On Solving the Perturbed Multi-Revolution Lambert Problem: Applications in Enhanced SSA

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JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 38, No. 9, September 2015

New Solutions for the Perturbed Lambert Problem Using Regularization and Picard Iteration

Outline

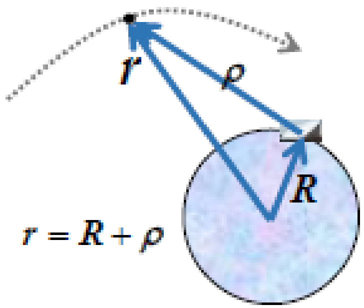
- Two Motivations
  - Advanced SSA Data Association for Radar Sensing
  - Extremal Field Maps For Multi-Revolution Orbit Transfer
    - Impulsive, Low-Thrust, and Hybrid

- Recent Advances in Solving the Fundamental Initial and Two-Point Boundary Value Problem of Astrodynamics: *Integral Path Iteration Methods*
  - Comparisons to State-of-Practice Algorithms

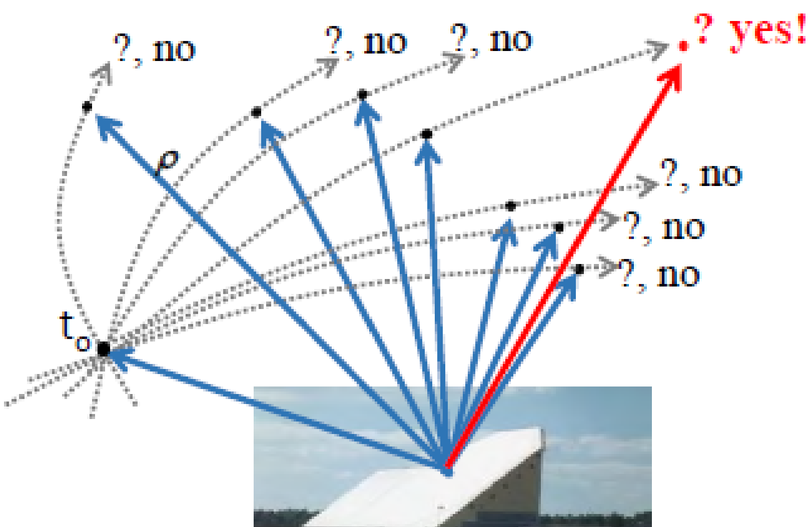
- Regularization, Insights & Consequences Thereof
  - KS Regularization
  - Regularized Orbit Elements

- Some Examples & Impacts

Data Association 101 for Radar Data



"See" *an object where predicted on prelim orbit* at future measurement times?



2 measurements sets ⇒ **$(v_x\ v_y\ v_z)_o$ =?**

3 constraints **Lambert's Problem** **3 unknowns**

To test each data association hypothesis: Solve Lambert's Problem using hypothesized pairing of 2 data sets and then propagate to subsequent measurement times to test hypothesis: *Does the orbit agree with additional measurements +/- bound?*

- **Radar Data Association Problem has >N^2 complexity, if we test preliminary orbit hypotheses using all possible observation pairs.**
  - For short arc case (fraction of an orbit), the problem is *much easier* than multi-orbit case.
  - Computation time for each hypothesis test is dominated by orbit propagation cost implicit in Lambert solution process.
- **Three Coupled Important Challenges:**
  - For longer arcs, approximation errors in the Keplerian Lambert solution are larger than measurement errors => not good!
  - Lambert algorithms using state-of-practice numerical propagation & high fidelity force models leads to SSA computing bottleneck.
  - For multi-rev case, in general, more than one orbit solution satisfies two positions and times.
- **Wish List for Research to Meet Challenges:**
  - Means for much more efficient solution process for *perturbed Lambert problem.*
  - Means to resolve ambiguities due to uncertainty, and especially, due to *multiplicity of solutions for multi-revolution case.*
  - Desire a *parallelizable & scalable approach*
  - Seek higher fidelity hypothesis test with *~ 100x speedup*

Accelerated Picard Iteration

- Successive *path approximation method* for solving nonlinear differential eqns of the form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

Charles Émile Picard
(1856-1941)

- Can be rearranged without approximation to the following integral:

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))ds,$$

- Sequence of trajectory approximations (Picard Iteration) generated by:

$$\tilde{\boldsymbol{x}}^{n+1}(t) = \boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(s, \tilde{\boldsymbol{x}}^{n}(s))ds, \quad n = 1, 2, \ldots; \quad \tilde{\boldsymbol{x}}^{0}(\tau)) = \text{"warm start"}$$

Picard proved the general circumstances under which: $\tilde{\boldsymbol{x}}^{n}(t) \Rightarrow \boldsymbol{x}(t)$.

- Large domain of convergence, with a *geometric* convergence rate. In discussions with Atluri, we found dramatic *terminal convergence speedup* using an *integral eqn error feedback term*:

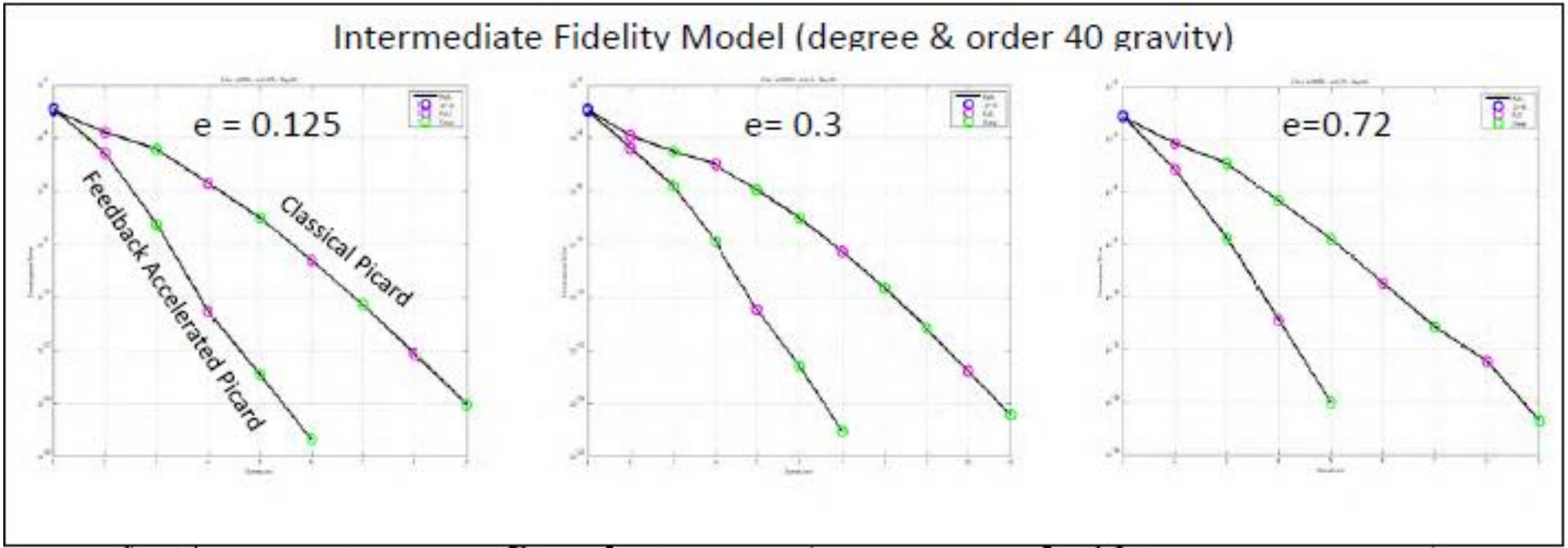
$$\boxed{\tilde{\boldsymbol{x}}^{n+1}(t) = \boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}^{n}(s))ds},$$

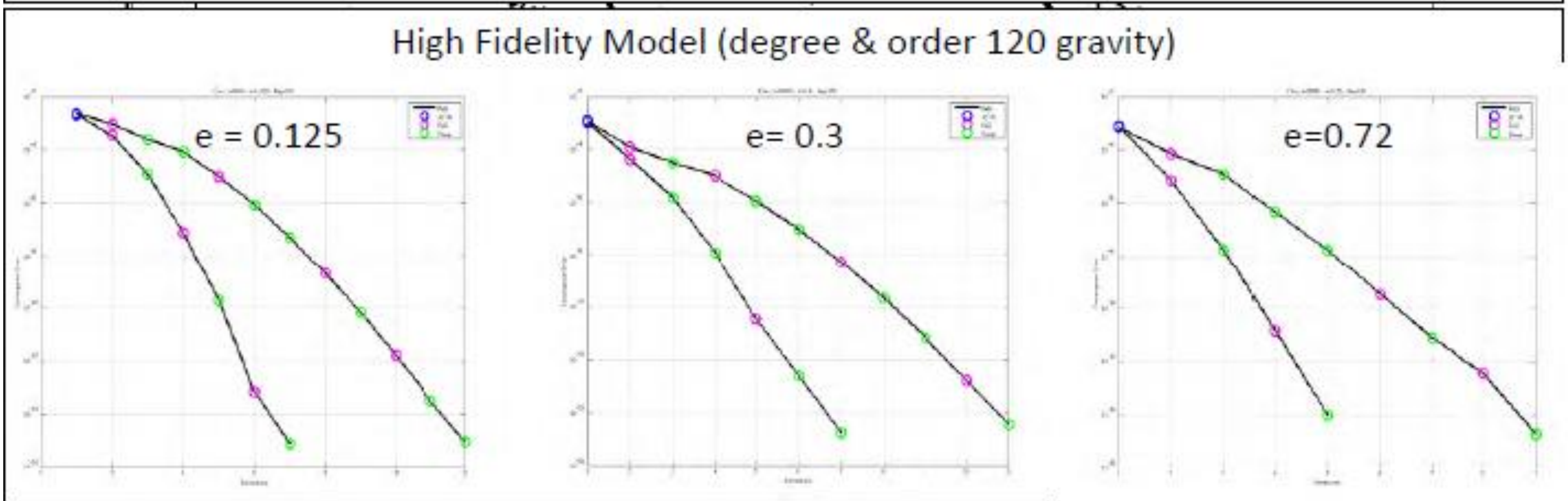
$$\boldsymbol{x}^{n+1}(t) = \tilde{\boldsymbol{x}}^{n+1}(t) + \int_{t_0}^{t} \{[\boldsymbol{J}(s, \boldsymbol{x}^{n}(s))]\overbrace{[\underbrace{\boldsymbol{x}(t_0) + \int_{t_0}^{s} \boldsymbol{f}(\boldsymbol{x}^{n}(\eta), \eta)d\eta}_{\tilde{\boldsymbol{x}}^{n+1}(s)} - \boldsymbol{x}^{n}(s)]}^{\text{“}n^{\text{th}}\text{ iteration equation error”}}\}ds, \quad [\boldsymbol{J}(t, \boldsymbol{x}^{n}(t))] \equiv \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}^{n}(t)}$$

$$\Rightarrow \quad \boxed{\boldsymbol{x}^{n+1}(t) = \tilde{\boldsymbol{x}}^{n+1}(t) + \int_{t_0}^{t} \{[\boldsymbol{J}(s, \boldsymbol{x}^{n}(s))][\tilde{\boldsymbol{x}}^{n+1}(s) - \boldsymbol{x}^{n}(s)]\}ds} \quad \Leftrightarrow$$

very signigicantly accelerates Picard Iteration

Impact of Integral Error Feedback





KS Regularizing Transformation

Cartesian Coordinates

$$\left.\begin{aligned}\ddot{x} &= -\frac{\mu}{r^3}x + a_{d_x}\\ \ddot{y} &= -\frac{\mu}{r^3}y + a_{d_y}\\ \ddot{z} &= -\frac{\mu}{r^3}z + a_{d_z}\end{aligned}\right\}$$

KS Coordinates ODEs (a *rigorous linearization* of 2-body prob):

$$\boxed{\boldsymbol{u}'' + \frac{1}{4}\boldsymbol{u} = \frac{1}{4\mu}\left(r + 4\boldsymbol{u}'^T\boldsymbol{u}'\right)\left(rI + 4\boldsymbol{u}'\boldsymbol{u}'^T\right)[L(\boldsymbol{u})]^T\boldsymbol{a}_d},\quad \Leftarrow \left.\begin{matrix}\text{new form of KS}\\ \text{eqs of motion}\end{matrix}\right\}, \text{ where}$$

$[L_4(\boldsymbol{u})]\boldsymbol{u}' = 0$ is an exact integral, even for generally perturbed motion, and

$$\frac{dt}{dE} = r\left[\frac{1}{2\mu}[r + 4\boldsymbol{u}'^T\boldsymbol{u}']\right]^{1/2}$$

Initial conditions must satisfy:

$$\left\{\begin{aligned}&[L(\boldsymbol{u}(0))]\boldsymbol{u}(0) = \boldsymbol{r}(0) \Rightarrow \infty \text{ of } \boldsymbol{u}(0) \text{ solutions}\\ &\boldsymbol{u}'(0) = \frac{1}{2\sqrt{\mu\alpha(0)}}[L(\boldsymbol{u}(0))]^T\dot{\boldsymbol{r}}(0)\end{aligned}\right.$$

The KS Transformation is:

$$\left.\begin{aligned}&\boldsymbol{r} = L(\boldsymbol{u})\boldsymbol{u}\\ &\frac{dt}{dE} \equiv t' = \frac{1}{\sqrt{\mu\alpha}}r\\ &\alpha = \frac{2}{r} - \frac{\dot{\boldsymbol{r}}^T\dot{\boldsymbol{r}}}{\mu} \equiv \frac{1}{a}\\ &\quad \equiv 2[r + 4\boldsymbol{u}'^T\boldsymbol{u}']^{-1}\end{aligned}\right\}, \textit{ where}$$

$$\boldsymbol{r} = \begin{Bmatrix}x\\ y\\ z\\ 0\end{Bmatrix},\quad \dot{\boldsymbol{r}} = \begin{Bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ 0\end{Bmatrix},\quad \boldsymbol{r}' = \begin{Bmatrix}x'\\ y'\\ z'\\ 0\end{Bmatrix},\quad \boldsymbol{u} = \begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix},\quad L(\boldsymbol{u}) \equiv \begin{bmatrix}u_1 & -u_2 & -u_3 & u_4\\ u_2 & u_1 & -u_4 & -u_3\\ u_3 & u_4 & u_1 & u_2\\ u_4 & -u_3 & u_2 & -u_1\end{bmatrix}.$$

Important Properties : $L^T(\boldsymbol{u}) = rL^{-1}(\boldsymbol{u}),\quad r = \boldsymbol{u}^T\boldsymbol{u}.$

Note the zero 4th element of $\boldsymbol{r}$ in the KS transformation $\boldsymbol{r} = L\boldsymbol{u}$ gives the identity

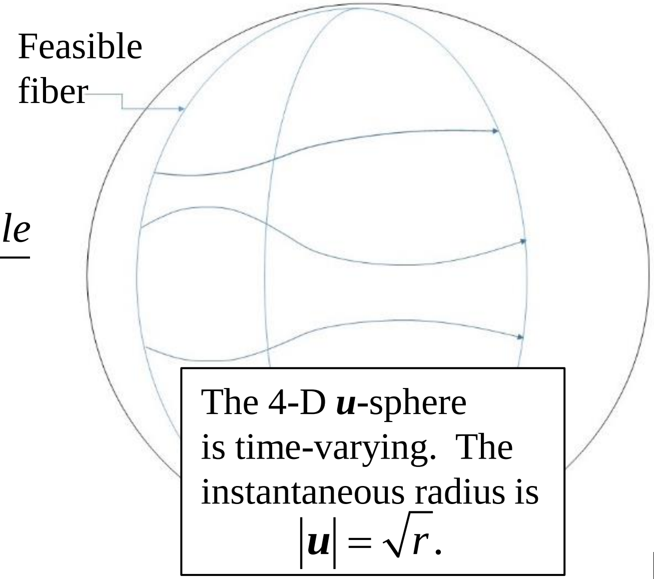
$$L_4(\boldsymbol{u})\boldsymbol{u} = [u_4 \quad -u_3 \quad u_2 \quad -u_1]\boldsymbol{u} = u_4u_1 - u_3u_2 + u_2u_3 - u_1u_4 \equiv 0.$$

If we have a vector $\boldsymbol{v}$ that satisfies $L_4(\boldsymbol{u})\boldsymbol{v} = 0$, then you can show $\boxed{L(\boldsymbol{u})\boldsymbol{v} \equiv L(\boldsymbol{v})\boldsymbol{u}.}$

Question: Will Picard Iteration (MCPI) converge better using the KS equations of motion?

Ans: Yes, MCPI convergences faster and over a ~3x larger time interval for both IVP & BVP.

KS Uniqueness Challenges
(for the perturbed Lambert problem)



Kustaanheimo - Stiefel Transformation Uniqueness Theorem

For any given $\{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$, there is an infinity of <u>*geometrically feasible*</u> $\boldsymbol{u}$-vectors that must lie on a 4D feasible $\boldsymbol{u}$-space curve known as a *fiber*.

Once a particular feasible $\boldsymbol{u}$-point is selected from some point on the feasible fiber, then the transformed velocity

$\boldsymbol{u}' = (1/2)(\mu/a)^{-1/2} L^T(\boldsymbol{u})\dot{\boldsymbol{r}}$ is unique and satisfies $L_4(\boldsymbol{u})\boldsymbol{u}' = 0$.

Important : The inverse transformation $\{\boldsymbol{r}(t) = L(\boldsymbol{u})\boldsymbol{u},\ \ \dot{\boldsymbol{r}}(t) = \dfrac{2(\mu/a)^{1/2}}{r} L(\boldsymbol{u})\boldsymbol{u}'\}$ *from* ***all infinity of feasible trajectories*** $\{\mathbf{u}(E), \mathbf{u}'(E)\}$ *that ensue from a feasible fiber initial state* $\{\mathbf{u}(0), \mathbf{u}'(0)\}$ <u>*give the same unique Cartesian space trajectory*</u> $\{\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t)\}$.

What does it mean for solving initial value problems?

It means $\{\boldsymbol{u}(E), \boldsymbol{u}'(E)\}$ from <u>*any* geometrically feasible initial $\boldsymbol{u}$-position</u>, with $\boldsymbol{u}' = (1/2)(\mu/a)^{-1/2} L^T(\boldsymbol{u})\dot{\boldsymbol{r}}$ will generate (upon inverse transformation) the <u>correct physical solution of a general initial value problem</u>.

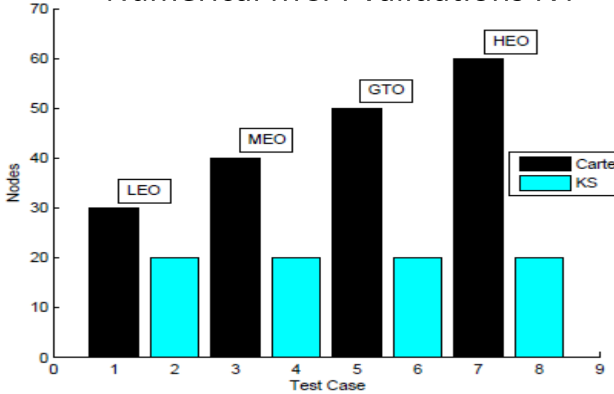
=> *This truth is well-known in the literature and is the main use of the KS transformation*.

However, to solve a TPBVP Lambert problem u-space …There are an infinity of feasible choices of geometrically compatible terminal (boundary) position coordinates on the two fibers in $\boldsymbol{u}$-space. **Question:** When I select one geometrically feasible point on the left (initial) fiber, which one do we select on the right (final) fiber *such that both of the specified initial and final* ***u*** *vectors lie on the* <u>*same dynamical path*</u> *in* ***u****-space*?

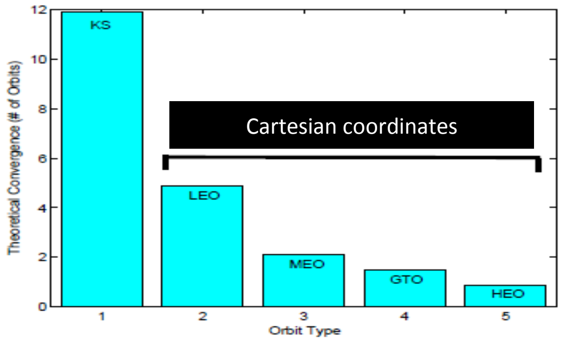
MCPI BVP Algorithm Has ~3x Increased Domain of Convergence using the KS ODEs Compared to Cartesian ODEs

Convergence is Independent of Eccentricity

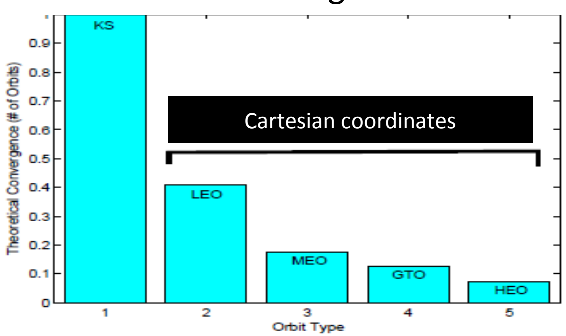
Numerical MCPI Validations IVP

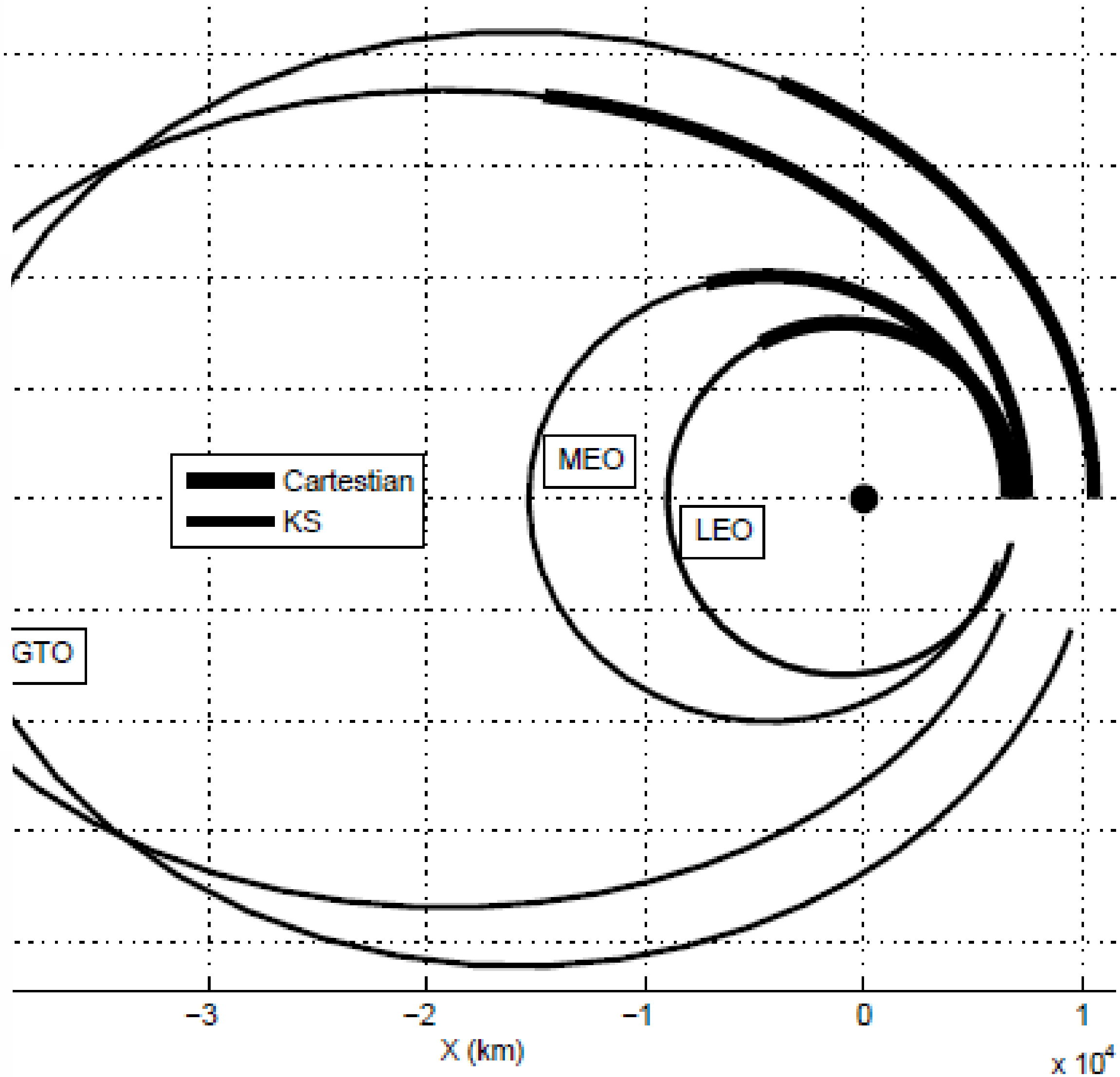


Theoretical Convergence for IVP

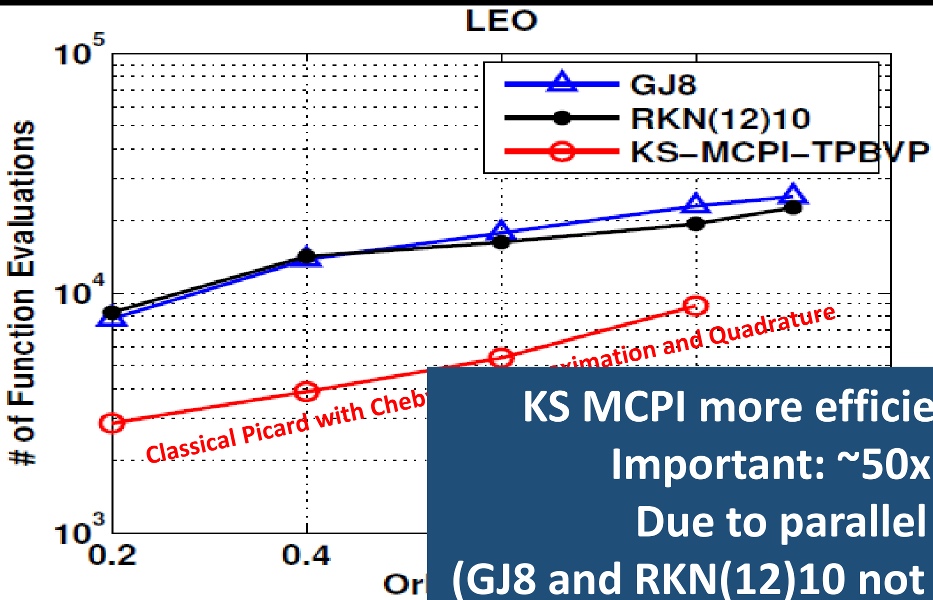


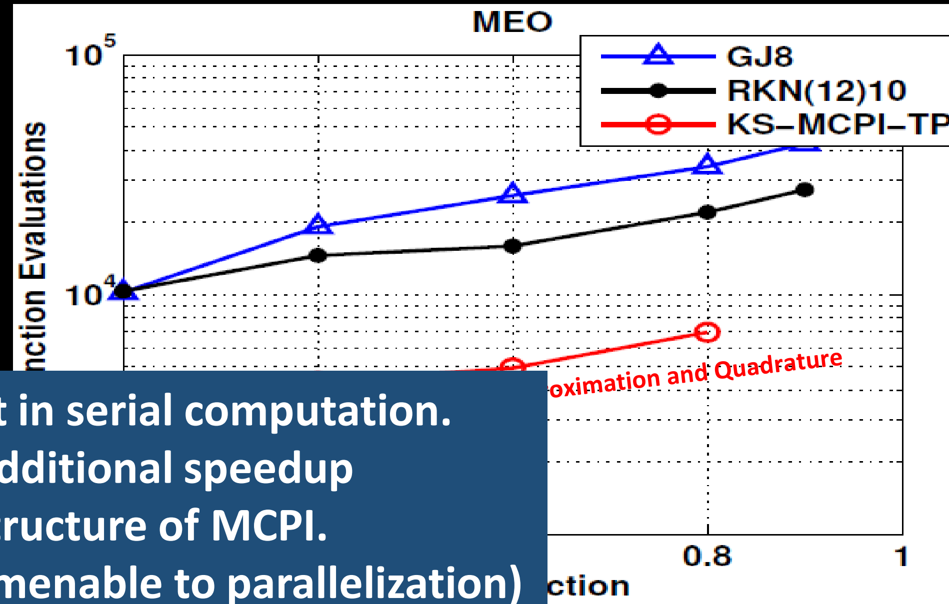
Theoretical Convergence for BVP



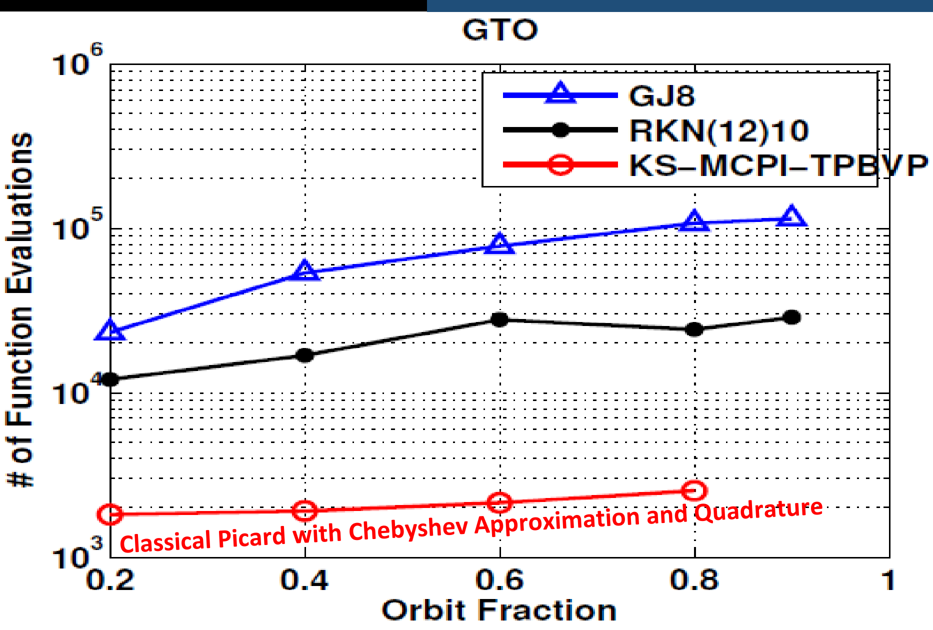


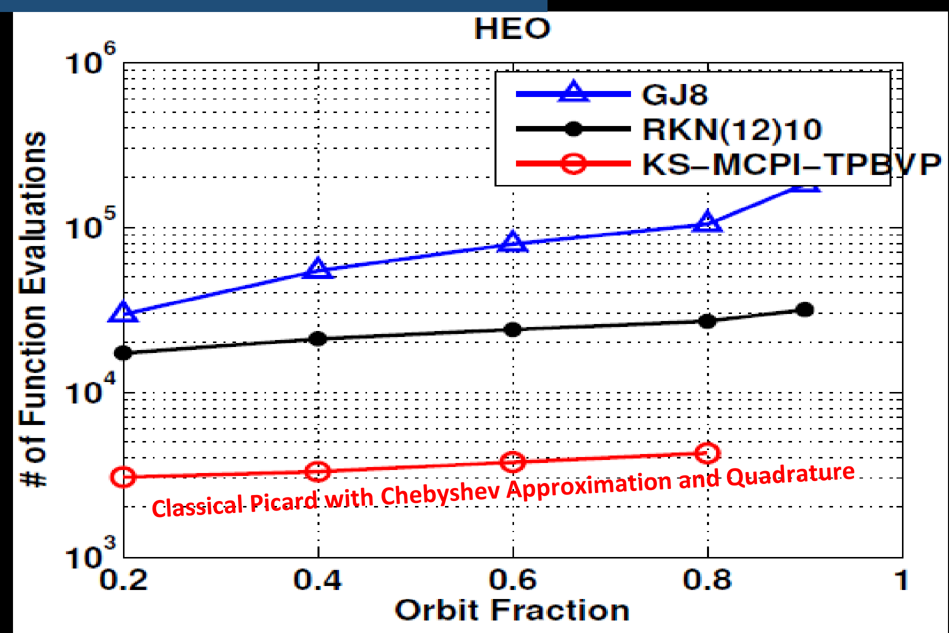
Efficiency of KS Lambert MCPI vs State of the Practice
for Fractional Orbit Case (40,40 Spherical Harmonic Gravity)



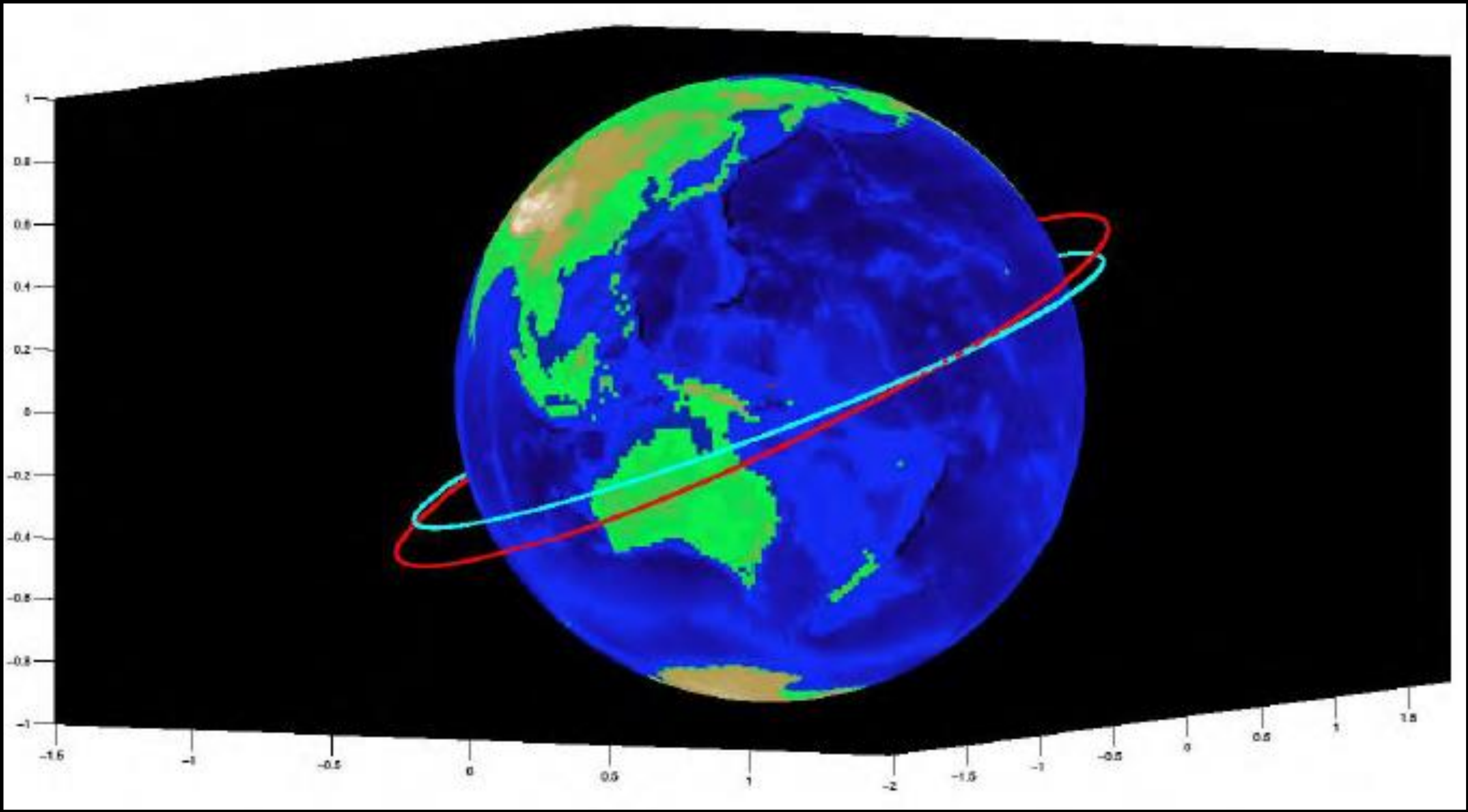


KS MCPI more efficient in serial computation.
Important: ~50x additional speedup
Due to parallel structure of MCPI.
(GJ8 and RKN(12)10 not amenable to parallelization)





A Specific Orbit Transfer….



Extremal Field Maps (EFMs): Multiple Solutions

of the Perturbed Lambert Problem => “Visibility of Reachability”

(40,40 Spherical Harmonic Gravity Field)

Near Real Time EFMs Have Important Utility:

=> Near Real Time Mission Planning (… on-board?)

=> Obtain Quick SSA Answers… Questions Such As

- Who can reach whom? For how much ΔV? When?

Equinoctal Variation of Parameters

$$\frac{dp}{dt} = \frac{2pa_\theta}{w}\sqrt{\frac{p}{\mu}}$$

$$\frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left\{a_r \;\; sin(L) + \frac{[(w+1)cos(L)+f]a_\theta}{w}\right.$$

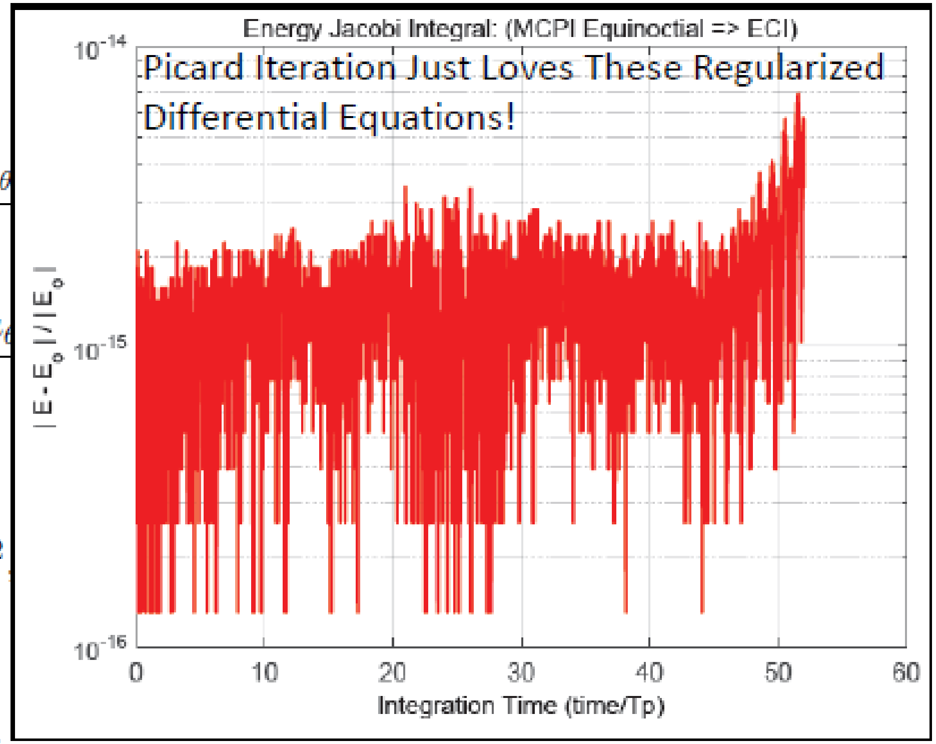
$$\frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left\{-a_r cos(L) + \frac{[(w+1)sin(L)+g]a_\theta}{w}\right.$$

$$\frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2 a_h}{2w}cos(L)$$

$$s^2 = 1 + h^2 + k^2$$

$$\frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2 a_h}{2w}sin(L)$$

$$\frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \sqrt{\frac{p}{\mu}}\frac{[hsin(L) - kcos(L)]a_h}{w}$$



Transformation from Equinoctal to classical elements:

$$e = \sqrt{f^2+g^2} \qquad a = \frac{p}{(1-e^2)} \qquad \nu = L - \bar{\omega}$$

$$i = 2tan^{-1}(\sqrt{h^2+k^2}) \qquad \Omega = tan^{-1}\left(\frac{k}{h}\right), \qquad \bar{\omega} \equiv \omega + \Omega = tan^{-1}\left(\frac{g}{f}\right), \quad \omega = \bar{\omega} - \Omega$$

$$r = a[1 - gsin(F) - fcos(F)] \qquad F = E + \omega + \Omega \qquad L = \nu + \omega + \Omega$$

Optimal Multi-Rev Low-Thrust Orbit Transfers

We seek to minimize:

$$J = \int_{t_0}^{t_f} 1 dt = t_f - t_0$$

Subject to:

$$\frac{d\mathbf{e}}{dt} = \mathbf{M}\frac{T}{m}\mathbf{u} + \mathbf{D}$$, with $\mathbf{D} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \sqrt{\mu p}\left(\frac{w}{p}\right)^2]^T$, and $\mathbf{e}_0$ and $\mathbf{e}_f$ specified.

The Hamiltontion is ….

$$H = 1 + \lambda^T \left[\mathbf{M}\frac{T}{m}\mathbf{u} + \mathbf{D}\right]$$

The optimal steering unit vector that minimizes H is

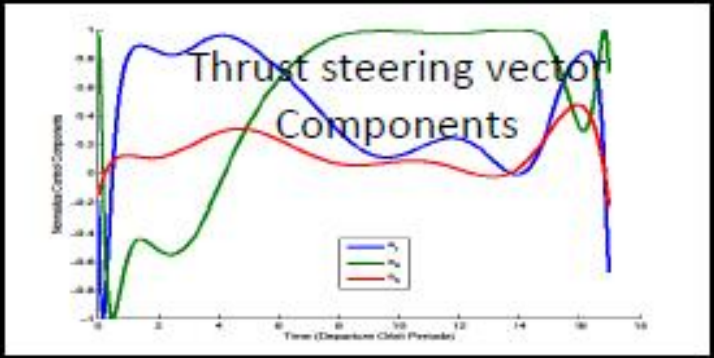
$$\mathbf{u} = -\frac{[\mathbf{M}^T\lambda]}{\sqrt{\lambda^T \mathbf{M}\mathbf{M}^T\lambda}} = -\hat{\mathbf{m}}$$

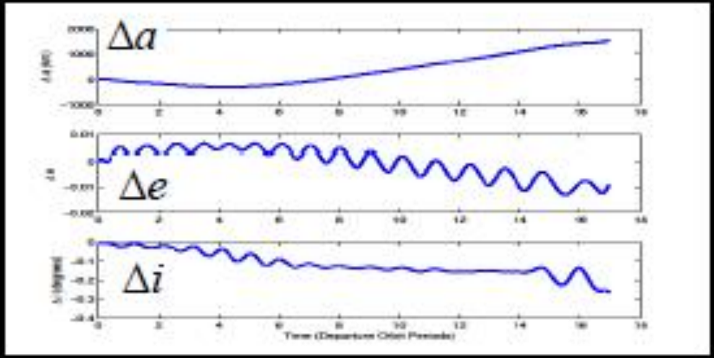
The co-state is governed by the differential equation:

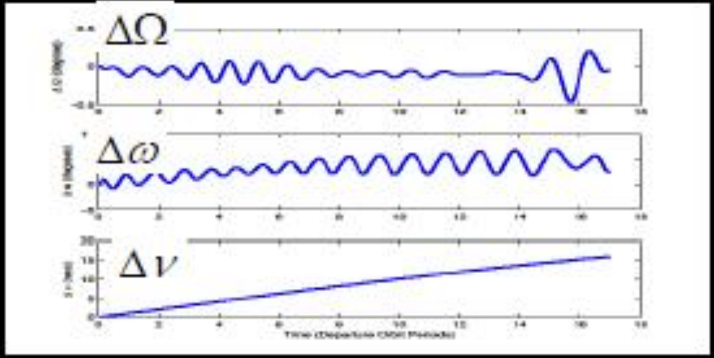
$$\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{e}} = -\left(\lambda^T \frac{\partial \mathbf{M}}{\partial \mathbf{e}}\frac{T}{m}\mathbf{u} + \lambda^T \frac{\partial \mathbf{D}}{\partial \mathbf{e}}\right), \quad \lambda(t_0) \text{ is unknown.}$$

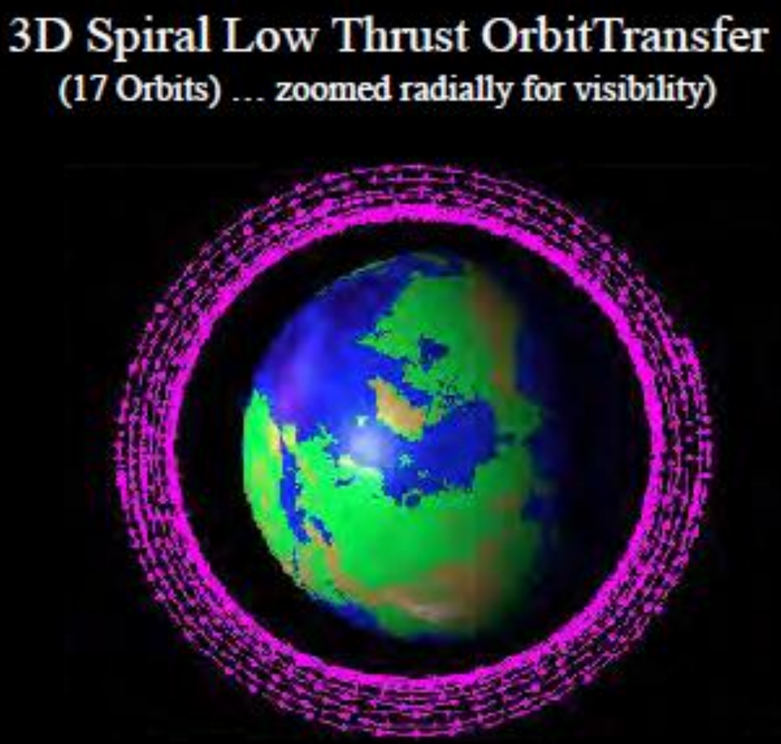
The 6 unknown initial co-states and the unknown final time are determined to satisfy the terminal state $\mathbf{e}(t_f) = e_f$, and $H(t_f)=0$. This two-point boundary value problem is highly nonlinear and requires a good starting solution …. determined by a direct optimization method.

Near Minimum Time Low thrust Orbit Transfer



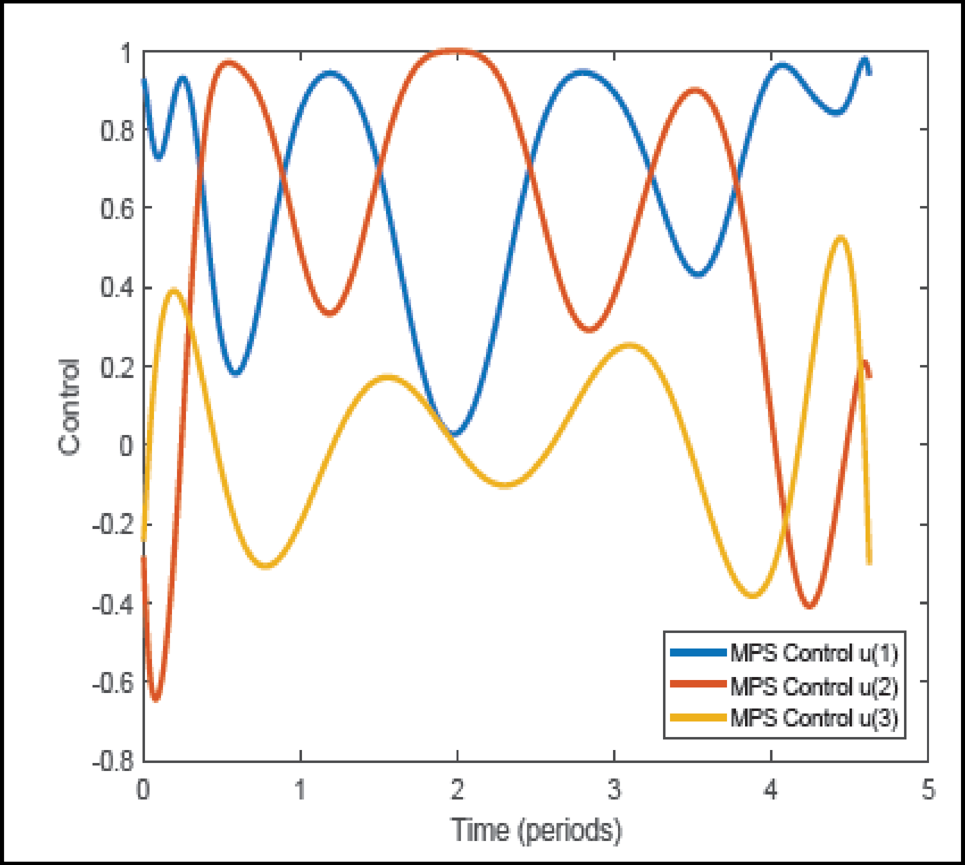


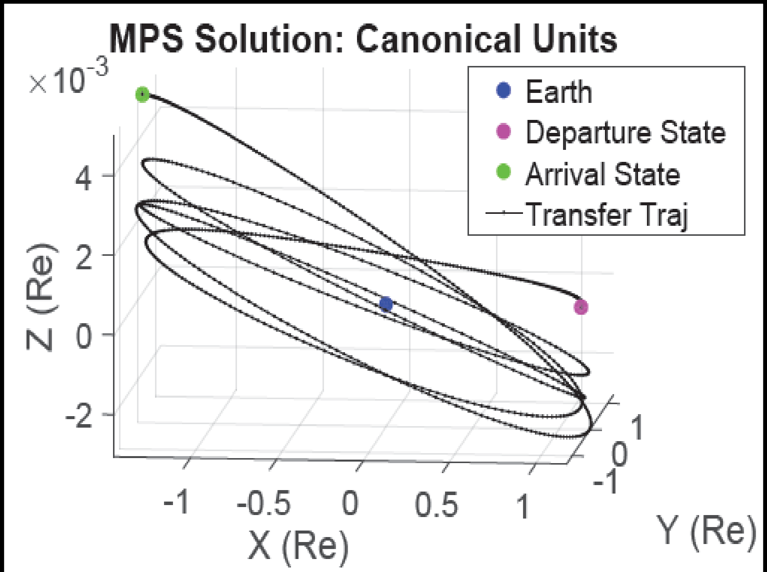


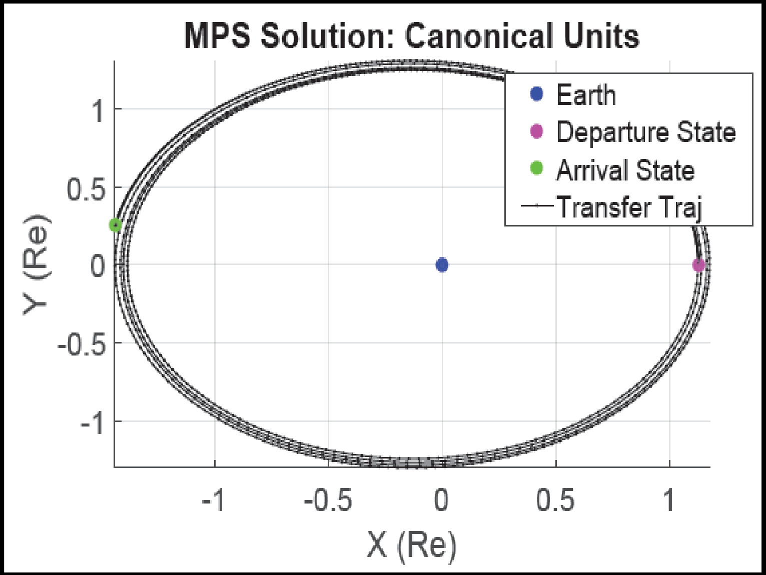


Near Planar Min Time Low Thrust Orbit Transfer
(much smaller plane change, shorter maneuver time)

Optimal Control Steering Vector Components
(radial, transverse, and orthogonal)







Outline

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  - Advanced SSA Data Association for Radar Sensing
  - Extremal Field Maps For Multi-Revolution Orbit Transfer
    - Impulsive, Low-Thrust, and Hybrid

- Recent Advances in Solving the Fundamental Initial and Two-Point Boundary Value Problem of Astrodynamics:
  Integral Path Iteration Methods
  - Comparisons to State-of-Practice Algorithms
    - Established significant new insights, formulations and computational methods to accelerate fundamental astrodynamics computations: orbit propagation and solution of Lambert’s problem.
    - >3x efficiency for serial implementation, ~10x to 50x further increase feasible via parallelization.

- Regularization, Insights & Consequences Thereof
  - KS Regularization
    - Found important insights to resolve ambiguities in solving the KS Lambert Problem.
  - Regularized Orbit Elements
    - Regularization enhances Picard Iteration for both initial and two-point boundary value problems.

- Some Examples & Impacts
  - Demonstrated a parallelizable and Scalable Approach for near real time computing of Extremal Field Maps for enhanced SSA, Mission Planning, etc.
    - Impulsive orbit transfers
    - Continuous low thrust orbit transfers
    - Hybrid thrust orbit transfers

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