A Distributional Monte Carlo Method for the Boltzmann Equation

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Outline

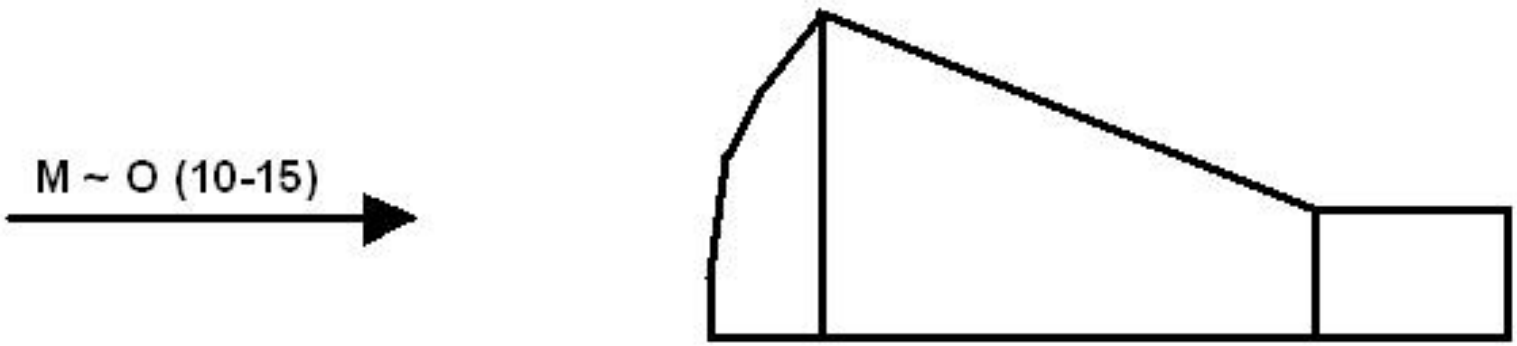
- Background
  - Kinetic Theory
  - Boltzmann Equation
  - Direct Simulation Monte Carlo
- Kernel Density Estimation
- Distributional Monte Carlo
  - DMC-KDE
  - DMC-BGK
- A Fast Solver for the Boltzmann Equation

Background

- The Direct Simulation Monte Carlo (DSMC) method is popular for treatment of flows with continuum break down
- Such flows are important in hypersonic aerodynamics and micro-scale flows
- DSMC has limitations due to its nonphysical representation of the velocity distribution function (point measure)
- We seek to develop distributional methods for stronger convergence and reduced variances

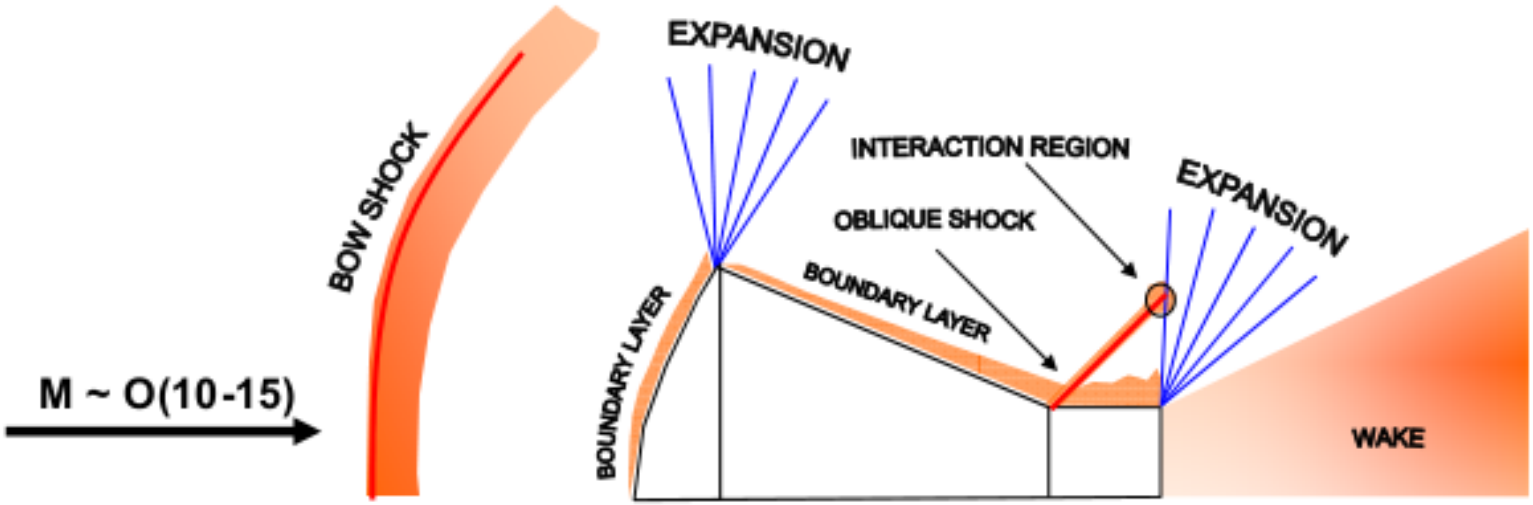
Blunt Body Example

- Two ways to physically describe a gas
  - Macroscopic $\longrightarrow$ Continuum Fluid Equations
  - Microscopic $\longrightarrow$ Kinetic Theory
- Consider the hypersonic flow field around a blunt body



Why Kinetic Theory?

- Regions where continuous equations may be invalid:



- Large portions of the flow field may be in local thermodynamic equilibrium, but the non-equilibrium effects of the small regions ripple throughout the domain
- Microscopic representation of flow field is required to accurately capture the physics

The Kinetic Description of Gas

- Gas consists of particles that most of the time do not interact.
- Each particle is associated with a velocity and a position.
- The state of gas is described using the *molecular velocity distribution function* $f(t, \vec{x}, \vec{v})$.

$\Rightarrow$ $f(t, \vec{x}, \vec{v}) d\vec{x} d\vec{u}$ gives the number of molecules contained in a box of size $d\vec{x} \times d\vec{u}$ of the physical space

Macroscopic Considerations

- Most thermodynamic quantities can be written as expectation values (weak form) of bounded and continuous functions, or moments of the velocity distribution function; e.g. density, bulk-velocity, temperature, stress-tensor, and energy flux.
- The availability of the velocity distribution values (strong form) may facility faster and more accurate deterministic solutions

Relaxation Process

- As molecules collide and exchange energy, distribution of their velocities approaches the Maxwellian equilibrium distribution

$$f_M(\vec{v}) = \frac{n}{(2\pi RT)^{3/2}} \exp\left(-\frac{|\vec{v} - \vec{u}(t,\vec{x})|^2}{2RT}\right)$$

- The gas is **in the state of continuum** if its velocity distribution approaches the Maxwellian distribution.
- The Maxwellian distribution is defined by density n, bulk velocity $\vec{u}$, and temperature T.

Boltzmann Equation

- Governing equation of Kinetic Theory
- Describes the evolution of particle energy PDF
- For unexcited monatomic gas, may examine velocity PDF

$$\frac{\partial}{\partial t} f\left(\vec{r}, \vec{c}, t\right) + \vec{c} \cdot \frac{\partial}{\partial \vec{r}} f\left(\vec{r}, \vec{c}, t\right) + \vec{F} \cdot \frac{\partial}{\partial \vec{c}} f\left(\vec{r}, \vec{c}, t\right) = J\left[f\right]\left(\vec{r}, \vec{c}, t\right)$$

where $f\left(\vec{r}, \vec{c}, t\right)$ is the probability of finding a particle of gas at time t, at point $\vec{r}$, moving with velocity $\vec{c}$.

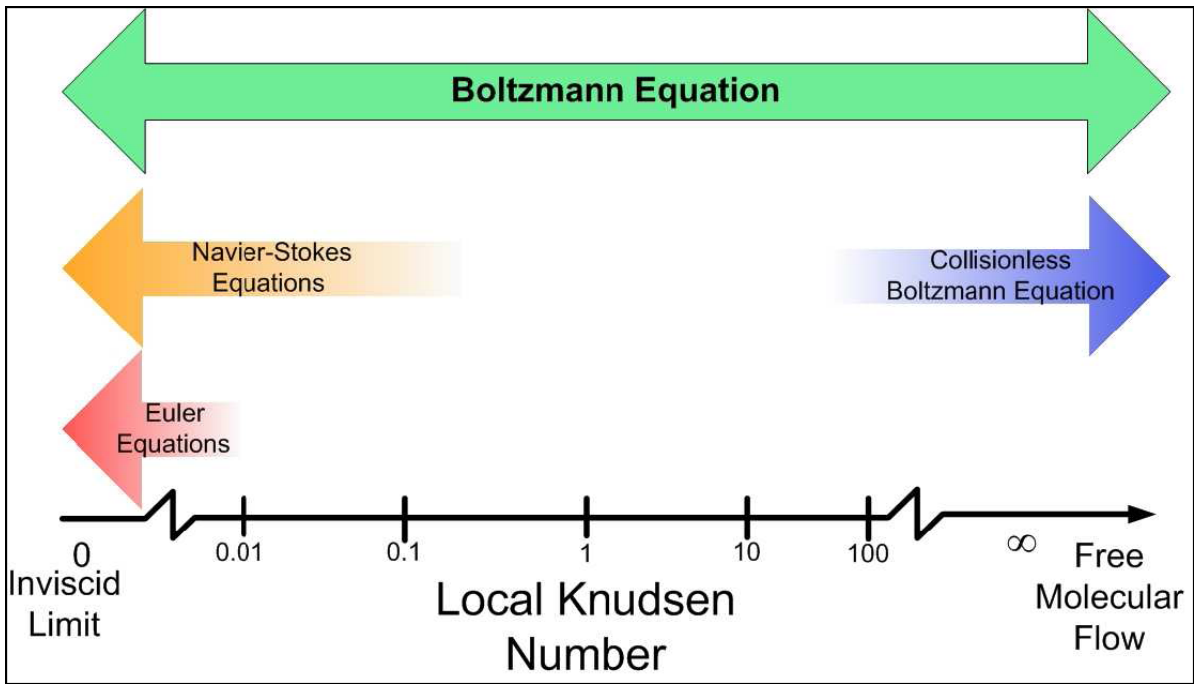
Relation to Continuum Eqs.

- Taking appropriate moments of the BE yields conservation equations for mass, momentum, and energy
- Hilbert expansion: $f = f_M(1 + \Phi + \Phi^2 + \cdots)$
  - For $f = f_M$, $\rightarrow$ Euler Equations
  - Including 1^{st} perturbation $\rightarrow$ Navier-Stokes Equations
  - Higher order terms yield Burnett, Super Burnett, etc.

The Spectrum

Metric of Validity of Continuum and Equilibrium Hypotheses:

$$Kn = \frac{\lambda}{L}$$



Problem with Hilbert's 6th Problem?

- Hilbert called to develop "mathematically the limiting processes ... which lead from the atomistic view to the laws of motion of continua"
- Contemporary research interpretation: passage from the kinetic theory of Boltzmann to the continuum theory of Euler as the Knudsen number $K_n \to 0$.
- References:
  - "From Boltzmann to Euler," M. Slemrod, Computers and Math with Appl, 2012
  - "Hilbert's 6th Problem," Gorban and Karlin, Bulletins of Am Math Soc., (2014)
  - "The Problem with Hilbert's 6th Problem," M. Slemrod, Math Model. Nat. Phenom. (2015)
  - "Famous Fluid Equations are Incomplete," Mathematical Physics Review, Quanta Mag., Jul 21, 2015

Direct Simulation Monte Carlo

- Developed by Bird c.1963, consistency with BE established in 1980s, applicable to dilute gases
- Stochastic simulation of a fraction of the particles, with each representing N/N_{sim} actual particles
- Uncoupling of molecular convection and collisions over small time steps
- Representative collision pairs selected and collisions computed
- Molecules convected along their velocity vector
- Simulation evolved multiple times to reduce statistical scatter

Uncoupling Principle

- Boltzmann Equation in absence of external forces:

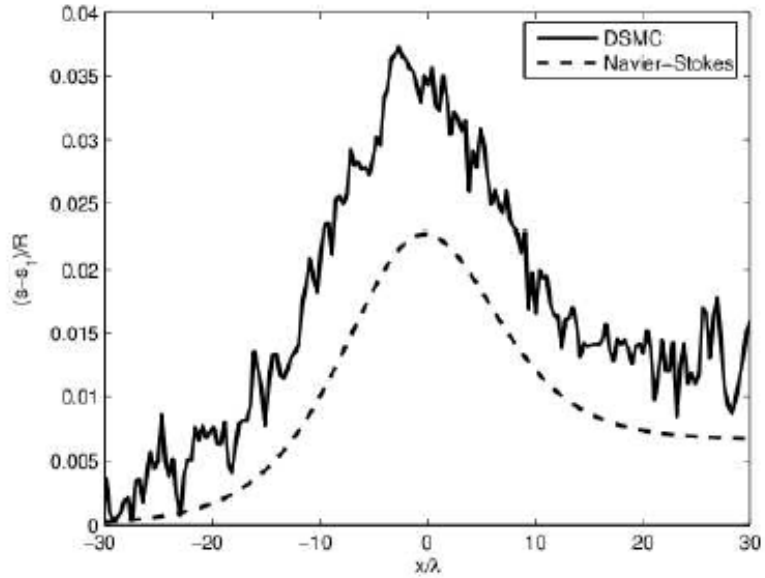
$$\frac{\partial}{\partial t}[f] = -Df + Jf$$

- The uncoupling of convection and collision over Δt

$$f = (1 - \Delta t D)(1 + \Delta t J) f_0 + \mathcal{O}(\Delta t^2)$$

Challenges with DSMC

- Significant Variation in Solution (Scatter)



- Error Estimation
- Computational Expense

DSMC Representation of the VDF

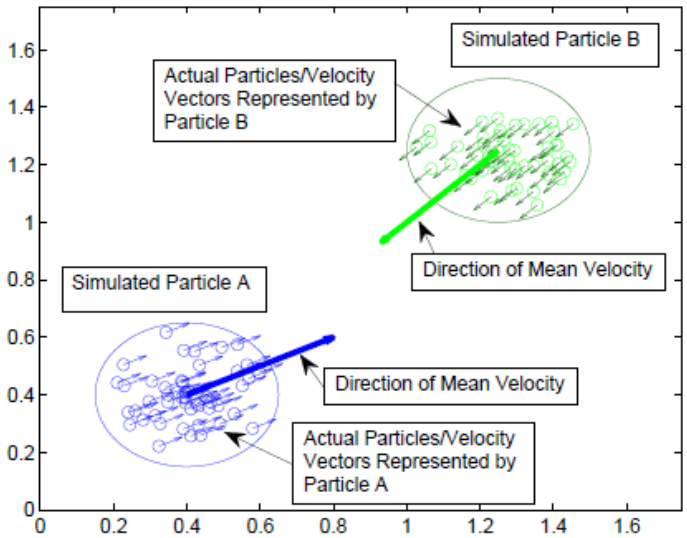
- As each simulated particle has a single velocity, within a given cell the VDF is represented as a point measure approximation

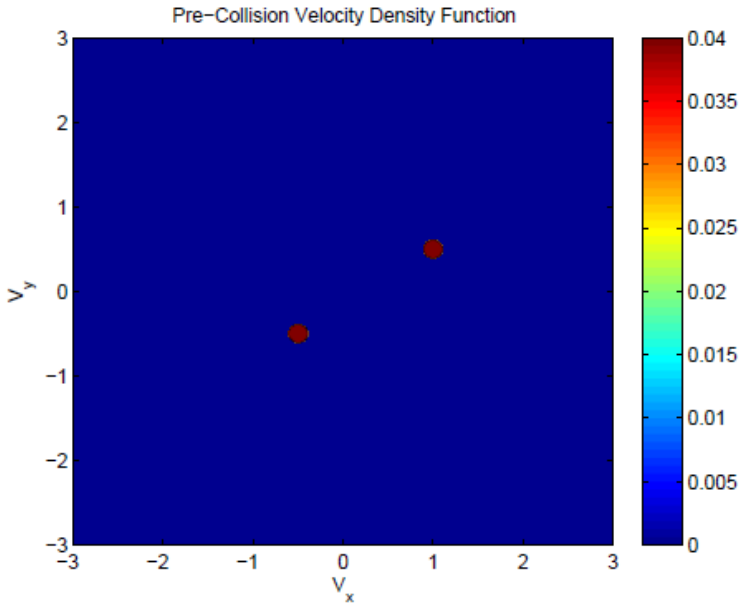
$$f(\vec{v}) = \frac{1}{N_p} \sum_{i}^{N_p} \delta(\vec{v} - \vec{v}_i)$$

- The assumption $W = N/N_p = O(10^8)$ particles each possess the same velocity is nonphysical
- In regions of low density this assumption exaggerates the effect of collisions and increases the fluctuations in the solution
- Assumption limits DSMC to weak converge

A DSMC 2-D Example

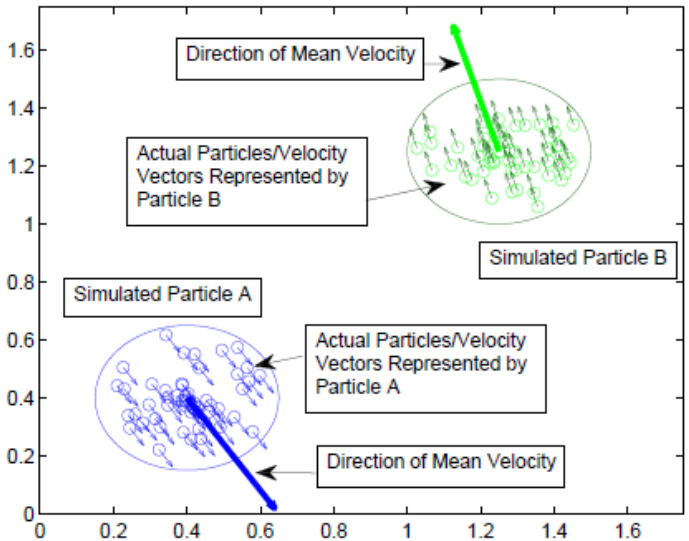
Pre-collision Illustration

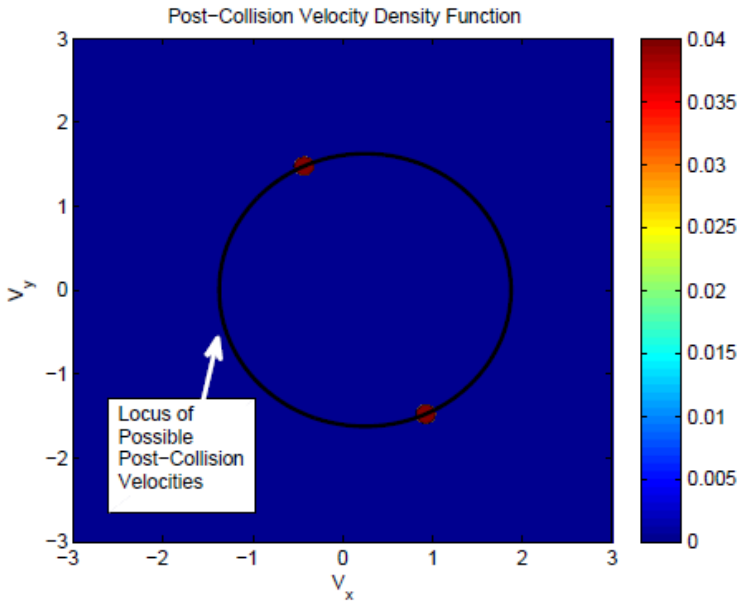




DSMC Simulated Collision

Post-collision Illustration



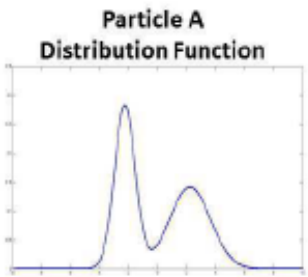


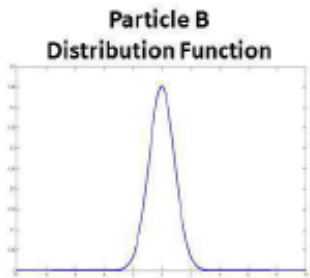
Distributional Monte Carlo

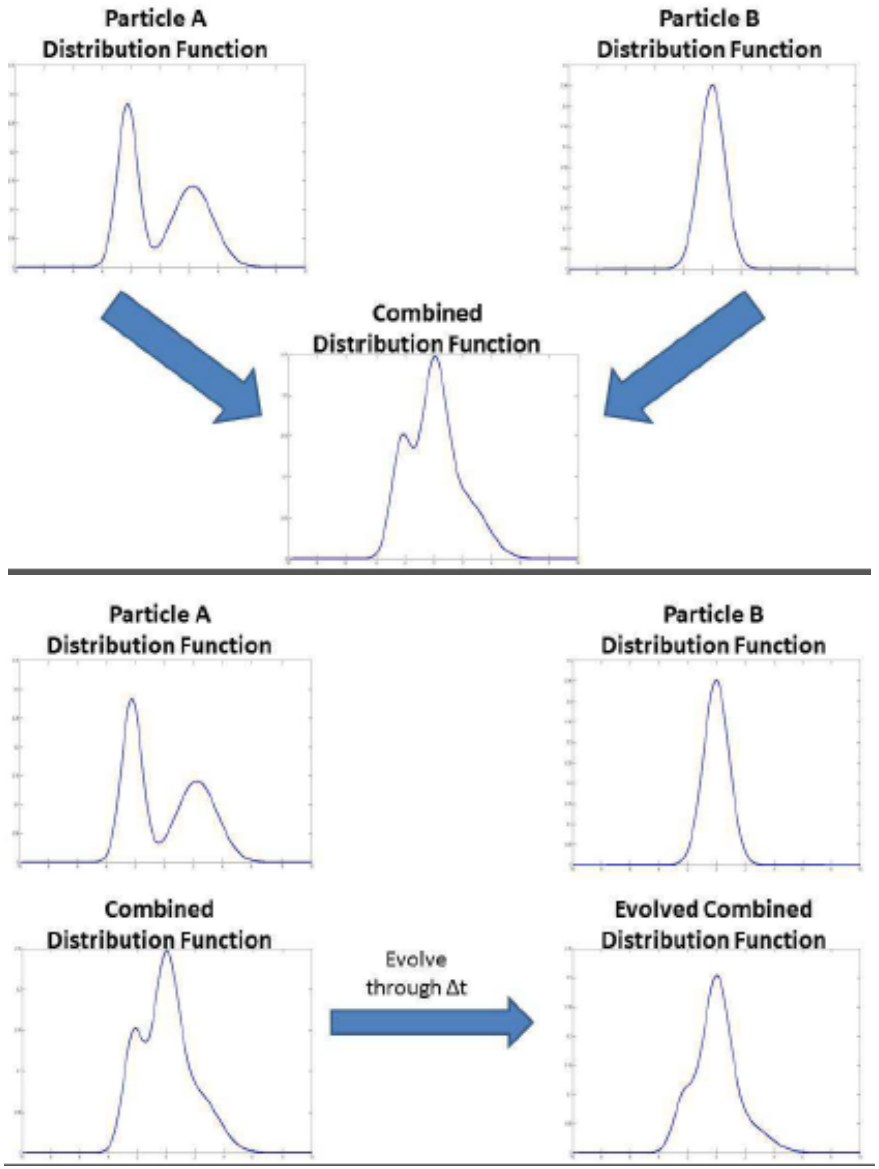
- As each simulated particle represents a large number of actual particles, allow each particles velocity to be distributed and possess its own VDF, f_i
- Overall VDF given by

$$f(\vec{c}) = \frac{1}{N_p} \sum_{i=1}^{N_p} f_i(\vec{c})$$

- Treat collisions between particle pairs as a space homogeneous relaxation of the combined 2W actual particles they represent






Particle A
Distribution Function
Particle B
Distribution Function
Combined
Distribution Function
Particle A
Distribution Function
Particle B
Distribution Function
Combined
Distribution Function
Evolve
through Δt
Evolved Combined
Distribution Function

Kernel Density Estimation (KDE)

- Process for estimating PDF from discrete samples

$$\hat{f}(x;h) = \frac{1}{Nh}\sum_{i=1}^{N} K\left(\frac{x - X_i}{h}\right)$$

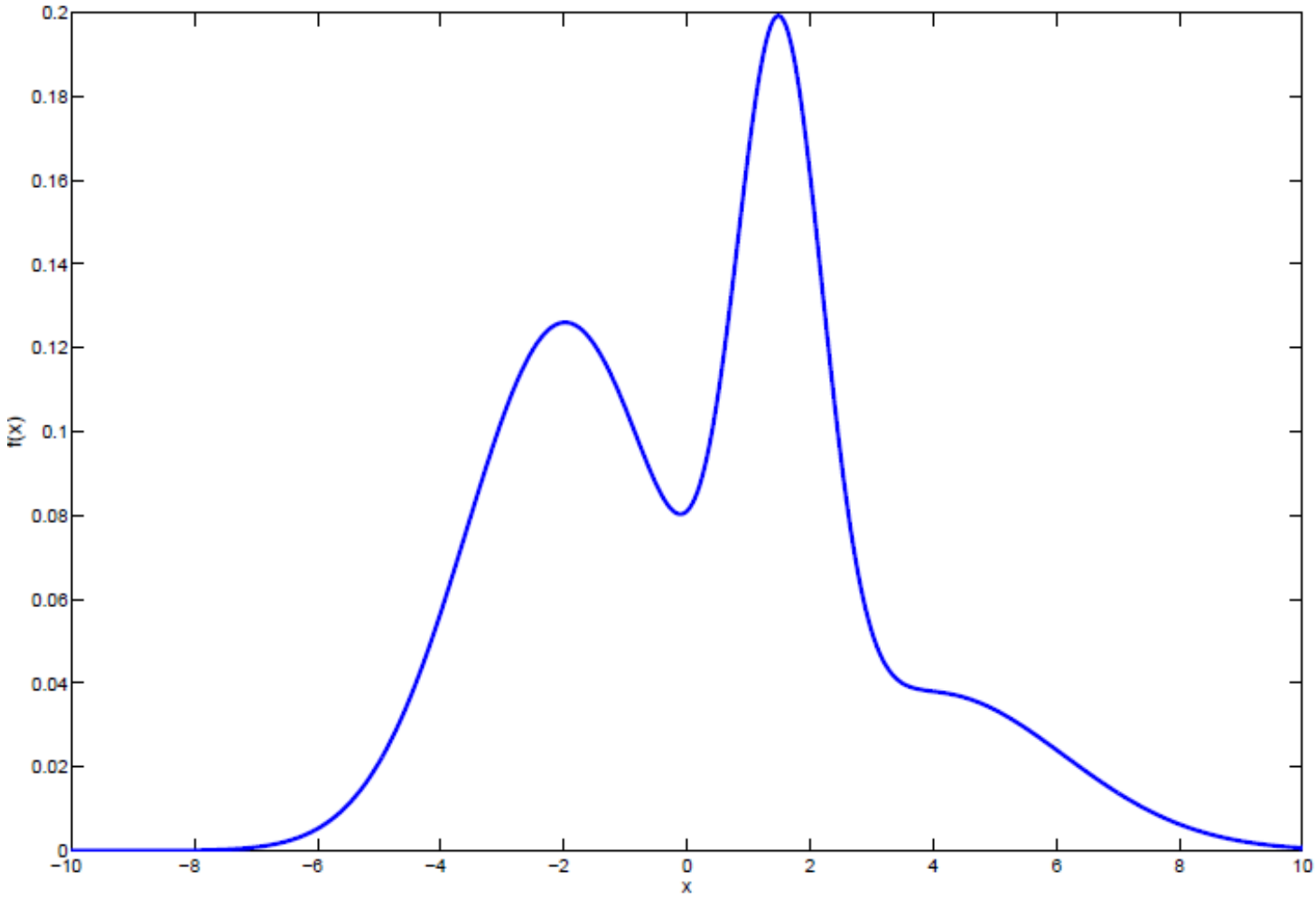
- The L^2 kernel function satisfying

$$\int K(x)\,dx = 1, \quad \int xK(x)\,dx = 0$$

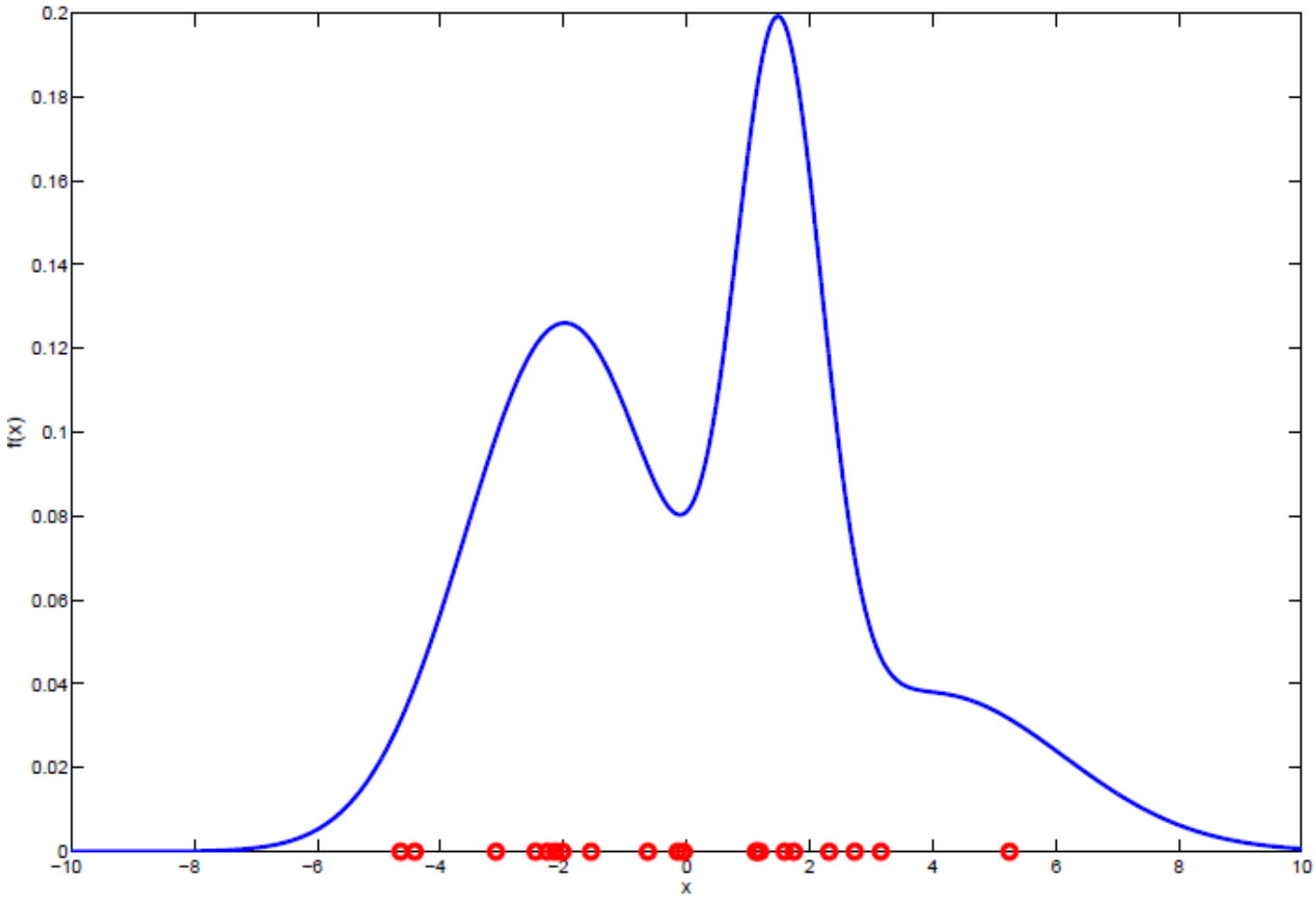
- Examples of kernel functions

$$K(x) = \begin{cases} \frac{1}{2} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}, \quad K(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

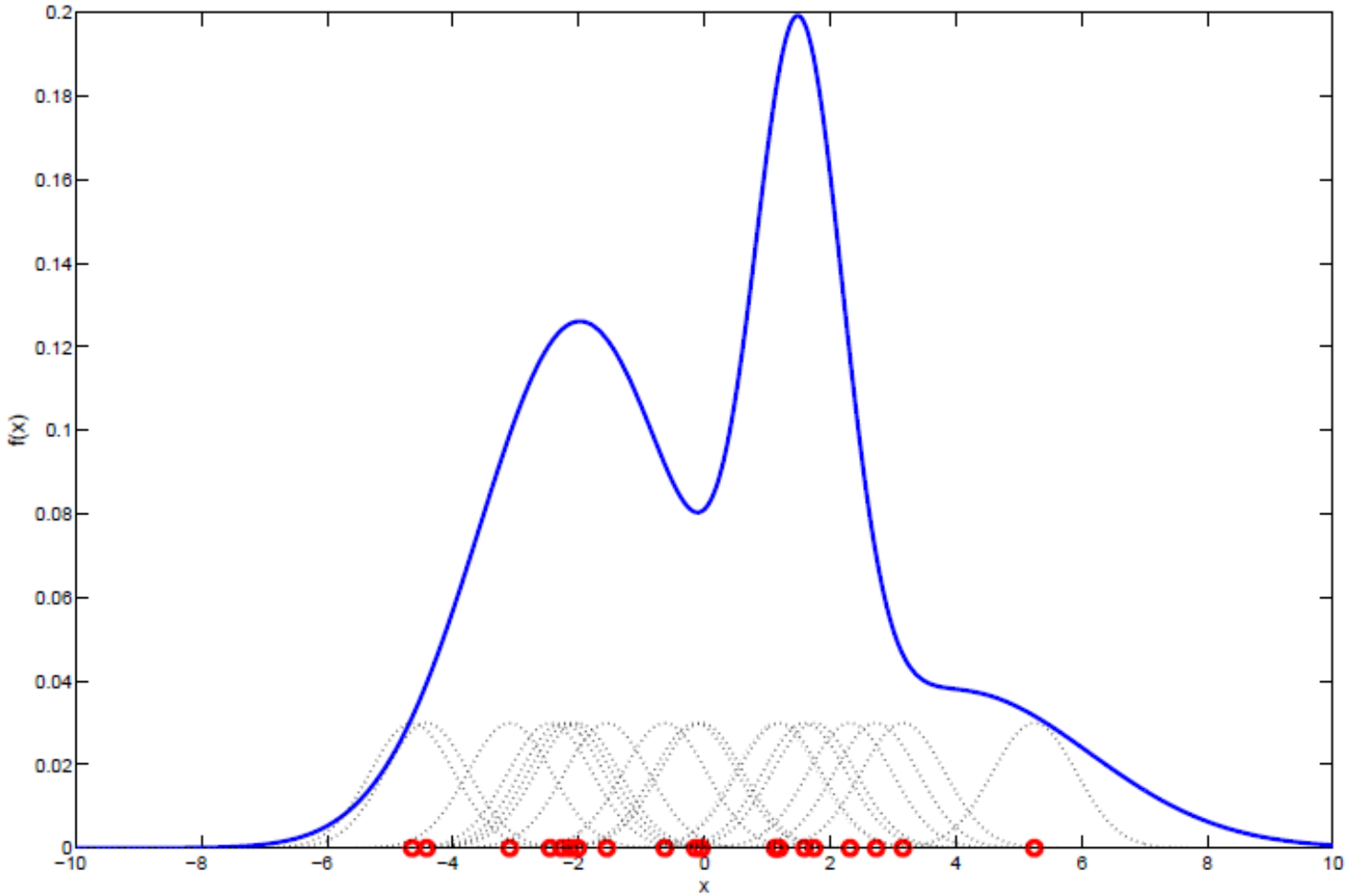
KDE Simulation 1/4



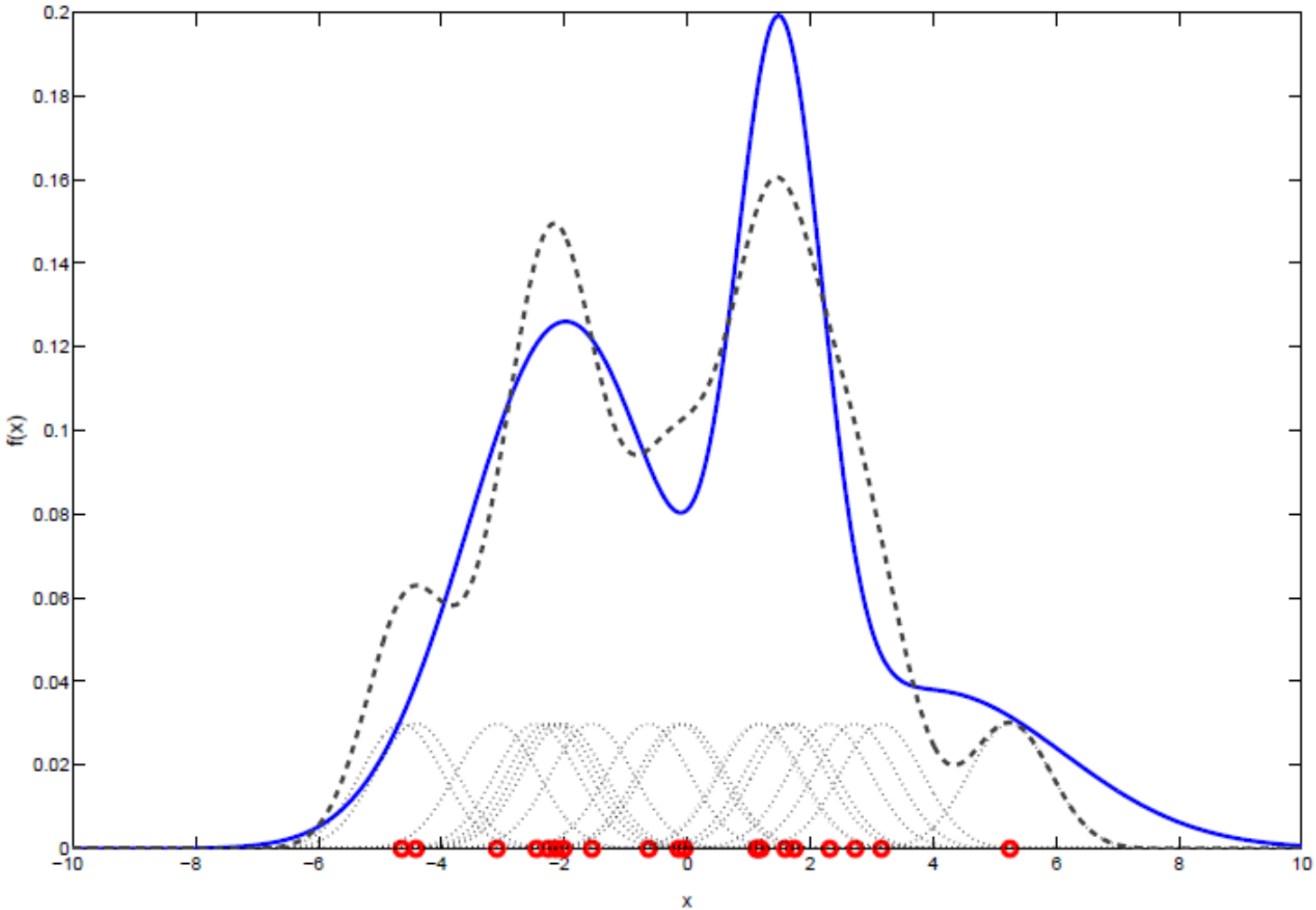
KDE Simulation 2/4



KDE Simulation 3/4



KDE Simulation 4/4



DMC-KDE

- Particle VDFs are of fixed functional form

$$f = \frac{1}{N_p} \sum_{i=1}^{N_p} f_i$$

$$f_i(\vec{v}) = \frac{1}{h^3} K\left(\frac{\vec{v} - \bar{\vec{v}}_i}{h}\right)$$

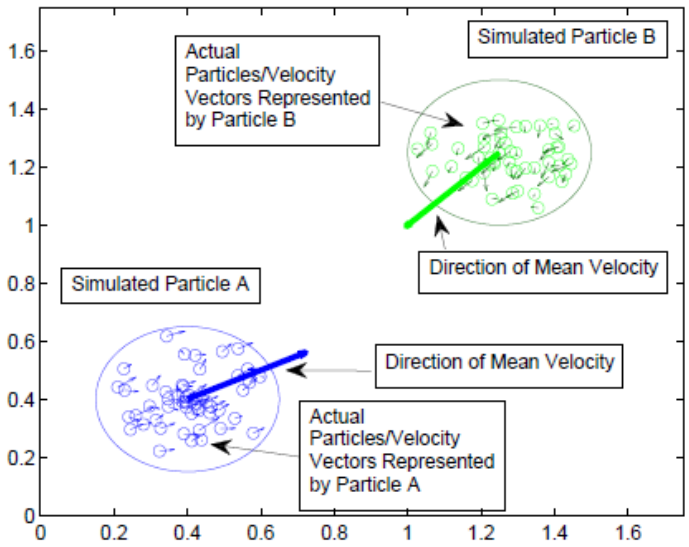
- Mean velocity allowed to vary, but kernel function/bandwidth fixed
- On the basis of physical reasoning we choose a Gaussian kernel for K

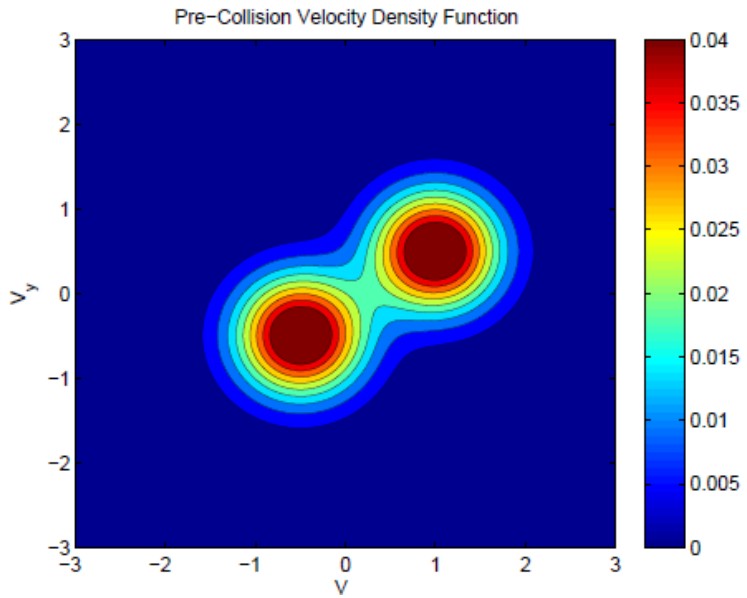
$$\begin{aligned} K(\vec{x}) &= (2\pi)^{-3/2} \exp\left(-\frac{\|\vec{x}\|^2}{2}\right) \\ h &= \left[\frac{32}{3\sqrt{2}N_p}\right]^{\frac{1}{5}} \sigma_{est} \end{aligned}$$

- Choosing $h(N_p)$ s.t. $\lim_{N_p \to \infty} h(N_p) = 0$

A DMC-KDE 2-D Example

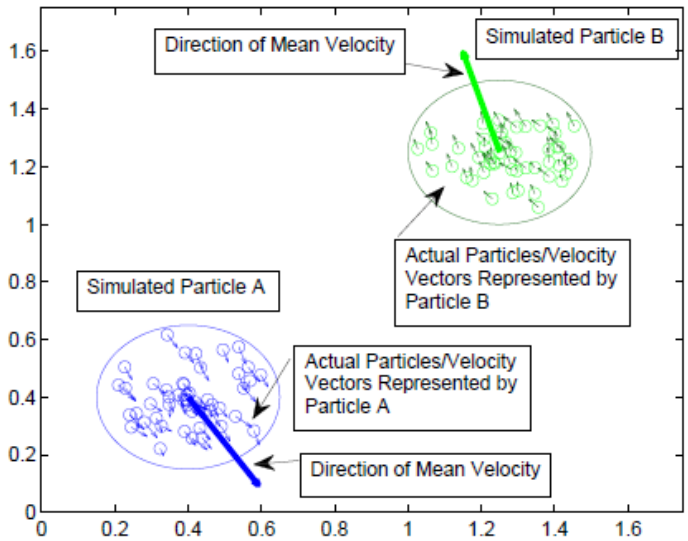
- Pre-collision: actual particles represented by one simulated particle no longer possess the same velocity, but are in translational equilibrium as a collection

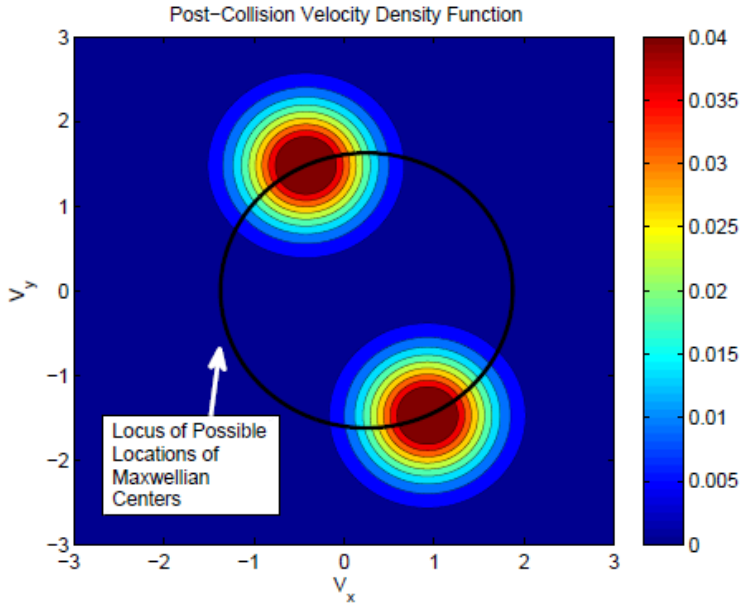




DMC-KDE Simulated Collision

- Post-collision mean velocities determined using same stochastic rules as Nanbus method





Convergence of DMC-KDE

- Preserves Weak Convergence in L^1 (probability measure)

$$\lim_{\Delta t \to 0} \lim_{N \to \infty} \int_{\mathbb{R}_3} \phi(\vec{c}) \tilde{f}(\vec{c}) \, d\vec{c} = \int_{\mathbb{R}_3} \phi(\vec{c}) f(\vec{c}) \, d\vec{c}$$

- Strong convergence in L^∞

$$\lim_{\Delta t \to 0} \lim_{N_p \to \infty} \left\| \tilde{f} - f \right\|_\infty = 0$$

- Pointwise convergence for bounded solutions

$$\lim_{\Delta t \to 0} \lim_{N_p \to \infty} \tilde{f}(\vec{c}) = f(\vec{c})$$

for all $\vec{c} \in \mathbb{R}^3$.

DMC-KDE Summary

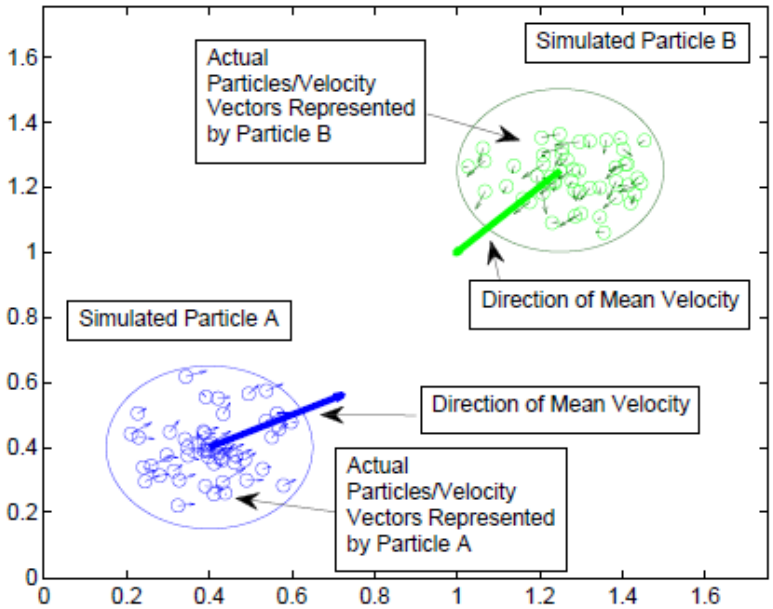
- Analogous to applying a kernel density estimator to DSMC
- Distribution function is directly obtained
  - *Stochastic Solver vs. Stochastic Simulator*
  - *Cost of evaluating distribution function at M points is $O(MN_p)$*
  - *Fast Gaussian Transform can be used to reduce to $O(M + N_p)$*
- Strong convergence is achieved
- DMC-KDE employs fixed PDFs thus is not fully distributional

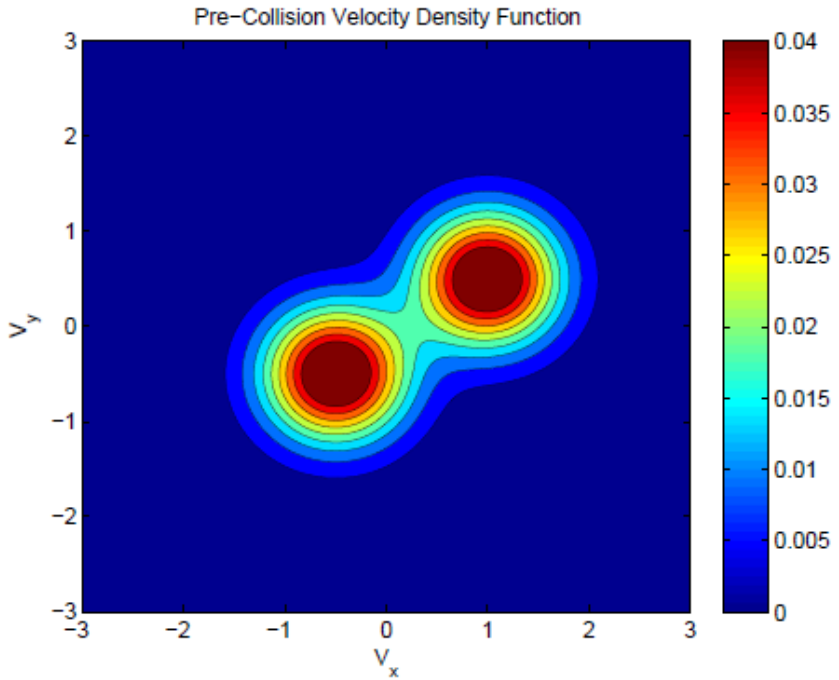
Essence of DMC

- Seek a fully distributional method with collision selection criteria and collision modeling that fully incorporate non-singular particle velocity distributions
  - allow for non-Maxwellian particle distributions
- For *near-Maxwellian* distributions, the collision selection criteria from DMC-KDE provides fair approximations
- Treat collision interactions "distributionally"
  - apply space homogeneous relaxation process on the joint distribution function of the $2W$ particles the two simulated particles represent
  - evolve as appropriate

A DMC 2-D Example

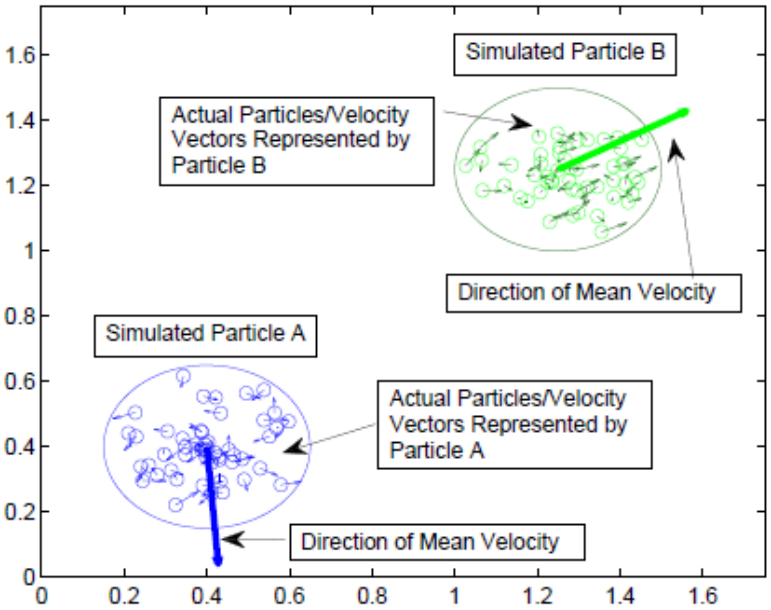
Each particle possesses a VDF

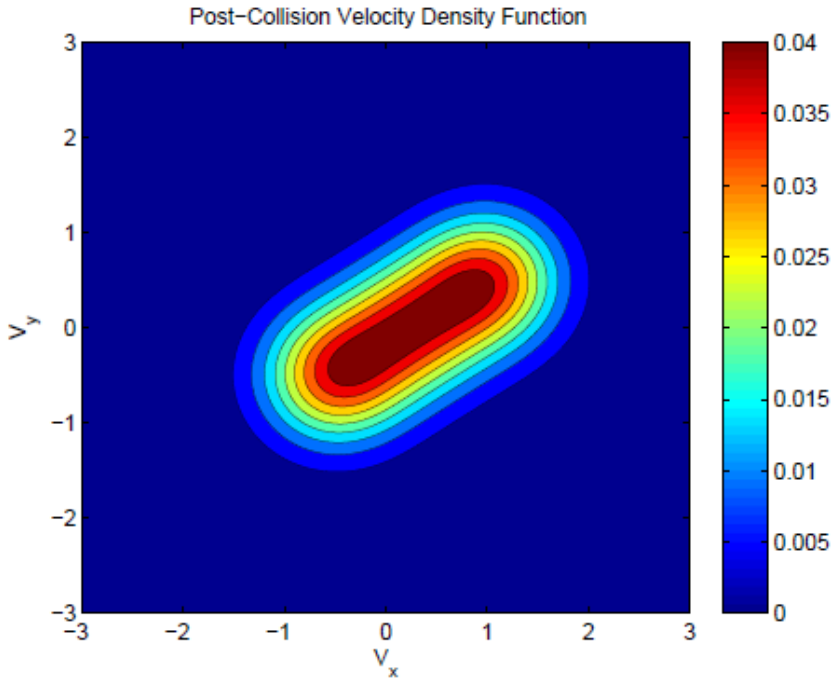




DMC Simulated Collision

Collision interactions computed as a relaxation toward equilibrium (for the simulated pair)





DMC Strong Convergence

If $f \in L^\infty$ is a solution of the space homogeneous BE at $t = t^k$, then $\check{f}^k$ of DMC converges in L^∞ to f^k:

$$\lim_{N_p \to \infty} \|\check{f} - f\| = 0.$$

A DMC Bhatnagar-Gross-Krook Approximation

- To facilitate numerical demonstration of the DMC framework, a scheme employing the BGK approximation was developed
- Distribution computed using space homogeneous BGK equation
- Deterministic collision modeling facilitated by distributed velocities
  - Point measure based SPM (e.g. DSMC) could not sample all possible outcomes
- Potential drastic effects on variance w/ removal of one stochastic element

Theorem: Let f be the solution of Boltzmann equation with smooth initial data f_0, $\hat{f}$ the DMC solution. Then $\|\hat{f} - f\|_{L^1} \to 0$, as $N_{sim} \to 0$, $\Delta t \to 0$

The Bobylev-Krook-Wu Solution

- First known closed form solution to the space homogeneous BE
- Specific to Maxwell molecule
- Spherically symmetric solution

$$f(\vec{v},t) = \frac{1}{\left(1\pi\tau(t)\right)^{-3/2}} \exp\left(-\frac{\|\vec{v}^2\|}{2\tau(t)}\right)\left[1+\frac{1-\tau(t)}{\tau(t)}\left(\frac{\|\vec{v}\|^2}{2\tau}-\frac{3}{2}\right)\right]$$

where,

$$\tau(t) = 1-\theta e^{-\lambda t}$$

$$\theta \in \left[0,\frac{2}{5}\right]$$

$$\lambda = \frac{\pi}{2}\int_0^{2\pi} B(\theta)\left(1-\cos(\theta)^2\right)d\theta$$

- Present work utilizes $\theta = \frac{2}{5}, \lambda = \frac{1}{6}$.

Normalized Moments

- Bobylev solution

$$z_n(t) = \left(1 - \frac{2}{5}e^{-\lambda t}\right)^{n-1}\left[1 + \frac{2}{5}(n-1)\,e^{-\lambda t}\right]$$

- DSMC:

$$z_n(t) = \frac{1}{N_p\,(2n+1)!!}\sum_{i=1}^{N_p}\vec{c}_i^{\,2n}(t)$$

- DMC-KDE:

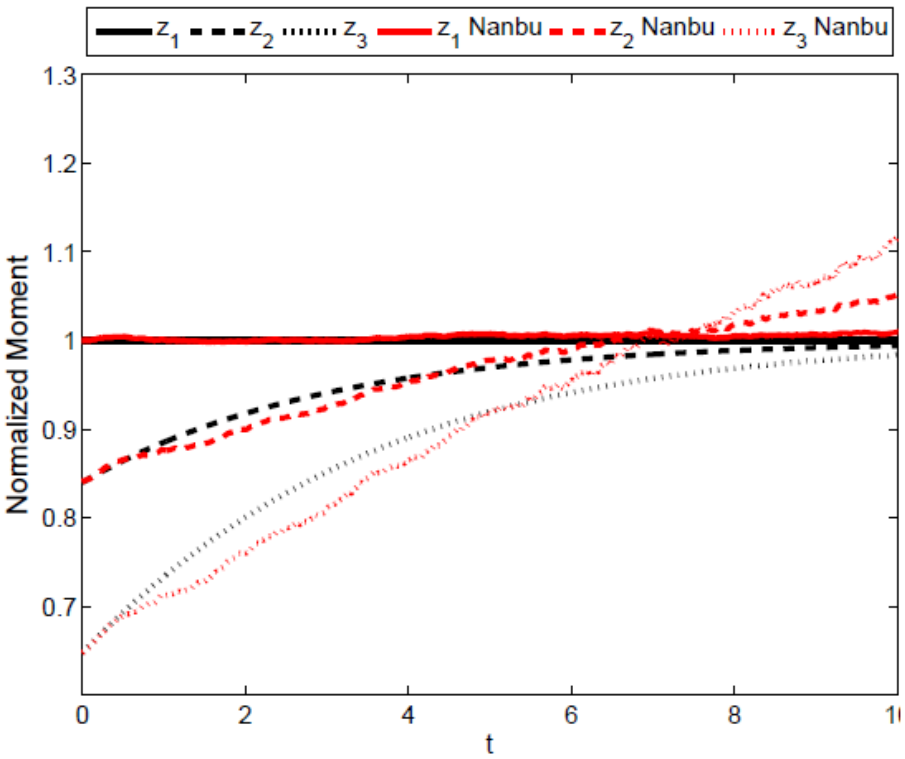
$$z_n(t) = \frac{1}{N_p\,(2n+1)!!}\sum_{i=1}^{N_p}\vec{c}_i^{\,2n}(t) + c_n h^{2n}$$

- DMC-BGK moments computed using simple quadrature

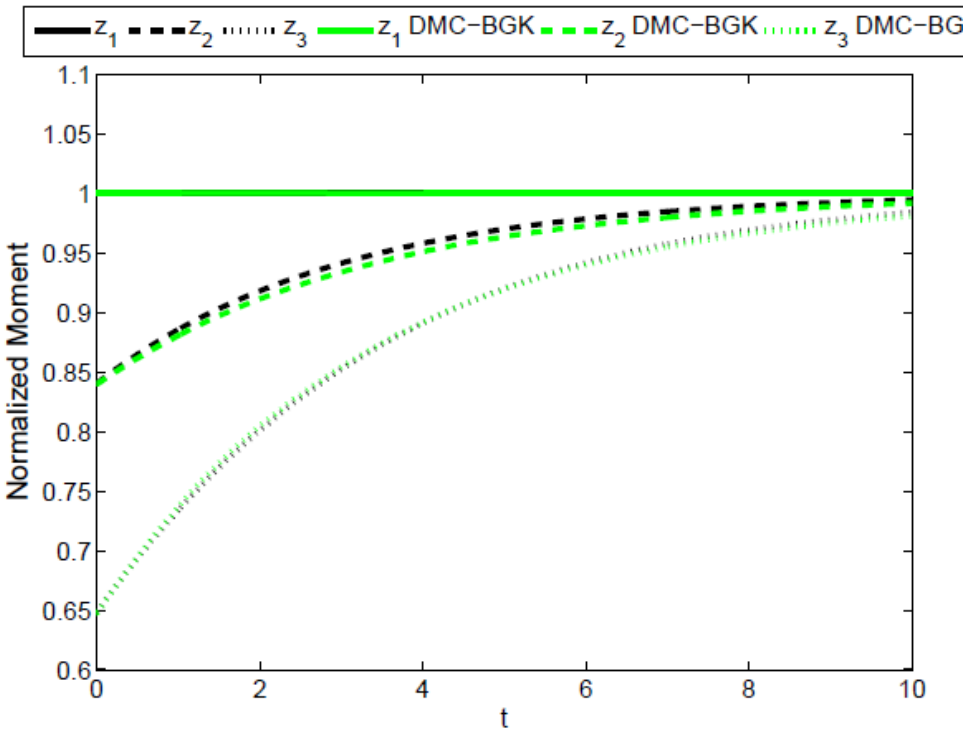
Normalized Moments Compared

- 100 simulated particles, 600 run ensemble, $\Delta v = 2/3$

Nanbu DSMC



DMC-BGK



Total Variation

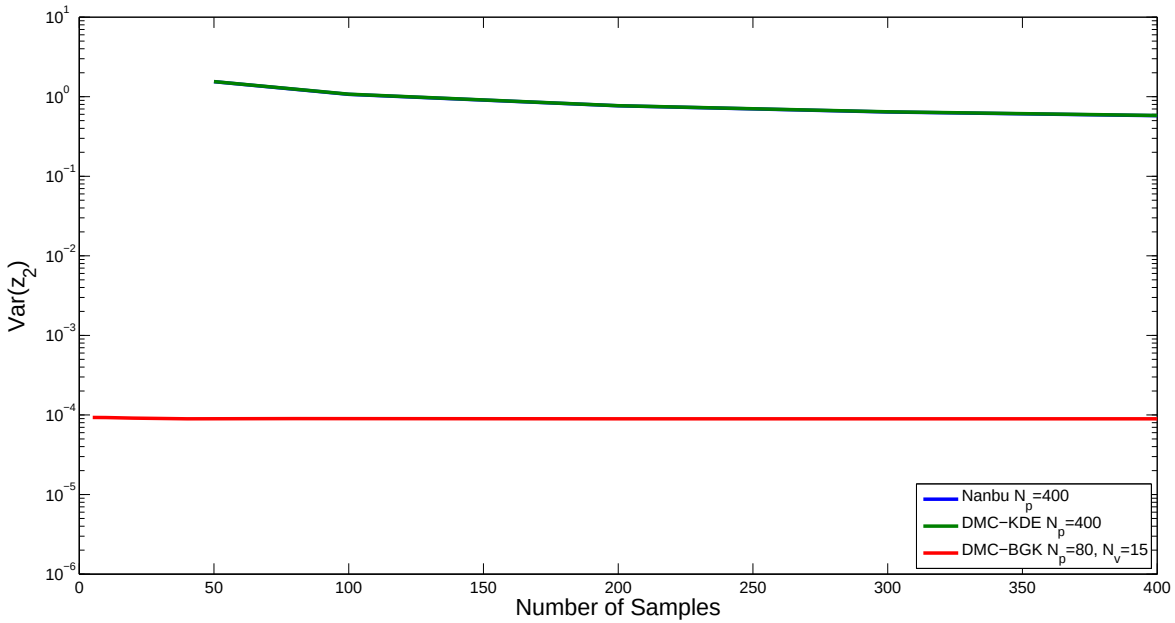
- Total variation:

$$V(z_n) = \int_0^{10} \left| z_n'(t) \right| dt$$

- Compared to Nanbu-DSMC and DMC-KDE, DMC-BGK shows
  1. marked reduction of variance
  2. superior energy conservation

Variation of Second NM

- Nanbu-DSMC/DMC-KDE/DMC-BGK

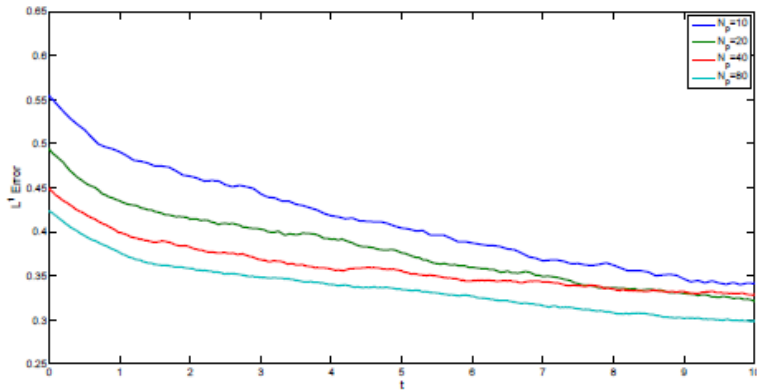


- Four orders of magnitude reduction in total variation obtained for z_1 through z_4

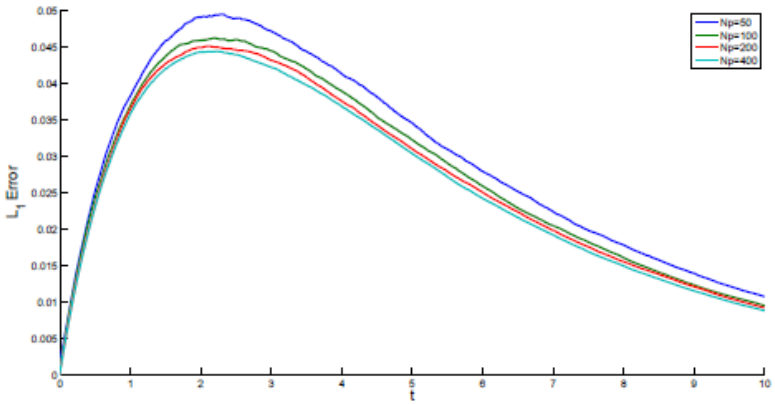
L^1 Error Compared

- 100 run ensemble, $\triangle v = 2/3$

DMC-KDE



DMC-BGK



DMC Summary

- DMC provides 1^{st} stochastic particle method utilizing distributed velocities
  - No restriction on method of computing collision outcomes
- DMC-KDE turns a Boltzmann simulator into a Boltzmann solver
  - Strong convergence
  - Direct computation/visualization of distribution function
  - No variance reduction
- DMC-BGK employs a deterministic method for collision outcomes
  - Hybrid stochastic-deterministic method
  - Substantial improvement in accuracy and variance reduction

Solving the Boltzmann Collision Integral

Development of a fast deterministic solver for the Boltzmann equation

Challenges in solving the Boltzmann Equation numerically

- Direct numerical evaluation takes $O(N^{\frac{11}{3}})$ operations each time step, where N is the number of velocity points.
- **A 2 GHz processor will use about** 50 **seconds if** $N = 10^3$ **and about** $1.59E{+}3$ **years if** $N = 10^6$**.**
- Approaches to solving the Boltzmann equation:
  - Use HPCs.
  - Use efficient numerical techniques (velocity discretization, integration, and methods for massive parallelization.)
  - Approximations to the Boltzmann equation must be used when appropriate.
  - Fastness must balance numerical errors.

Galerkin discretization in velocity variable

- Choose a rectangular region K and chop it uniformly into little cells K_j.
- The Galerkin approximation to the solution of the Boltzmann equation on each cell K_j is of the form

$$f(t,\vec{x},\vec{v})|_{K_j} = \sum_{i=1}^{s} f_{i;j}(t,\vec{x})\phi_i^j(\vec{v})\,.$$

- Substituting this into $J[f]$ and hit the result by a basis function and integrate over K_j:

$$\frac{\partial}{\partial t} f_{i;j}(t,\vec{x}) + \vec{v}_i^j \cdot \vec{\nabla}_x f_{i;j}(t,\vec{x}) = I_{\phi_i^j}$$

Galerkin projection of the collision operator

- where:

$$I_{\phi_i^j} = \int_3 \int_3 f(t, \vec{x}, \vec{v}) f(t, \vec{x}, \vec{v}_1) A(\vec{v}, \vec{v}_1; \phi_i^j) d\vec{v}_1 \, d\vec{v} \,,$$

- and the term

$$A(\vec{v}, \vec{v}_1; \phi_i^j) = \frac{|\vec{g}|}{2} \int_0^{2\pi} \int_0^{b_*} (\phi_i^j(\vec{v}') + \phi_i^j(\vec{v}_1') - \phi_i^j(\vec{v}) - \phi_i^j(\vec{v}_1)) b \, db \, d\varepsilon \,. \tag{1}$$

- $A(\vec{v}, \vec{v}_1; \phi_i^j)$ is independent of time and can be **precomputed**.

Properties of the collision kernel

- **Theorem 1.** Let operator $A(\vec{v}, \vec{v_1}; \phi_i^j)$ be defined by (1) with all gas particle having the same mass and potential of the particle interaction being spherically symmetric. Then $A(\vec{v}, \vec{v_1}; \phi_i^j)$ is symmetric with respect to $\vec{v}$ and $\vec{v_1}$, that is

$$A(\vec{v}, \vec{v_1}; \phi_i^j) = A(\vec{v_1}, \vec{v}; \phi_i^j) \quad \forall \vec{v}, \vec{v_1} \in \mathbb{R}^3$$

  Also,

$$A(\vec{v}, \vec{v}; \phi_i^j) = 0 \quad \forall \vec{v} \in \mathbb{R}^3$$

- **Theorem 2.** Let operator $A(\vec{v}, \vec{v_1}; \phi_i^j)$ be defined by (1) and let potential of molecular interaction be dependent only on the distance between the particles. Then $\forall \vec{\xi} \in \mathbb{R}^3$,

$$A(\vec{v} + \vec{\xi}, \vec{v_1} + \vec{\xi}; \phi_i^j(\vec{u} - \vec{\xi})) = A(\vec{v}, \vec{v_1}; \phi_i^j)$$

  I.e, $A(\vec{v}, \vec{v_1}; \phi)$ is invariant with respect to a shift.

Derivation of the methods

1. Approximate $f(t, \vec{x}, \vec{v})$ as a sum of Maxwellian distributions.

$$f(t, \vec{x}, \vec{v}) = \sum_{i=1}^{p} f_M(t, \vec{x}, \vec{v}) \,.$$

The density estimation of solutions $f(t, \vec{x}, \vec{v})$ can be done by applying Expectation Maximization Algorithm (EMA).

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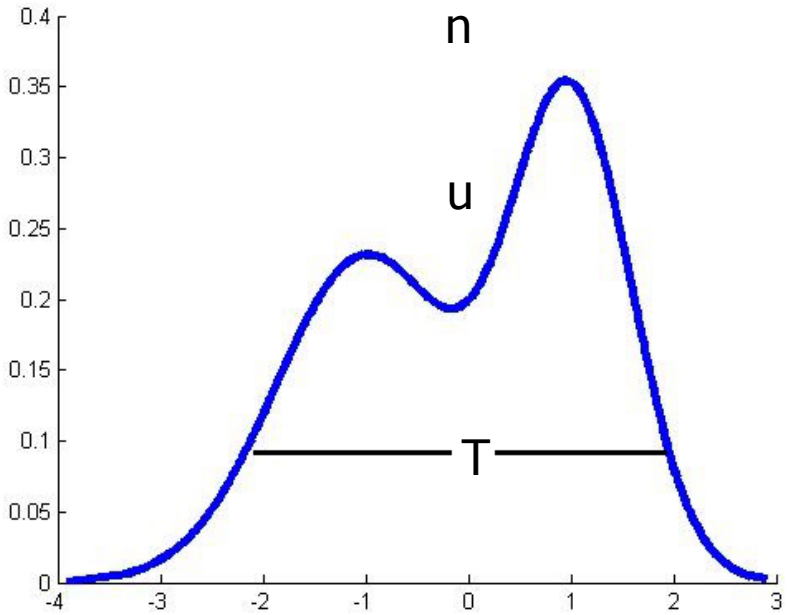

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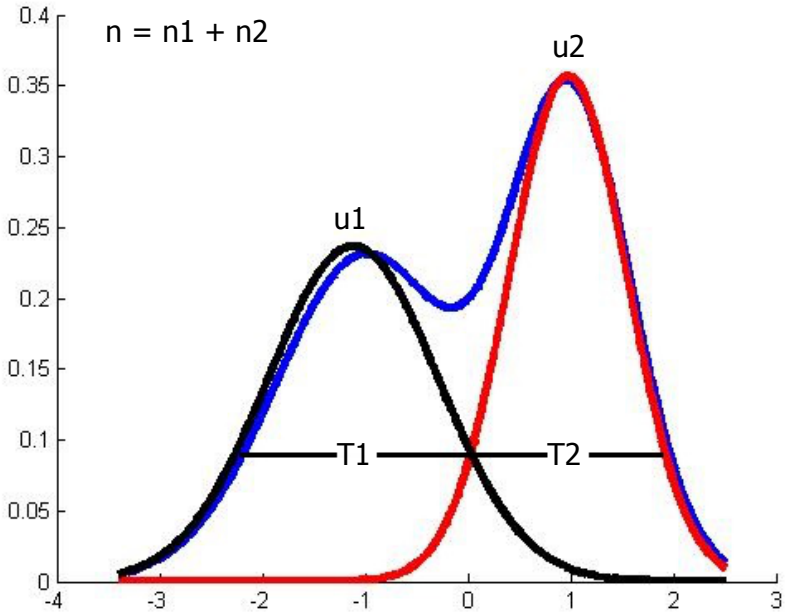

.

The density estimation of solutions $f(t, \vec{x}, \vec{v})$ can be done by applying Expectation Maximization Algorithm (EMA).

Derivation of the methods (cont.)

2. Rewrite DG projection of the B.C.I in convolution form.

Using the invariant property of $A(\vec{v}_1, \vec{v}; \phi_i^j)$ with respect to a shift, we can re-write the collision integral as:

$$I_i(\vec{\xi}) = \int_3 \int_3 f(t, \vec{x}, \vec{v} - \vec{\xi}) f(t, \vec{x}, \vec{v}_1 - \vec{\xi}) A(\vec{v}_1, \vec{v}; \phi_i^c) d\vec{v} d\vec{v}_1 \, .$$

where $\vec{\xi} = \vec{\xi^j}$ is the vector that connects the centers of the element K_c (cell in the center) and the element K_j.

Derivation of the methods (cont.)

3. Replacing the density estimation expression of $f(t, \vec{x}, \vec{v})$ into the formulation of $I_i(\vec{\xi})$, we obtain:

$$
\begin{aligned}
I_i(\vec{\xi}) &= \frac{8}{\omega_i \Delta \vec{v}} \int_3 \int_3 \left(\sum_{i'=1}^{p} f_{M_{i'}}(t, \vec{x}, \vec{v} - \vec{\xi}) \right) \left(\sum_{i''=1}^{p} f_{M_{i''}}(t, \vec{x}, \vec{v}_1 - \vec{\xi}) \right) A(\vec{v}_1, \vec{v}; \phi_i^c) d\vec{v} d\vec{v}_1 \\
&= \frac{8}{\omega_i \Delta \vec{v}} \sum_{\substack{i' \neq i'' \\ i', i''=1}}^{p} \int_3 \int_3 f_{M_{i'}}(t, \vec{x}, \vec{v} - \vec{\xi}) \, f_{M_{i''}}(t, \vec{x}, \vec{v}_1 - \vec{\xi}) A(\vec{v}_1, \vec{v}; \phi_i^c) \, d\vec{v} d\vec{v}_1 \\
&= \frac{8}{\omega_i \Delta \vec{v}} \sum_{\substack{i' \neq i'' \\ i', i''=1}}^{p} \left(I_i^{i' i''}(\vec{\xi}) \right),
\end{aligned}
$$

where $I_i^{i' i''}(\vec{\xi}) = \int_3 \int_3 f_{M_{i'}}(t, \vec{x}, \vec{v} - \vec{\xi}) \, f_{M_{i''}}(t, \vec{x}, \vec{v}_1 - \vec{\xi}) A(\vec{v}_1, \vec{v}; \phi_i^c) \, d\vec{v} d\vec{v}_1$.

Derivation of the methods (cont.)

4. Diagonalization of $A(\vec{v}, \vec{v}_1; \phi_i^c)$.
   - In the discrete velocity space, kernel $A(\vec{v}, \vec{v}_1; \phi_i^c)$ can be represented by an $N \times N$ symmetric matrix. Thus we can compute its Eigen-Decomposition using available software such as LAPACK and/or PROPACK:

$$A(\vec{v}, \vec{v}_1; \phi_i^c) = \sum_{k=1}^{N} \lambda_k \Psi_k(\vec{v}) \Psi_k^T(\vec{v}_1) + \varepsilon_{DG}$$

Derivation of the methods (cont.)

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   - We can further approximate $A(\vec{v}, \vec{v}_1; \phi_i^c)$ using M-pairs of eigenvalues and eigenvectors:

$$A(\vec{v}, \vec{v}_1; \phi_i^c) = \sum_{k=1}^{M} \lambda_k \Psi_k(\vec{v}) \Psi_k^T(\vec{v}_1) + \varepsilon_{DG} + \varepsilon_{truncation}$$

   where $\varepsilon_{truncation} = \mathrm{O}\left(\sqrt{\frac{\sum_{i=M+1}^{N} |\lambda_i|^2}{\sum_{i=1}^{N} |\lambda_i|^2}}\right)$

Derivation of the methods (cont.)

5. Substituting the Eigen-Decomposition of $A(\vec{v}, \vec{v}_1; \phi_i^c)$ into $I_i^{i'i''}$:

$$I_i^{i'i''}(\vec{\xi}) = \int_3 \int_3 f_{M_{i'}}(t, \vec{v} - \vec{\xi})\, f_{M_{i''}}(t, \vec{v}_1 - \vec{\xi}) \left(\sum_{k=1}^{M} \lambda_k \Psi_k(\vec{v}) \Psi_k(\vec{v}_1) \right) d\vec{v} d\vec{v}_1$$

$$= \sum_{k=1}^{M} \lambda_k \left(\int_3 f_{M_{i'}}(t, \vec{v} - \vec{\xi}) \Psi_k(\vec{v})\, d\vec{v} \right) \left(\int_3 f_{M_{i''}}(t, \vec{v}_1 - \vec{\xi}) \Psi_k(\vec{v}_1)\, d\vec{v}_1 \right).$$

Derivation of the methods (cont.)

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Therefore,

$$I_{\phi_i^j} = \sum_{\substack{i' \neq i'' \\ i', i'' = 1}}^{p} \sum_{k=1}^{N} \lambda_k \left(\int_3 f_{M_{i'}}(t, \vec{v} - \vec{\xi}) \Psi_k(\vec{v})\, d\vec{v} \right) \left(\int_3 f_{M_{i''}}(t, \vec{v}_1 - \vec{\xi}) \Psi_k(\vec{v}_1)\, d\vec{v}_1 \right). \quad (2)$$

Derivation of the methods (cont.)

5. Substituting the Eigen-Decomposition of $A(\vec{v}, \vec{v}_1; \phi_i^c)$ into $I_i^{i'i''}$:

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6. Goal: To compute the convolution integrals accurately!

Pre-computed Ansatz: $F_k(\vec{\xi}, T)$

- To optimize the evaluation of (2), we design the following object:

$$F_k(\vec{\xi}, T) = \frac{1}{(\sqrt{\pi T})^3} \int_3 \exp\left(-\frac{|\vec{w} - \vec{\xi}|^2}{T}\right) \Psi_k(\vec{w}) d\vec{w}\,.$$

- Therefore,

$$\int_3 f_M(t, \vec{v} - \vec{\xi}) \Psi_k(\vec{v}) d\vec{v} = n\, F_k(\vec{\xi} + \bar{\vec{u}}, T)\,.$$

  That is, to obtain the values of the convolution integrals, we just need to perform interpolations for $F_k(\vec{\xi}, T)$.

- The values of $F_k(\vec{\xi}, T)$ can be computed accurately using adaptive quadrature with high Gauss order.

Evaluating B.C.I takes $\mathcal{O}(N^2)$ flops each iteration

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2. These operations are repeated for each of the N eigenvectors used in relation (2).

Overall, it takes $O(N^2)$ operations.

Note: The time of performing the projection algorithm $<<$ the time of performing DG interpolations for N eigenvectors as N is large.

Advantages & Disadvantages of the new method

Advantages:

- Fast method of order $O(N^2)$, where N is the total number of velocity points.
- Can be implemented using parallelization.

Disdvantages:

- Interpolation errors.
- Requires large storage $O(MN^2)$, where M is a fraction of N, to store the values of the pre-computed ansatz F_k