

STEKLOFF EIGENVALUES AND INVERSE SCATTERING THEORY

David Colton, Peter Monk
University of Delaware

Fioralba Cakoni
Rutgers University

Research supported by AFOSR Grant FA 9550-13-1-0199



Scattering by an Inhomogeneous Medium

The **direct scattering problem** under consideration is to find u such that



$$\begin{aligned}\Delta u + k^2 n(x)u &= 0 && \text{in } \mathbb{R}^3 \\ u &= u^s + u^i \\ \lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) &= 0\end{aligned}$$

where $u^i(x) = e^{ikd \cdot x}$, $|d| = 1$, $n(x)$ is piecewise continuous and $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus \bar{D}$. There exists a unique solution u of the direct scattering problem for which u^s has the asymptotic behavior

$$u^s(x, d) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + O\left(\frac{1}{r^2}\right)$$

where $\hat{x} = x/|x|$. u_∞ is the **far field pattern** of u^s .

Transmission Eigenvalues

The **far field operator** $F : L^2(S^2) \rightarrow L^2(S^2)$ where $S^2 := \{x : |x| = 1\}$ is defined by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d)g(d)ds_d$$

and is injective unless k is a **transmission eigenvalue**, i.e. a value of k such that there exists a nontrivial solution to

$$\begin{array}{lll} \Delta w + k^2 n(x)w = 0 & \text{in} & D \\ \Delta v + k^2 v = 0 & \text{in} & D \\ w = v & \text{on} & \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on} & \partial D \end{array}$$

where ν is the unit outward normal to ∂D .

Transmission eigenvalues can be determined using the operator F and carry information about $n(x)$.

Transmission Eigenvalues

Let k_1 be the first transmission eigenvalue and suppose $n(x) > 1$ for $x \in \bar{D}$ or $n(x) < 1$ for $x \in \bar{D}$. Then, given k_1 and a knowledge of D , a constant n_0 can be determined such that the scattering problem for $n(x) = n_0$ also has k_1 as its first transmission eigenvalue. Then

$$\min_{\bar{D}} n(x) \leq n_0 \leq \max_{\bar{D}} n(x).$$

Flaws or voids in D can be detected by changes in k_1 and hence n_0 .

In our previous research initiative with Wright-Patterson AFB the above ideas were applied to anisotropic materials interrogated by microwaves, in which case the changes in the material structure is characterized by changes in the eigenvalues of the permittivity tensor. Two problems were considered:

- 1 Scattering by anisotropic dielectrics in free space,
- 2 Scattering by isotropic dielectrics on a perfectly conducting plane.

See RIMSS Task Order 0005, Final Report, August 2013.

Pros and Cons of Using Transmission Eigenvalues

Pros:

- 1 A number n_0 is obtained which for isotropic media is close to an average value of $n(x)$ and for anisotropic media is close to the arithmetic mean of the eigenvalues.
- 2 Transmission eigenvalues are a physical characterization of the media which corresponds to the non-scattering of special incident fields.

Cons:

- 1 The first transmission eigenvalue is determined by the material properties of the scatterer, i.e. one can not choose the interrogating frequency.
- 2 The method of transmission eigenvalues only applies to dielectrics or materials with very small absorption.

Modified Far Field Operator

An alternative approach to using transmission eigenvalues is to modify the far field operator and make use of **Stekloff eigenvalues**. In particular let ν be the unit outward normal to ∂D and h denote the solution of the **exterior impedance problem** (where $\Im(\lambda) \geq 0$):

$$\Delta h + k^2 h = 0, \quad x \in \mathbb{R}^3 \setminus \bar{D}$$

$$h(x) = e^{ikd \cdot x} + h^s(x)$$

$$\frac{\partial h}{\partial \nu} + \lambda h = 0 \quad \text{on } \partial D$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial h^s}{\partial r} - ikh^s \right) = 0.$$

Modified Far Field Operator

If we now replace the far field operator F by the **modified far field operator** $\mathcal{F} : L^2(S^2) \rightarrow L^2(S^2)$ defined by

$$(\mathcal{F}g)(\hat{x}) := \int_{S^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)] g(d) ds(d) \quad (*)$$

where h_∞ is the far field pattern of h then \mathcal{F} is injective unless λ is a **Stekloff eigenvalue** i.e. a value of λ such that there exists a nontrivial solution of

$$\begin{aligned} \Delta w + k^2 n(x)w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} + \lambda w &= 0 \quad \text{on } \partial D. \end{aligned}$$

Modified Far Field Operator

Now for $z \in D$ let

$$\Phi(x, z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$$

and let Φ_∞ be the far field pattern for $\Phi(x, z)$, i.e.

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z}, \quad \hat{x} = \frac{x}{|x|}.$$

Consider the modified equation

$$\mathcal{F}g = \Phi_\infty(\hat{x}, z)$$

where \mathcal{F} is the modified far field operator defined by $(*)$ and for $g \in L^2(S^2)$ define the **Heglotz wave function** with kernel g by

$$v_g(x) := \int_{S^2} \exp(ikx \cdot d) g(d) ds(d).$$

Modified Far Field Operator

Theorem

If λ is not a Stekloff eigenvalue then for every $\epsilon > 0$ there exists $g_\epsilon^z \in L^2(S^2)$ such that for $z \in D$

$$\|\mathcal{F}g_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} < \epsilon. \quad (**)$$

If w is the solution to

$$\begin{aligned} \Delta w + k^2 n(x)w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} + \lambda w &= \frac{\partial \Phi(\cdot, z)}{\partial \nu} + \lambda \Phi(\cdot, z) \quad \text{on } \partial D. \end{aligned}$$

then w can be uniquely decomposed as $w = w^i + w^s$ where $w^i \in H^1(D)$ satisfies $\Delta w + k^2 n(x)w = 0$ in D and $w \in H_{loc}^1(\mathbb{R}^3)$ satisfies the radiation condition. The function g_ϵ^z in $(**)$ is such that $v_{g_\epsilon^z}$ satisfies

$$\|w^i - v_{g_\epsilon^z}\|_{H^1(D)} = O(\epsilon).$$

Modified Far Field Operator

Theorem

- 1 Assume that λ is not a Stekloff eigenvalue and let g_ϵ^z satisfy (**). Then if $v_{g_\epsilon^z}$ is the Herglotz wave function with kernel g_ϵ^z for every $z \in D$ we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ is bounded as $\epsilon \rightarrow 0$.
- 2 Assume that λ is a Stekloff eigenvalue and let $g_\epsilon^z \in L^2(S^2)$ satisfy (**). Then for almost every $z \in D$ we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ cannot be bounded as $\epsilon \rightarrow 0$.

Remark If λ is a Stekloff eigenvalue then (**) is valid unless the Stekloff eigenfunction can be uniquely continued as a solution of $\Delta w + k^2 n(x)w = 0$ into all of \mathbb{R}^3 .

Stekloff Eigenvalues

Theorem

Assume that $n(x)$ is real valued. Then Stekloff eigenvalues exist, are real and are discrete.

The case when $n(x)$ is complex valued (i.e. the scattering object is absorbing) is more difficult since then the eigenvalue problem is no longer self-adjoint.

Theorem

Assume that $n(x) = n_1(x) + i \frac{n_2(x)}{k}$ when $n_1 > 0$ and $n_2 > 0$. Then

- 1 There exist infinitely many Stekloff eigenvalues in the complex plane and they form a discrete set without finite accumulation points.
- 2 Except for a finite number of eigenvalues, all the Stekloff eigenvalues lie in a wedge of arbitrarily small angle with lower edge on the negative x -axis.

Stekloff Eigenvalues

The advantage of using Stekloff eigenvalues as a target signature are the following:

- 1 The interrogation frequency can be chosen arbitrarily.
- 2 The method of Stekloff eigenvalues in principle applies to both absorbing and non-absorbing media.

Remark: The Stekloff eigenvalue problem for Maxwell's equations is an open problem. In particular, when D is a ball it can be shown that if λ is a Maxwell's Stekloff eigenvalue then so is $1/\lambda$ i.e. the use of compact operators to establish the existence and discreteness of Stekloff eigenvalues is problematic.

Stekloff Eigenvalues and Nondestructive Testing

In nondestructive testing one is interested in small changes in the inhomogeneity $n(x)$. In particular, support $n(x)$ is perturbed by δn giving rise to a change in the Stekloff eigenfunction $w \in H^1(D)$ by δw and Stekloff eigenvalue by $\delta\lambda$. Define

$$(f, g) := \int_D f \bar{g} dx, \quad \langle f, g \rangle := \int_{\partial D} f \bar{g} ds.$$

Then, neglecting quadratic terms, we have that

$$\delta\lambda \approx \frac{k^2 (\delta n w, w)}{\langle w, w \rangle}$$

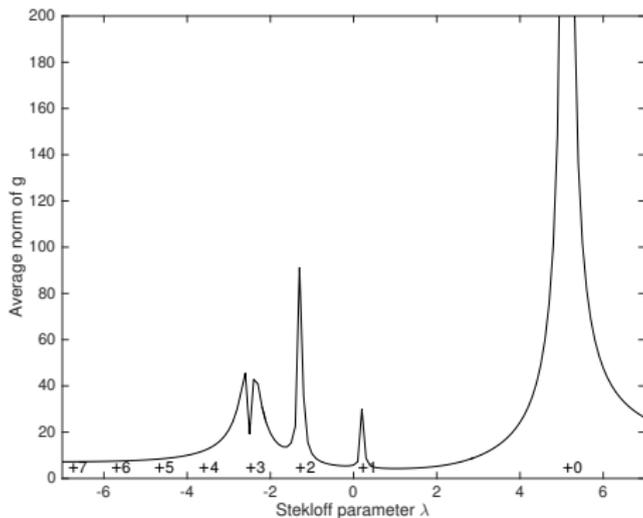
We will illustrate the applicability of this formula in the numerical examples that follows

Remarks on Numerical Examples

- The preceding formula suggests that some eigenvalues are more susceptible to changes in $n(x)$ than others.
- The first two or three Stekloff eigenvalues (ordered by absolute value) are the ones best approximated using the far field data for the wave number we have used.
- Examples are given using far field data. However, near field data can also be used just as easily.

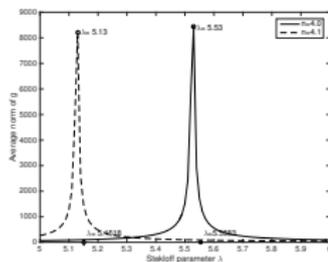
Stekloff eigenvalues: Unit Disk

- D is the unit disk
- $n(x) = 4, k = 1$
- Arbitrary 51 incoming waves
- No extra noise on the data
- Eigenvalues are exact and shown by + in the graph.

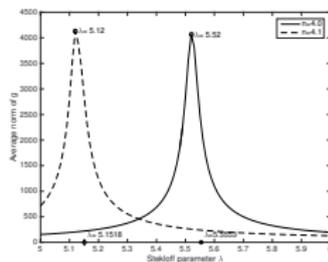


Sensitivity to noise

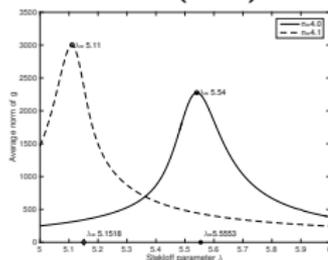
- Same example as before but $n(x) = 4$ or $n(x) = 4.1$
- Noise added pointwise
- Percentage is the relative ℓ^2 norm



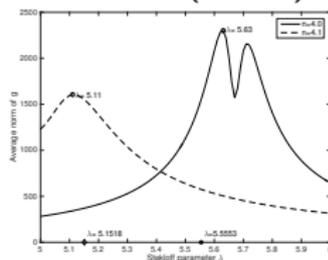
$\epsilon = 0(0\%)$



$\epsilon = 0.01(.57\%)$



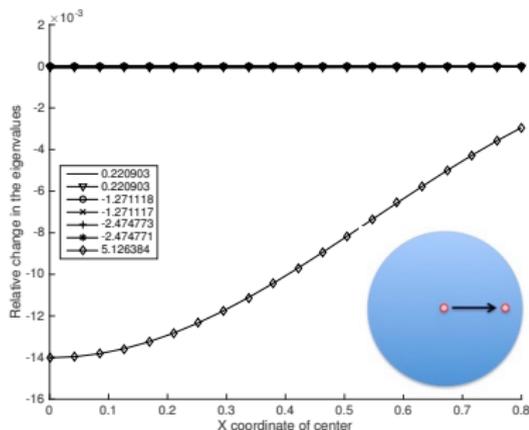
$\epsilon = 0.05(2.9\%)$



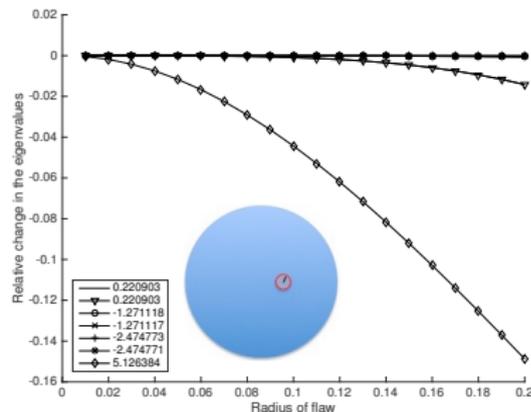
$\epsilon = 0.15(8.6\%)$

Sensitivity of Eigenvalues: Unit Disk with Flaw

The “flaw” is a circular region of radius r_c centered at $(x_c, 0)$ with $n(x) = 1$ inside the flaw. Noise $\epsilon = 0.01$. Wavenumber $k = 1$.



Changing x_c , $r_c = 0.05$

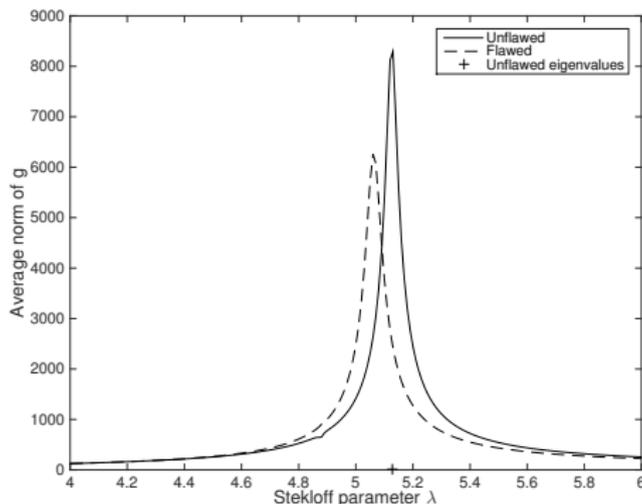


Changing r_c , $x_c = 0.3$.

Plot $(\lambda_{j^*}^c - \lambda_j)/|\lambda_j|$, $j = 1, \dots, 7$ against geometric parameters.

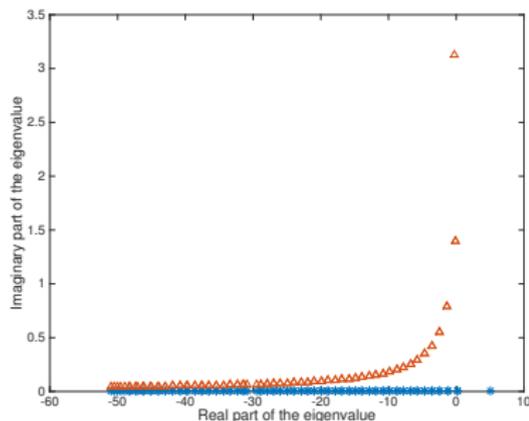
Changes in eigenvalues: Unit Disk with Flaw

Flaw is radius $r_c = 0.05$ centered at $(0.3, 0)$. All parameters as in the previous examples.

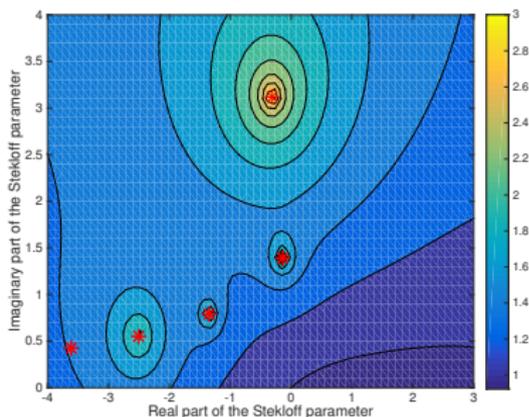


Complex Eigenvalues: Unit disk $n(x) = 4 + 4i$

Complex eigenvalues can be detected by the same procedure as before but now searching in a region in the complex plane.



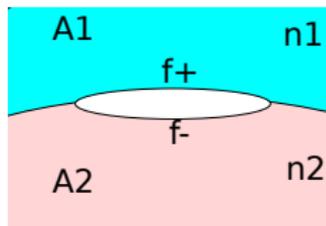
Comparison of eigenvalues
for $n(x) = 4$ (blue)
and $n(x) = 4 + 4i$ (red)



Contours of $\log_{10}(\|g\|)$.
Exact Stekloff eigenvalues
are shown as *.

Other Projects - Delamination

- **Non-destructive testing of the interface between two different materials.** This project is motivated by the possibility of using elastic waves to inspect whether the solid propellant in a rocket has separated from its body¹.



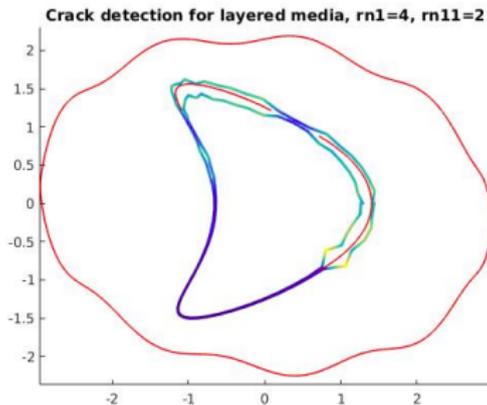
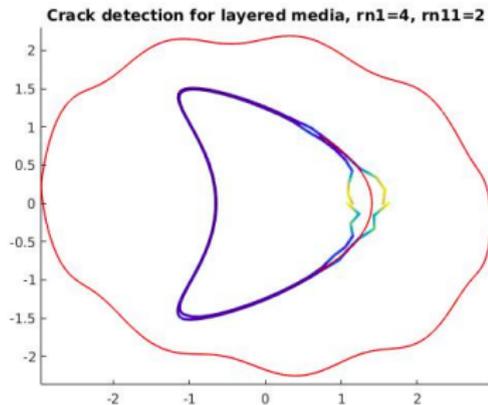
A_1, n_1 and A_2, n_2 are the **constitutive parameters** of material 1 and 2, respectively. f_1 and f_2 indicate the profile of the opening on the interface.

- Derive an approximate model based on asymptotic analysis.
- Use the linear sampling method to detect the defective part of the interface.
- Analyze the corresponding transmission eigenvalues to obtain additional information about the opening.

¹F. CAKONI, I. DE TERESA TRUEBA, H. HADDAR AND P. MONK,

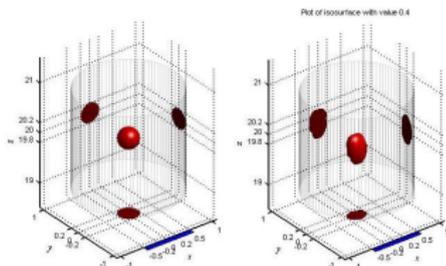
Other Projects - Delamination

Examples of reconstruction of the delaminated part of the interface using the linear sampling method based on the approximate asymptotic model.



Other projects: Electromagnetic Waveguides

- Use frequency domain data to image objects in a perfectly conducting waveguide. This project investigates the frequency domain LSM for finding perfectly conducting objects in a waveguide².



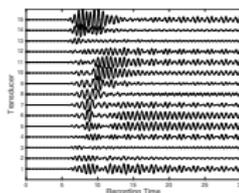
Reconstruction of a sphere of radius 0.2 in a unit radius waveguide. Measurements only above the scatterer.

- Variational analysis of the forward problem.
- Method of fundamental solutions to solve the forward problem.
- Analysis of the Linear Sampling method in this application.

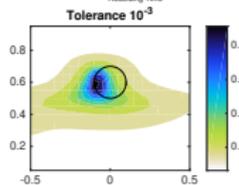
²FAN YANG, PhD thesis

Other projects: Time domain inverse problem in a waveguide

- Use time domain data directly to image objects in a waveguide. This project investigates the time domain LSM for finding sound soft objects in a sound hard waveguide³.



Time domain near field data



Scatter of radius .1. Unit height waveguide.

Reconstruction of a sound soft scatterer.

- Laplace domain techniques to analyze the forward problem.
- Convolution Quadrature time domain boundary integral forward solver.
- Analyze the Time Domain Linear Sampling method in this application.

³P. MONK AND V. SELGAS, submitted for publication,

Thanks

Thanks to our students:

- 1 Isaac Harris (now at TAMU)
- 2 Shizu Meng
- 3 Jake Rezak
- 4 Irene De Teresa Trueba
- 5 Fan Yang (now at Cal. Poly. San Luis Obispo)