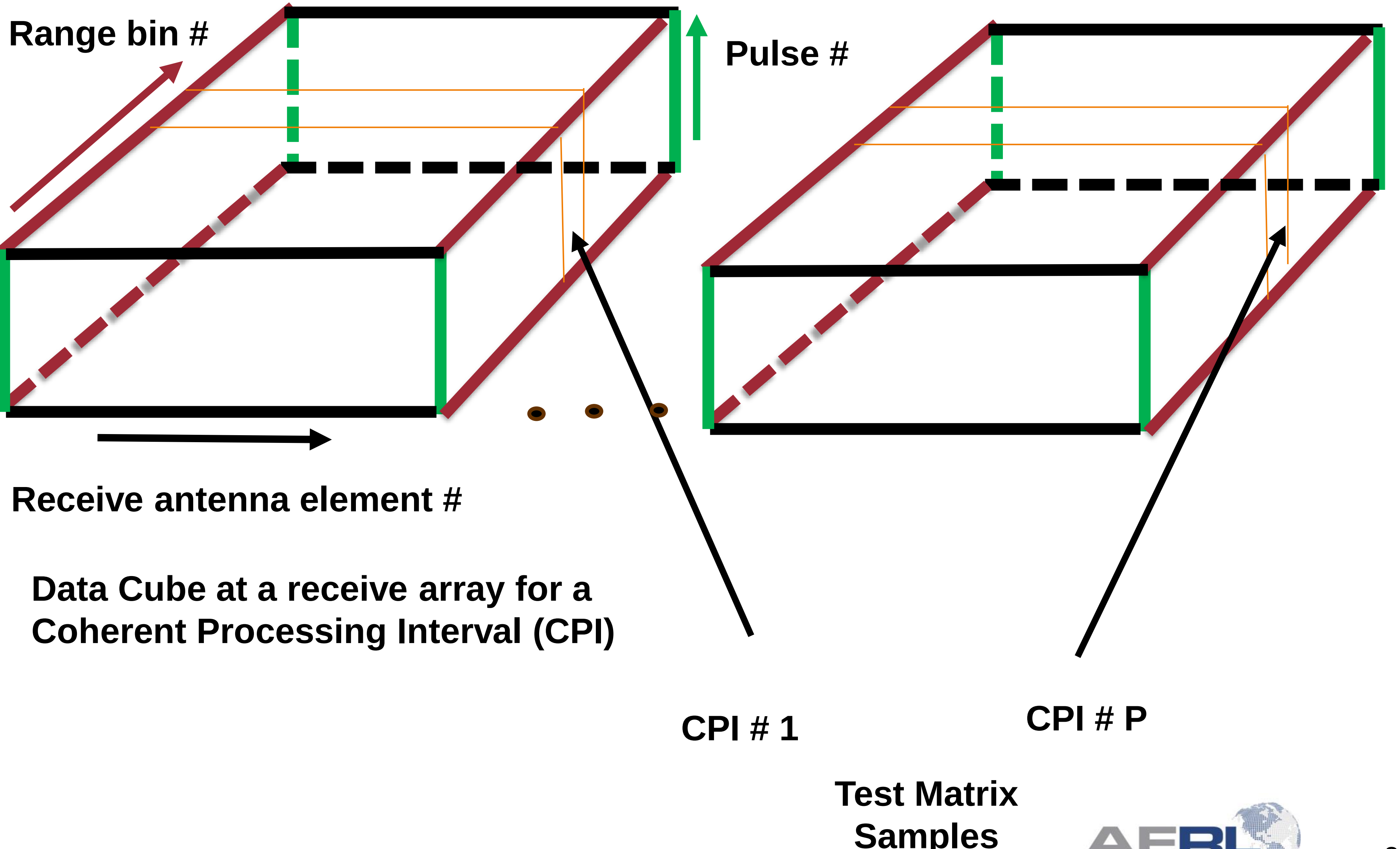
Subspace Version of Wilks'-Lambda Test and Application to Space-Time Adaptive Processing

Dr. Ram Raghavan

Distributed RF Sensing Branch
Multispectral Sensing and Detection Division
Air Force Research Laboratory
Wright Patterson AFB

January 8, 2025

Airborne detection geometry and Signal Subspace



UNCLASSIFIED

Data Format

Received I/Q samples from data cube rearranged in the following manner

Test Matrix:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \quad ; \quad \mathbf{Z}_1 \in \mathrm{C}^{M \times P} \quad ; \quad \mathbf{Z}_2 \in \mathrm{C}^{(N-M)\times \mathrm{P}}$$

Interference Training data:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \quad ; \quad \mathbf{Y}_1 \in \mathrm{C}^{M \times K} \quad ; \quad \mathbf{Y}_2 \in \mathrm{C}^{(N-M)\times K}$$

N : Length of column vectors (Number of degrees of freedom)

M : Dimension of signal subspace

P : Number of observations

K: Number of independent training vectors

Statistical Model for Test and Training Data

$$\mathbf{Z} \in C^{N\times P};\ \mathbf{Z}_1 \in C^{M\times P}\ ;\ \mathbf{Z}_2 \in C^{(N-M)\times P} \quad \text{(Test Matrix)}$$

$$\mathbf{Y} \in C^{N\times K};\ \mathbf{Y}_1 \in C^{M\times K}\ ;\ \mathbf{Y}_2 \in C^{(N-M)\times K} \quad \text{(Training Vectors)}$$

Data Vectors

$$Vec[\mathbf{Z}] \sim \begin{cases} N_c\left(\mathbf{0}\,,\mathbf{I}_P \otimes \mathbf{R}\right) & : \quad H_0 \\ N_c\left(Vec\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{0}_{(N\text{-}M)\times P} \end{bmatrix}, \mathbf{I}_P \otimes \mathbf{R}\right) & : \quad H_1 \quad Tr[\mathbf{A}_1\mathbf{A}_1^H] > 0 \end{cases}$$

$$Vec[\mathbf{Y}] \sim \begin{cases} N_c\left(\mathbf{0}\,,\mathbf{I}_\mathbf{K} \otimes \mathbf{R}\right) & : \quad H_0 \\ N_c\left(\mathbf{0}\,,\mathbf{I}_\mathbf{K} \otimes \mathbf{R}\right) & : \quad H_1 \end{cases}$$

Statistical Model

Wilks' Lambda Test for signal detection in unknown interference

Test Matrix:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z_1} \\ \mathbf{Z_2} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{X_1} \\ \mathbf{X_2} \end{bmatrix} & : H_0 \\ \begin{bmatrix} \mathbf{X_1} \\ \mathbf{X_2} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} & : H_1 \end{cases}$$

$$\mathbf{A}_1 \in \mathbb{C}^{M\times P} \quad ; \quad Tr[\mathbf{A}_1\mathbf{A}_1^H] > 0 \quad ; \quad \mathbf{A}_2 = \mathbf{0}_{(N-M)\times P}$$

Interference Training data:

$$\mathbf{Y} = \begin{bmatrix} \boldsymbol{Y_1} \\ \boldsymbol{Y_2} \end{bmatrix} \quad : \quad \boldsymbol{H_0 \; and H_1}$$

Wilks' Lambda Test:

$$t_{WL} = \left[\frac{|\mathbf{YY}^\dagger + \mathbf{ZZ}^\dagger|}{|\mathbf{YY}^\dagger|}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta_{WL}$$

Distribution of Wilks' Lambda Statistic

$$t_{WL}^{-1} \overset{\text{dist}}{\sim} \prod_{m=1}^{N} x_\beta(K+1-m, P) \ ; \ \text{if } H_0,$$

Product of statistically independent central beta distributed random variables

$$\mathbf{A} = \mathbf{s}\mathbf{a}^\dagger \qquad \mathbf{s} \in \mathbb{C}^{N\times 1} \qquad \mathbf{a} \in \mathbb{C}^{P\times 1}$$

Rank 1 signal model for Alternative Hypothesis

$$t_{WL}^{-1} \overset{\text{dist}}{\sim} x_\beta(K, P; c_0) \left[\prod_{m=2}^{N} x_\beta(K+1-m, P)\right] \ ; \ \text{if } H_1.$$

$$c_0 = ||\mathbf{a}||^2 \mathbf{s}^\dagger \mathbf{R}^{-1} \mathbf{s}.$$

Signal-to-interference-plus-noise ratio

Subspace version of the Wilks' Lambda Test

In STAP applications, the number of degrees of freedom (N) is typically large compared to signal subspace dimension (M)

Direct application of Wilks' Lambda test to STAP is not very useful

Need a signal subspace version of the Wilks' Lambda Test

All data resolved into signal subspace component and interference-plus-noise only components (contains no signal)

Use the training data and the test data from the interference-plus-noise only subspace to estimate and suppress the interference component in the signal subspace

Subspace version of Wilks' Lambda Test

$$y_{SWL} = \frac{|\mathbf{S}_{1.2} + (\mathbf{Z}_{1.2}(\mathbf{I}_P + \mathbf{Z}_2^{\dagger}\mathbf{S}_{22}^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_{1.2}^{\dagger})|}{|\mathbf{S}_{1.2}|} \underset{H_0}{\overset{H_1}{\gtrless}} \eta.$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1\mathbf{Y}_1^{\dagger} & \mathbf{Y}_1\mathbf{Y}_2^{\dagger} \\ \mathbf{Y}_2\mathbf{Y}_1^{\dagger} & \mathbf{Y}_2\mathbf{Y}_2^{\dagger} \end{bmatrix},$$

$$\mathbf{S}_{1.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

Schur complement of $\mathbf{S}_{22}$
In S

$$\mathbf{Z}_{1.2} = \mathbf{Z}_1 - \underbrace{\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{Z}_2}$$

Conditional mean of Z1 given Z2

Distribution of subspace version of Wilks' Lambda Statistic

$$[t_{SWL}^{-1}|H_0] \overset{\text{dist}}{\sim} \prod_{m=1}^{M} x_\beta(K-N+M+1-m, P),$$

$$[t_{SWL}^{-1}|\rho, c_0, H_1] \overset{\text{dist}}{\sim} x_\beta(K-N+M, P; c_0\rho) \times \left[\prod_{m=2}^{M} x_\beta(K-N+1+M-m, P)\right]$$

$$\rho = \hat{\mathrm{a}}^\dagger(\mathrm{I}_P + \mathrm{Z}_2^\dagger \mathrm{S}_{22}^{-1}\mathrm{Z}_2)^{-1}\hat{\mathrm{a}}, \qquad \hat{\mathrm{a}} = \mathrm{a}/||\mathrm{a}||$$

$$c_0 = ||\mathrm{a}||^2 \mathrm{s}^\dagger \mathrm{R}^{-1}\mathrm{s}.$$

Signal-to-interference-plus-noise ratio

Analytical expressions for PD and PFA

$$[y_{SWL}^{-1}|H_0] = \prod_{m=1}^{M} x_\beta(K - N + M + 1 - m, P)$$

$$f_{y_{SWL}^{-1}}(x) = \sum_{n=0}^{J_{max}} p_n(M) f_\beta(x; L, MP + n)$$

$$f_\beta(x; m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1}(1-x)^{n-1}$$

$$f_\beta(x; m, n|c) = e^{-cx} \sum_{k=0}^{m} \binom{m}{k} \frac{c^k \Gamma(m+n)}{\Gamma(m+n+k)} f_\beta(x; m, n+k)$$

$$; 0 \le x \le 1.$$

R. S. Raghavan, "Performance comparison of two subspace based GLRTs for rank-1 signal detection in unknown interference with multiple observations," **IEEE Transactions on Aerospace and Electronic Systems, Vol. 59, No.1, pp. 418 - 433, Feb. 2023.**

Invariance to transformations of data

Detection Statistic must be invariant to following Data Transformation:

$$\begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{C}_{11}\mathbf{Z}_1 + \mathbf{C}_{12}\mathbf{Z}_2 \\ \mathbf{C}_{22}\mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\mathbf{C}_{11} \in \mathrm{C}^{M \times M} \;; \quad |\,\mathbf{C}_{11}\,| \neq 0$$

$$\mathbf{C}_{22} \in \mathrm{C}^{(N-M)\times(N-M)} \;; \quad |\,\mathbf{C}_{22}\,| \neq 0$$

$$\mathbf{V}: \quad \text{Unitary Matrix}$$

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{C}_{11}\mathbf{Y}_1 + \mathbf{C}_{12}\mathbf{Y}_2 \\ \mathbf{C}_{22}\mathbf{Y}_2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}$$

$$\mathbf{U}: \quad \text{Unitary matrix}$$

Maximal Invariant Statistic

$$\mathbf{T}_{1.2} = \mathbf{Z}_{1.2}^{\dagger}\mathbf{S}_{1.2}^{-1}\mathbf{Z}_{1.2} \quad \in \mathbb{C}^{P\times P}$$

$$\mathbf{T}_{22} = \mathbf{Z}_{22}^{\dagger}\mathbf{S}_{22}^{-1}\mathbf{Z}_{22} \quad \in \mathbb{C}^{P\times P}$$

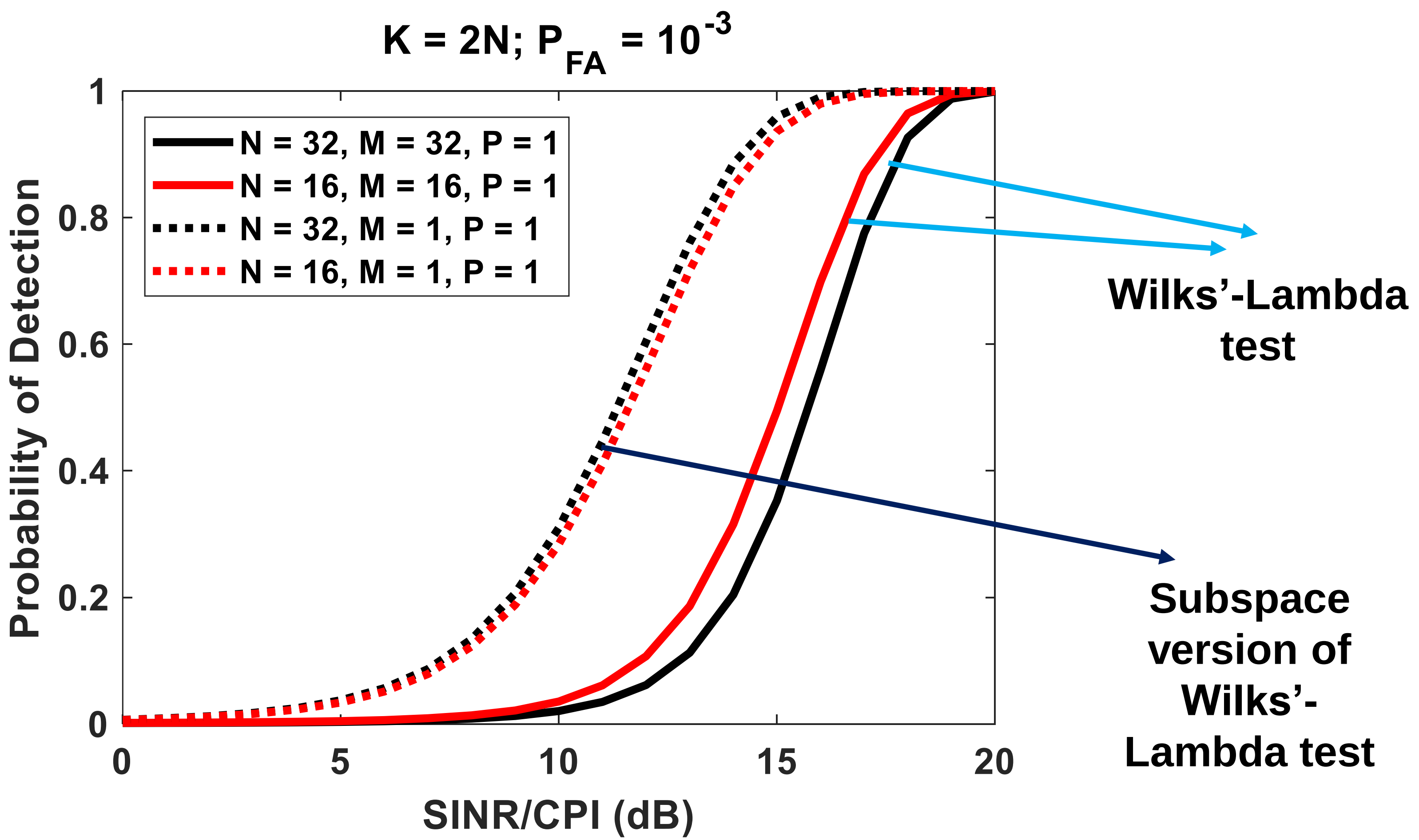
$$\mathbf{T} = \mathbf{Z}^{\dagger}\mathbf{S}^{-1}\mathbf{Z} \quad \in \mathbb{C}^{P\times P}$$

$$\mathbf{T} = \mathbf{T}_{1.2} + \mathbf{T}_{22}$$

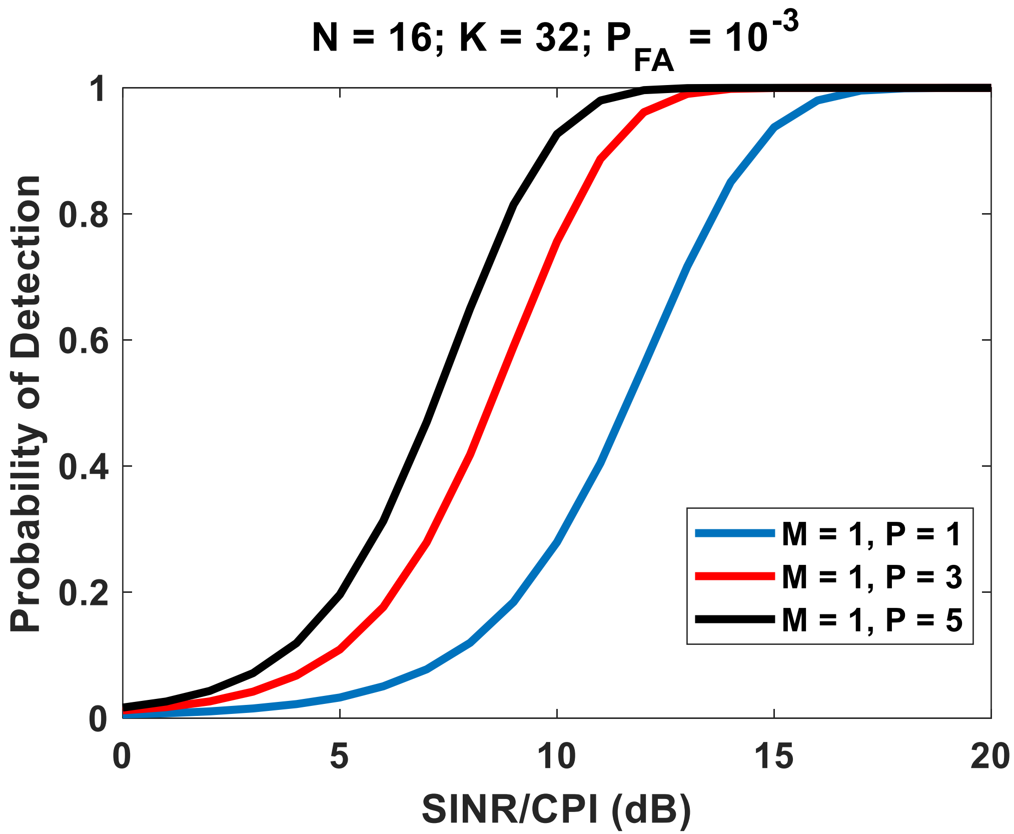
$$\begin{aligned} y_{SWL} &= |\mathbf{I}_M + \mathbf{S}_{1.2}^{-1}(\mathbf{Z}_{1.2}(\mathbf{I}_P + \mathbf{Z}_2^{\dagger}\mathbf{S}_{22}^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_{1.2}^{\dagger})| \\ &= |\mathbf{I}_P + [\mathbf{Z}_{1.2}^{\dagger}\mathbf{S}_{1.2}^{-1}\mathbf{Z}_{1.2}][\mathbf{I}_P + \mathbf{Z}_2^{\dagger}\mathbf{S}_{22}^{-1}\mathbf{Z}_2]^{-1}| \\ &= |\mathbf{I}_P + \mathbf{T}_{1.2}[\mathbf{I}_P + \mathbf{T}_{22}]^{-1}| \end{aligned}$$

R. S. Raghavan, “Maximal Invariant Statistic for subspace signal detection in Unknown Interference with multiple observations,” **IEEE Transactions on Aerospace and Electronic Systems, Vol. 60, No.2, pp. 2422 – 2427, April 2024.**

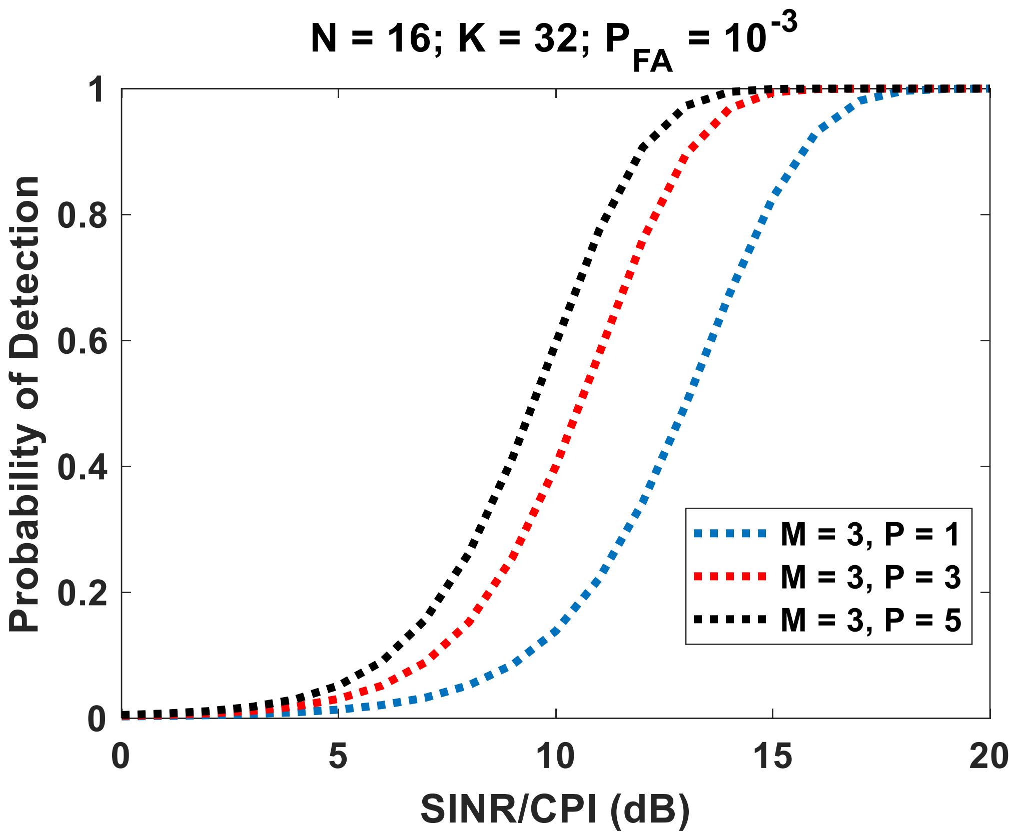
Sample Result



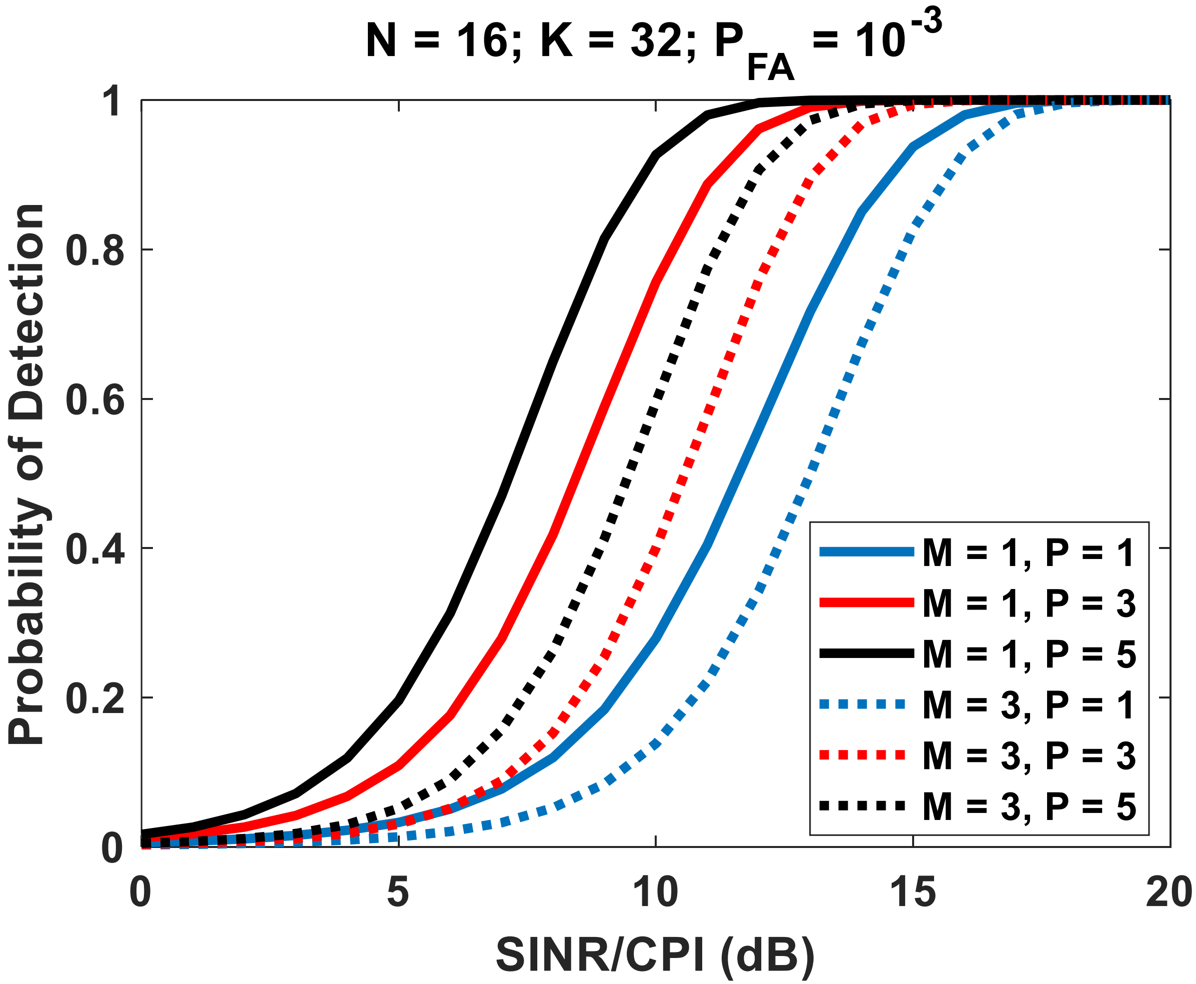
Sample Result



Sample Result



Sample Result



Summary

- Subspace version of Wilks-Lambda test derived for STAP ($N >> M$).
- Test used to detect a subspace signal in unknown interference for multiple observations.
- Test statistic shown to be expressed in terms of maximal invariant statistic - required for Constant False Alarm Rate (CFAR) property.
- Inverse of test statistic shown to be expressed as a product of statistically independent central beta random variables (rvs).
- Analytical expressions for probability of false alarm derived by expressing product of rvs as a mixture of independent central beta rvs.
- Similar approach used to derive expressions for probability of detection.

Referred Journal Publications

1. R. S. Raghavan, "Performance comparison of two subspace based GLRTs for rank-1 signal detection in unknown interference with multiple observations**," IEEE Transactions on Aerospace and Electronic Systems, Vol. 59, No.1, pp. 418 - 433, Feb. 2023.**

2. R. S. Raghavan, "Maximal Invariant Statistic for subspace signal detection in Unknown Interference with multiple observations," **IEEE Transactions on Aerospace and Electronic Systems, Vol. 60, No.2, pp. 2422 – 2427, April 2024.**