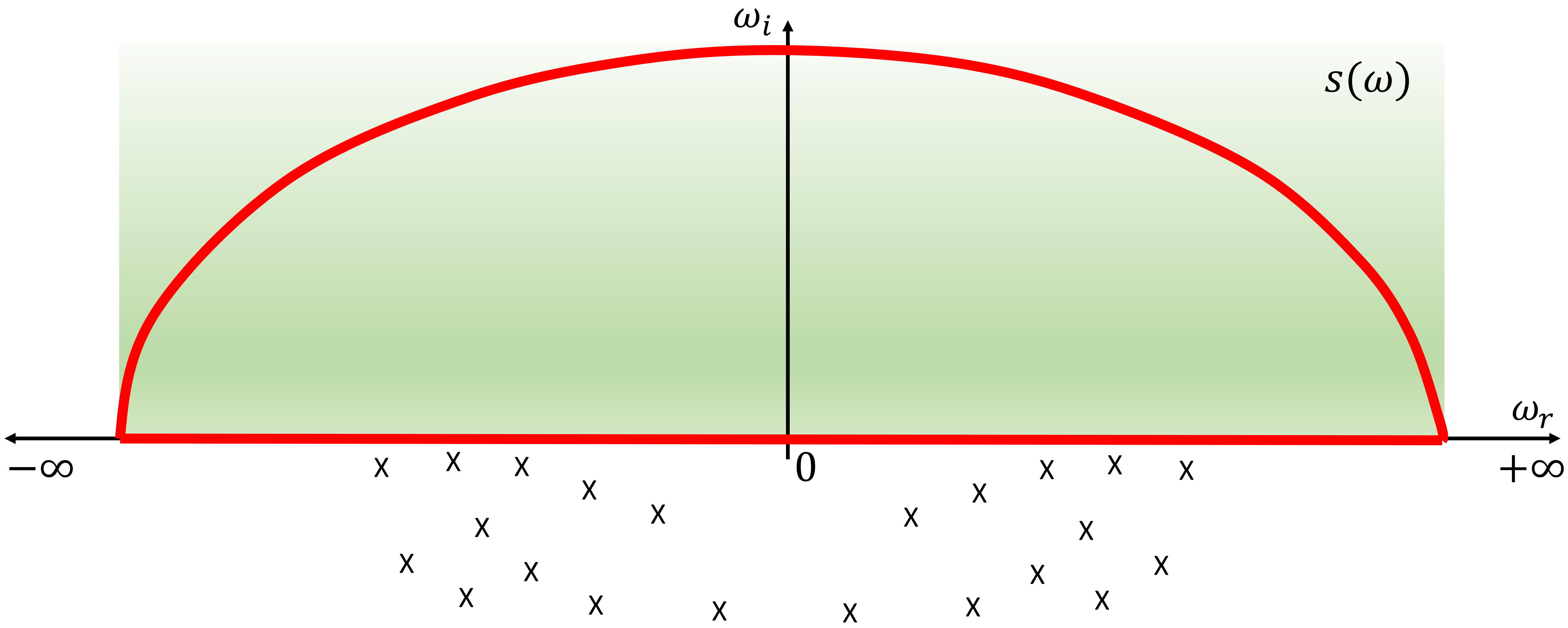
Causality-based sum rules and bounds



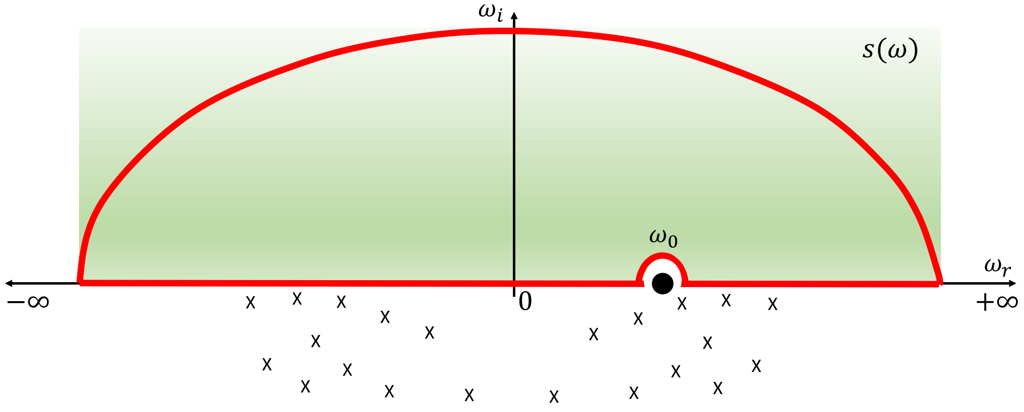
Causality-based sum rules and bounds



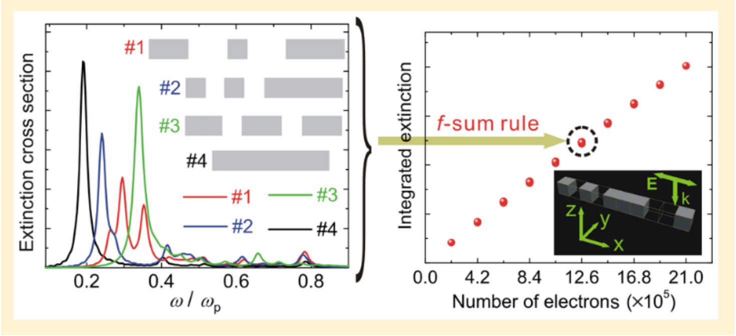
Causality-based sum rules and bounds



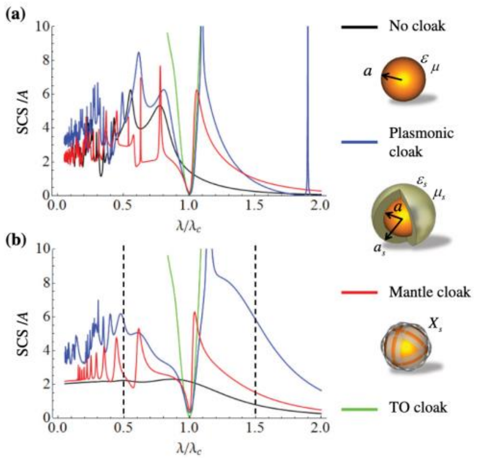
Causality-based sum rules and bounds



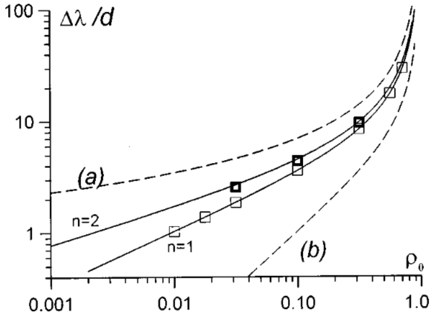
$$\int \sigma_{\text{ext}}(\omega)\,\mathrm{d}\omega = \frac{\pi e^2}{2\varepsilon_0 m_e c} N_e$$



R Gordon JCP 38, 1724 (1963)
ZJ Yong et al. Nano Lett. 15, 7633 (2015)



Monticone & Alu, “Do Cloaked Objects Really Scatter Less?,” Phys. Rev. X 3, 041005 (2013)



$$\left| \int_0^\infty \ln|\rho(\lambda)|\, d\lambda \right| \leq 2\pi^2 \mu_s\, d$$

Rozanov, IEEE TAP 48, 8 (2000)

Arbitrary anisotropic response function $\chi(\omega)$

$$\mathbb{X}(\omega) = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

Exemplar objective: maximum nonreciprocity, e.g. max $\int X_{12}(\omega)\,\mathrm{d}\omega$

$$\int_0^\infty \mathrm{Im}\, X_{ij}(\omega) d\omega = \frac{\pi\omega_p^2}{2}\delta_{ij}$$

$$X_{ij}(-\omega^*) = X_{ij}^*(\omega)$$

Altarelli et al. PRB 1972

Issues: no positivity in off-diagonal components, integrand oscillates, total integral equals zero, …

Resolution:

- Entrywise components are the wrong place to look for bounds
- Instead, separate the Hermitian and anti-Hermitian parts of the matrix:

$$\mathbb{X}(\omega) = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \mathbb{H} + i\mathbb{A} \qquad \mathbb{H} = \frac{\chi + \chi^\dagger}{2} \qquad \mathbb{A} = \frac{\chi - \chi^\dagger}{2i}$$

Crossing symmetry? $X_{ij}(-\omega^*) = X_{ij}^*(\omega)$

$$[\mathbb{H}(-\omega^*)]_{ij} = \frac{X_{ij}(-\omega^*) + X_{ji}^*(-\omega^*)}{2} = \frac{X_{ij}^*(\omega) + X_{ji}(\omega)}{2} \neq \left[\mathbb{H}^\dagger(\omega)\right]_{ij}$$

Resolution:

- Entrywise components are the wrong place to look for bounds
- ~~Instead, separate the Hermitian and anti-Hermitian parts of the matrix~~
- Instead, separate the reciprocal and nonreciprocal parts of the matrix

$$\mathbb{X}(\omega) = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \mathbb{R}(\omega) + \mathbb{N}(\omega) \qquad \mathbb{R} = \frac{\mathbb{X} + \mathbb{X}^T}{2} \qquad \mathbb{N} = \frac{\mathbb{X} - \mathbb{X}^T}{2}$$

Crossing symmetries:

$$\mathbb{R}(-\omega^*) = \mathbb{R}^\dagger(\omega)$$

$$\mathbb{N}(-\omega^*) = -\mathbb{N}^\dagger(\omega)$$

KK relations

$$\mathrm{Re}\,\mathbb{R}(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega' \mathrm{Im}\,\mathbb{R}(\omega')}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega'$$

Reciprocal part

$$\mathrm{Re}\,[\omega\mathbb{N}(\omega)] = \frac{2}{\pi}\int_0^\infty \frac{(\omega')^2\,\mathrm{Im}\,\mathbb{N}(\omega')}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega'$$

Nonreciprocal part

Sum rules

$$\int_0^\infty \omega\,\mathrm{Im}\,\mathbb{R}(\omega)\,\mathrm{d}\omega = \frac{\pi\omega_p^2}{2}\,\mathbb{I}$$

$$\int_0^\infty \omega^2\,\mathrm{Im}\,\mathbb{N}(\omega)\,\mathrm{d}\omega = \frac{\pi\omega_p^2\omega_c}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

(for example)

Passivity

$$\omega\,\mathrm{Im}\,\mathbb{R}(\omega) \geq 0$$

[real ω]

$$-\omega\,\mathrm{Im}\,\mathbb{R}(\omega) \leq \omega\,\mathrm{Im}\,\mathbb{N}(\omega) \leq \omega\,\mathrm{Im}\,\mathbb{R}(\omega)$$

KK relations

$$\mathrm{Re}\,\mathbb{R}(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega' \mathrm{Im}\,\mathbb{R}(\omega')}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega'$$

Reciprocal part

$$\mathrm{Re}\,[\omega\mathbb{N}(\omega)] = \frac{2}{\pi}\int_0^\infty \frac{(\omega')^2\,\mathrm{Im}\,\mathbb{N}(\omega')}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega'$$

Nonreciprocal part

Sum rules

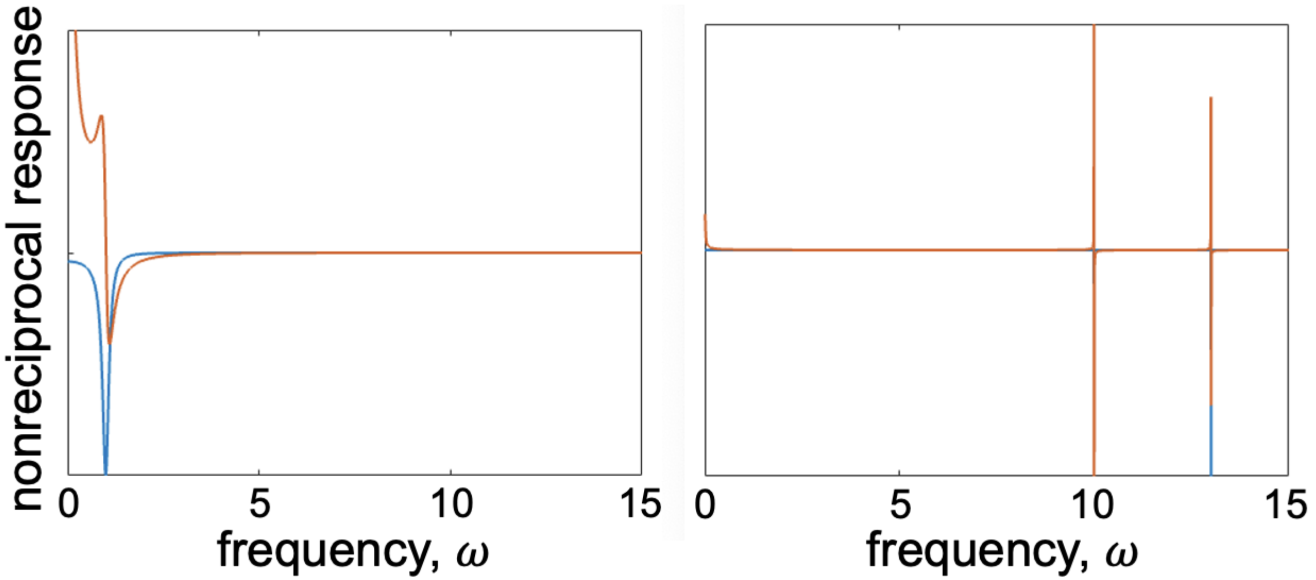
$$\int_0^\infty \omega^2\,\mathrm{Im}\,\mathbb{N}(\omega)\,\mathrm{d}\omega = \frac{\pi\omega_p^2\omega_c}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Extinction nonreciprocity:

$$\propto i\int [\mathrm{Im}\,\mathbb{N}(\omega)]_{12}\,\mathrm{d}\omega$$

Sum rule:

$$i\int \omega^2 [\mathrm{Im}\,\mathbb{N}(\omega)]_{12}\,\mathrm{d}\omega = \pm\frac{\pi\omega_p^2\omega_c}{2}$$



High-Power Fiber Amplifiers

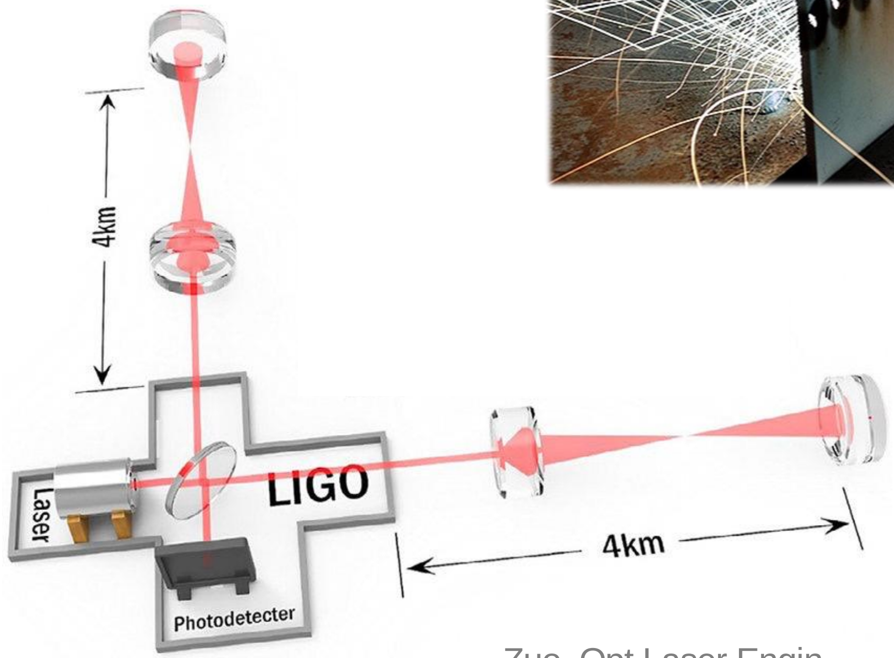
Nonlinearities problematic?

Applications

Laser processing

IPG Photonics

Laser interferometer

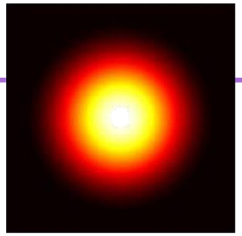


Zuo, Opt Laser Engin, 106187 (2020)

NEED

- High average power
- Narrow linewidth
- **Good beam quality**

(Easy to focus, collimate, shape)



Defense

Lockheed Martin

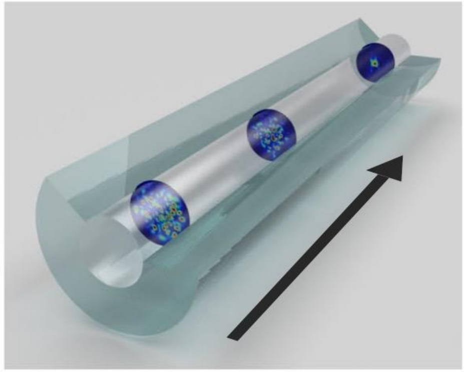
Nonlinear Scattering in Multimode Fibers

❑ Electronic vs mech/thermal nonlinearities are dominant under different excitation conditions

- Pulsed excitation, broadband, high peak power => electronic nonlinearities (Kerr) dominate
- Regime of eventual thermalization and spatial beam cleaning
- CW narrowband excitation in amplifying fiber => **mechanical (SBS) and thermo-optic (TMI) nonlinearities dominate.**
- These nonlinearities don't conserve momentum and lead to propagation instabilities
- Most applications thought to require single-mode fiber (beam quality)
- Propose using multimode fiber and wavefront shaping to mitigate instabilities

Spatial self beam cleaning

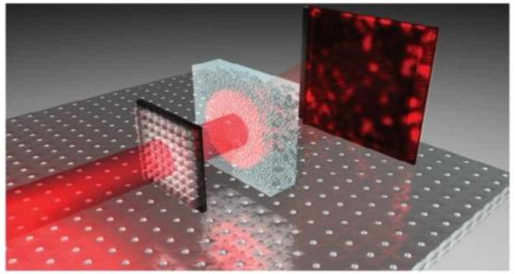
a

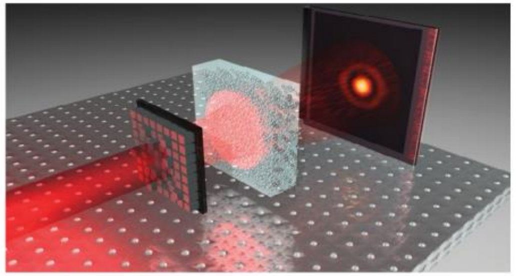


Wu et al. *Nature Photonics*, *13*(11), pp.776-782.

- WFS with SLM studied extensively in linear scattering systems
- Less so in nonlinear cases => no transmission matrix

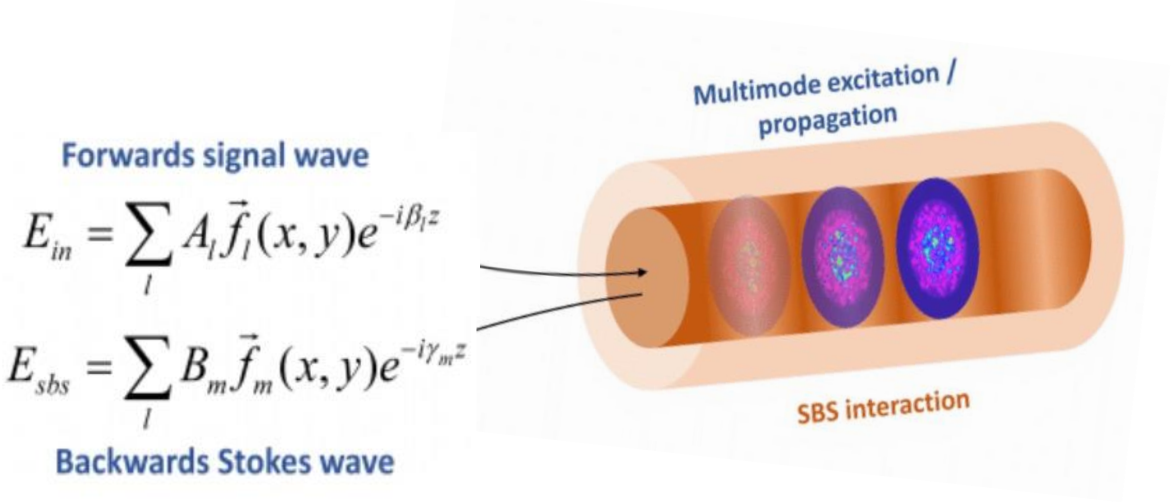
WFS in linear systems





Cao et al. Nature Physics 18, no. 9 (2022): 994-1007.

SBS Theory: Multimode Excitation



Assumptions: translational invariance in z-direction and paraxial approximation

We derive growth equations for B_m in terms of input wavefront and fiber properties

A series of well-justified approximations makes this multiscale problem semi-analytically tractable

Optical Equation

$$[\nabla^2 - \frac{n^2\omega^2}{c^2}]\vec{E} = \mu_0\omega^2(\overleftrightarrow{\chi}_N.\vec{E})$$

$$\overleftrightarrow{\chi}_N = \overleftrightarrow{\pi} : \nabla \otimes \vec{u}$$

coupled

Acoustic Equation

$$V_L^2\nabla(\nabla.\vec{u}) + V_s^2\nabla\times\nabla\times\vec{u} + \rho_0\Omega^2\vec{u} = \vec{F}$$

$$\vec{F} = -\frac{1}{2}\nabla.[\overleftrightarrow{\pi} : \vec{E}\otimes\vec{E}]$$

These equations simplify for isotropic fiber but tensor character remains in shear terms, small for Silica

Multimode Stokes Growth

Wisal et al., PRX 2024, detailed theory

$$\frac{dB_m(\Omega)}{dz} = \sum_{i,j,l} Y_{mlij}(\Omega) \underbrace{A_l A_i^*}_{\text{parameters}} B_j e^{i(\beta_i+\gamma_j-\beta_l-\gamma_m)z}$$

Linear Growth Equations

$$\cancel{-\frac{dA_l(\Omega)}{dz} = \sum_{i,j,m} Y_{mlij}(\Omega) B_m B_j^* A_i e^{-i(\beta_i+\gamma_j-\beta_l-\gamma_m)z}.}$$

$$\frac{dB_m(\Omega, z)}{dz} = \Bigg[\underbrace{\sum_l Y_{mlml}(\Omega) |A_l|^2}_{G_m} \Bigg] B_m(\Omega, z) \Rightarrow \frac{d\,P_s^m}{dz} = -G_m\, P_s^m$$

1) Solve for acoustic modes and integrate them out – cubic nonlinearity

2) Undepleted signal approximation

3) Phase Matching $(\beta_i + \gamma_j - \beta_l - \gamma_m = 0)$

4) G_m depends input signal power distribution (but not on phase) – phase can be used for WFS output!
5) Problem is unstable to SBS from zero input power – no onset threshold
6) Growth from noise => “threshold” is conventional

Multimode SBS theory: Phase Matched

- Exponential growth in Stokes Power in each mode

$$\frac{\square_s^m}{} = - \; _m \square_s^m$$

- Growth rate depends on signal power distribution and **SBS gain spectra**

$$_m(\Omega\{\ _l\}) = \sum_l \underbrace{\square_B^{(m,l)}}_{\text{Determined by fiber properties}} \underbrace{\square_l}_{\text{Controlled by input profile}}$$

- Resonant Gain spectrum with offset peaks, strength depends on acousto-optic modal overlap calculable

$$_B^{lm}(\Omega) = \ _0 \sum_k \frac{\Gamma_B | \ _{mlk}|^2}{(\Omega - \Omega_{mlk})^2 + \Gamma_B^2}$$

$$_{mlk} = \langle \vec{f}_l^{(1)} . \vec{f}_m^{(2)} \vec{g}_z^k \rangle$$

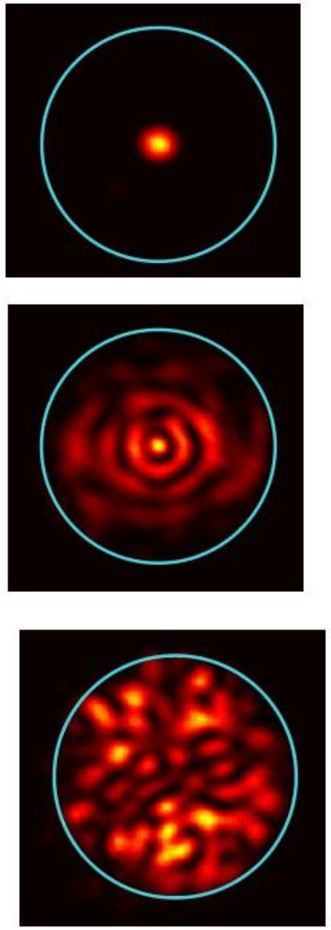
MM excitation: Experimental data, passive fiber

Far fi

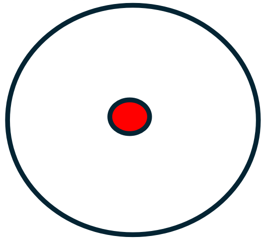
FM
-
onl
y

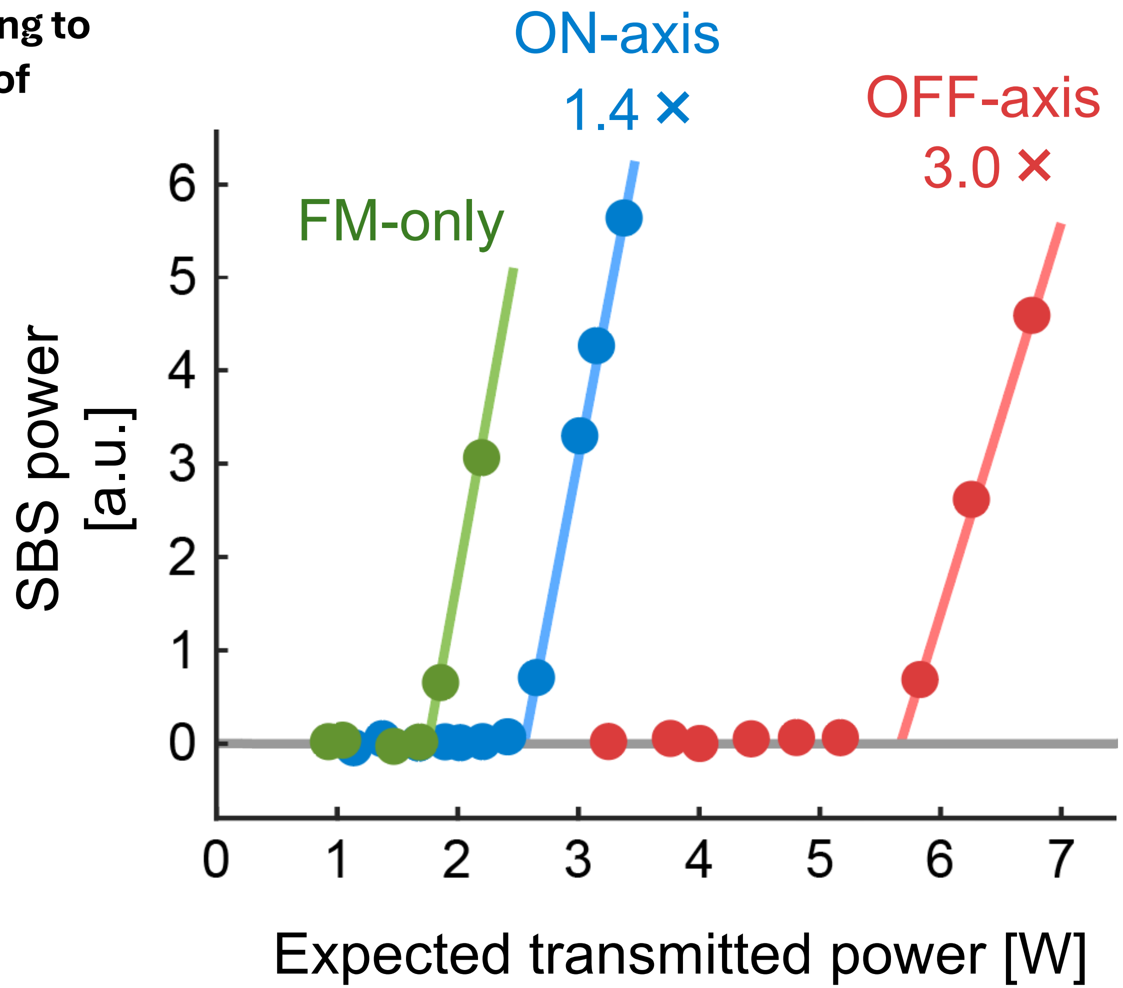
ON-axis focusing

OFF-axis focusing



Use input focusing to control number of modes





*Step-index fiber, core: 20 μm, NA = 0.3, length: 50 m

Chen et al, *Nature Communications* 14, no. 1 (2023): 7343.

Optimal Excitation

"Optimal input excitation for SBS suppression in multimode fibers",
K. Wisal, C-W Chen. Z. Kuang, S. Warrensmith, Owen Miller,
Hui Cao, A. Douglas Stone (Optica 11, 1663, Dec 2024)

$$P_s^m(0) = P_s^m(L)e^{G_m L} \qquad G_m(\Omega, \{P_l\}) = \sum_l G_B^{lm}(\Omega)\, P_l$$

Effective SBS Gain

$$G_B = \max_{m,\Omega} G_m(\Omega) = \max_{m,\Omega} \sum_l G_B^{lm}(\Omega) P_l$$

Objective

$$\min_{\{P_l\}} G_B = \min_{\{P_l\}} [\, \max_{m,\Omega} \sum_l G_B^{lm}(\Omega) P_l]$$

Constraints

$$\sum_l P_l = 1 \quad , \quad P_l \geq 0 \quad , \quad l = 1,2,3 \ldots M$$

Linear Programming

"Optimal input excitation for SBS suppression in multimode fibers",
K. Wisal, C-W Chen. Z. Kuang, S. Warrensmith, Owen Miller,
Hui Cao, A. Douglas Stone (Optica 11, 1663, Dec 2024)

$$\min_{\{P_l\}}[\max_{m,\Omega}\sum_l G_B^{lm}(\Omega)P_l]$$

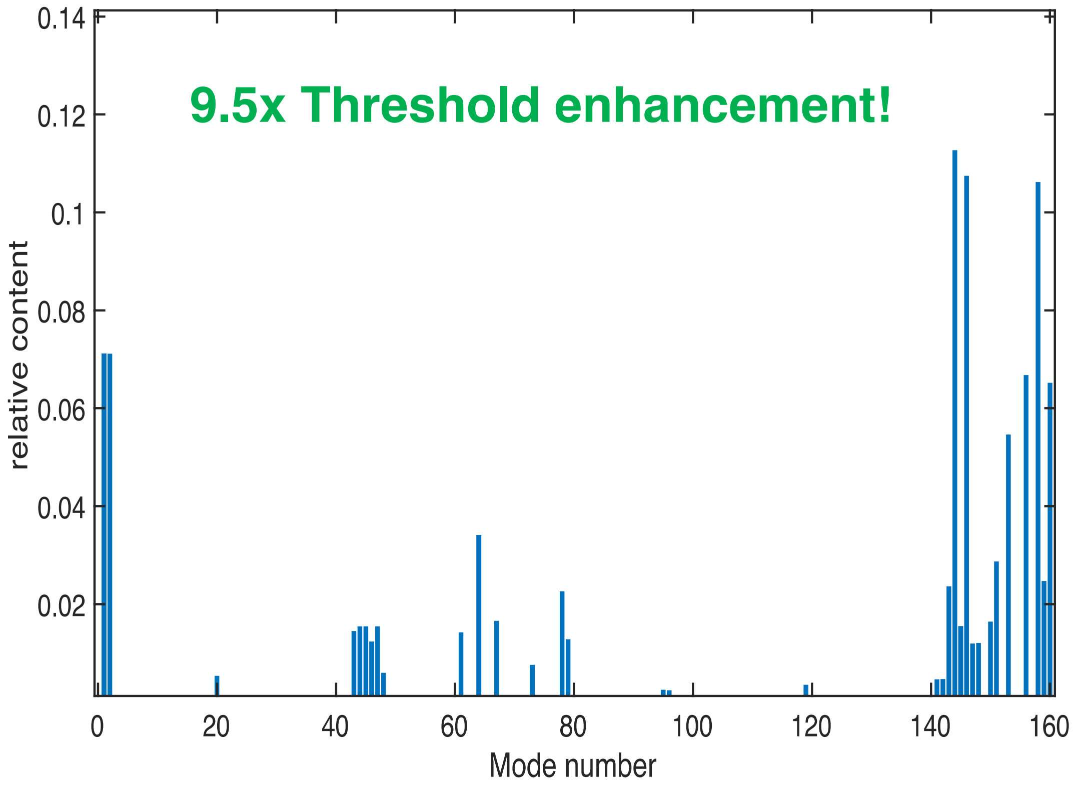
Introduce a slack variable t

$$\text{Minimize}_{P^1,P^2,\dots,P^N,t} \ t$$

$$\text{subject to } \sum_{l=1}^{N} P^l = P_0$$
$$\sum_{l=1}^{N} G_B^{lm} P^l \le t$$
$$P^l \ge 0$$

- **Power spread across multiple groups of modes**
- **More weight to Higher Order Modes**

9.5x Threshold enhancement!



Optimal Mode Content

Looking forward

Showed threshold improvement under SBS. Also found:

- Threshold improvements under transverse mode instabilities (TMI)
- Threshold improvements subject to SBS & TMI simultaneously
- Input robustness: phase-only SLM okay!

More generally:

- Rich interplay of spatial and spectral DOFs in multimode fibers
- Linear control of nonlinear thresholds
- Progress towards highest-power single-frequency fiber lasers

Optimal input excitations for suppressing nonlinear instabilities in multimode fibers
K. Wisal, C.-W. Chen, Z. Kuang, O. D. Miller, H. Cao, and A. D. Stone, *Optica* (Dec. 2024)