
Mitigating uncertainty in imaging

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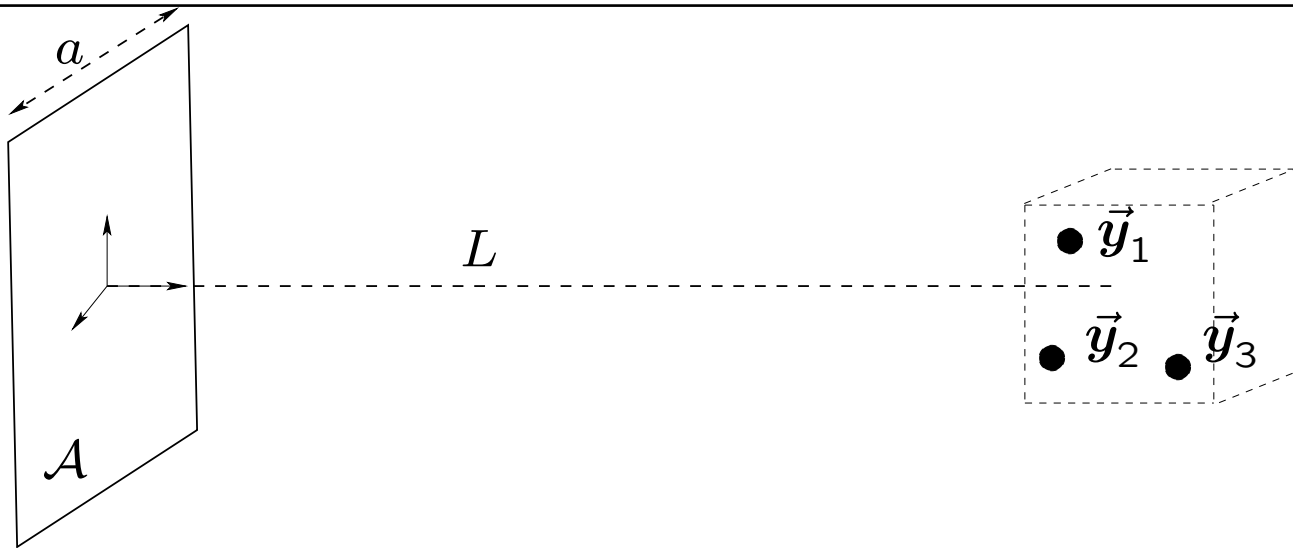
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Array imaging problem



Question: Can we achieve high resolution when imaging in the presence of **uncertainty**?

- **Strong noise:** Active array, time-harmonic E&M waves. Subspace projection (MUSIC) method. (With J. Garnier)
- **Random media:** Passive array, broad-band sonar sources. Convex optimization approach. (With I. Kocyigit)

Single frequency E&M active array imaging

- **Wave model:** $\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{x}; \vec{x}_s) - k^2 \frac{\epsilon(\vec{x})}{\epsilon_o} \vec{E} = ik \sqrt{\frac{\mu_o}{\epsilon_o}} \vec{j}_s \delta(\vec{x} - \vec{x}_s)$

with outgoing radiation condition, where k = wavenumber.

- Scatterers at \vec{y}_p modeled by* **reflectivity** $\rho_p = \alpha^3 \left(\frac{\epsilon_p}{\epsilon_o} - 1 \right) \mathbf{M}_p$

$\alpha^3 \sim$ scatterer volume/ λ^3 is small, $\mathbf{M}_p \in \mathbb{R}^{3 \times 3}$ is polarization tensor, depending on shape of scatterer and ϵ_p/ϵ_o , for $1 \leq p \leq P$.

Imaging problem: Determine $\{\vec{y}_p, \rho_p\}$ from scattered electric field at active array of N antennas.

Complete measurements: where antennas emit along all three directions and measure all components of electric field \rightsquigarrow data is $3N \times 3N$ complex matrix contaminated with noise.

Incomplete measurements: excitations and measurements of one or two components of electric field.

*Ammari, Lesselier...

Data model

- **Complete:** $\mathbf{D}_J = \mathbf{D}\mathbf{J} + \mathcal{W}$, where \mathbf{J} is excitation,

$$\mathbf{D} = \sum_{p=1}^P \mathcal{G}(\vec{y}_p) \rho_p \mathcal{G}^T(\vec{y}_p) + O(\alpha^4), \quad \mathcal{G}(\vec{y}_p) = \begin{pmatrix} \mathbf{G}(\vec{x}_1; \vec{y}_p) \\ \vdots \\ \mathbf{G}(\vec{x}_N; \vec{y}_p) \end{pmatrix} \in \mathbb{C}^{3N \times 3}$$

$$\mathbf{G}(\vec{x}, \vec{y}) = \left(\mathbf{I} + \frac{\nabla \nabla^T}{k^2} \right) \frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} = \text{dyadic Green tensor}$$

\mathcal{W} = complex Gaussian noise, mean zero, independent entries with standard deviation σ .

- **Incomplete:** $\text{diag}(\mathbf{S}^T, \dots, \mathbf{S}^T) \mathbf{D}_J \text{diag}(\mathbf{S}, \dots, \mathbf{S})$ for $\mathbf{S} = (\vec{e}_q)_{q \in \mathcal{S}}$

$3 \times |\mathcal{S}|$ sensing matrix for components q emitted and measured.

- **Good*** $\mathbf{J} \rightsquigarrow \widetilde{\mathbf{D}} := \mathbf{D}_J \mathbf{J}^{-1} = \mathbf{D} + \mathbf{W}$, Gaussian iid \mathbf{W} , st. dev. $\frac{\sigma}{\sqrt{3N}}$.

*Garnier, Solna: \mathbf{J} = complex Hadamard matrix.

SVD Analysis

$\mathcal{G}(\vec{y}) = \mathbf{H}(\vec{y})\mathbf{\Sigma}(\vec{y})\mathbf{V}^\dagger(\vec{y})$ is SVD of $3N \times 3$ matrix of Green func.

$\mathcal{R}_p = \mathbf{\Sigma}(\vec{y}_p)\mathbf{V}(\vec{y}_p)^\dagger \rho_p \left[\mathbf{\Sigma}(\vec{y}_p)\mathbf{V}(\vec{y}_p)^\dagger \right]^T = \mathbf{u}_p \mathbf{\mathfrak{S}}_p \mathbf{v}_p^\dagger$ is SVD of 3×3 matrix involving reflectivity ρ_p .

- Complete, noiseless data matrix

$$\mathbf{D} = \sum_{p=1}^P \mathbf{H}(\vec{y}_p) \mathbf{u}_p \mathbf{\mathfrak{S}}_p \mathbf{v}_p^\dagger \mathbf{H}(\vec{y}_p)^T$$

Imaging: $\text{range } \mathcal{G}(\vec{y}) = \text{range } \mathbf{H}(\vec{y}) \subset \text{range } \mathbf{D}$ iff $\vec{y} \in \{\vec{y}_1, \dots, \vec{y}_P\}$

- Similar for incomplete case but bad conditioning of matrices.
- Subspace of leading singular vectors is changed by noise.

Imaging using asymptotic results in random matrix theory

- From r significant noisy singular values $\tilde{\sigma}_j \geq 2\sigma$, estimate* true

$$\sigma_j \approx \frac{1}{2} [\tilde{\sigma}_j + \sqrt{\tilde{\sigma}_j^2 - (2\sigma)^2}]$$

- Singular vectors change direction $|\tilde{\mathbf{u}}_j^\dagger \mathbf{u}_j|^2 \approx \cos^2 \theta_j := 1 - \left(\frac{\sigma}{\sigma_j}\right)^2$ and $\tilde{\mathbf{u}}_j^\dagger \mathbf{u}_q \approx 0$, for $j, q = 1, \dots, r$ and $j \neq q$.

- For leading singular vector $\mathbf{h}_1(\vec{\mathbf{y}})$ of $\mathcal{G}(\vec{\mathbf{y}})$, if $\vec{\mathbf{y}} \in \{\vec{\mathbf{y}}_1, \dots, \vec{\mathbf{y}}_P\}$,

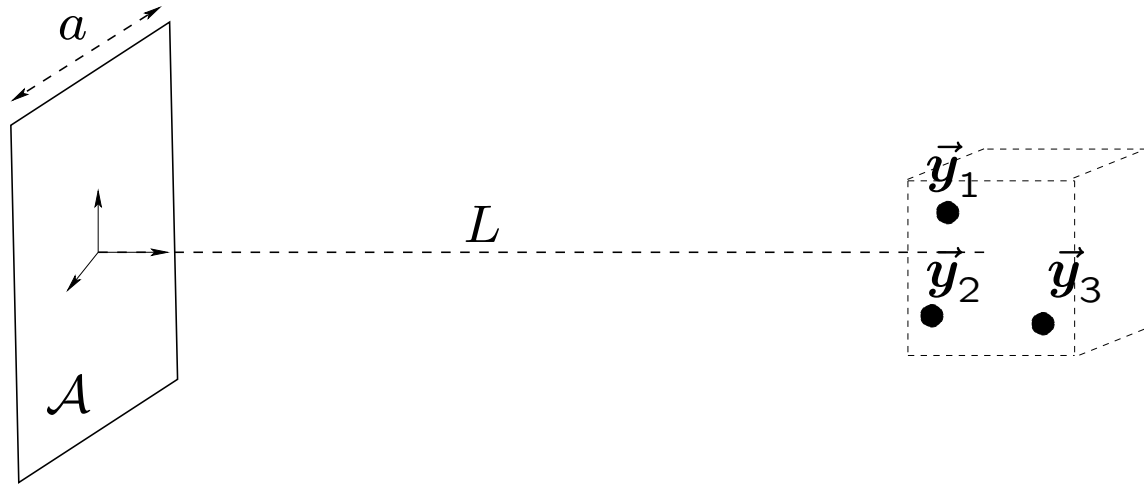
$$|\mathbf{u}_j^\dagger \mathbf{h}_1(\vec{\mathbf{y}})| \approx \frac{|\tilde{\mathbf{u}}_j^\dagger \mathbf{h}_1(\vec{\mathbf{y}})|}{|\cos \theta_j|}, \quad j = 1, \dots, r \leq 3P.$$

- Localize scatterers by peaks of $\mathcal{I}(\vec{\mathbf{y}}) = \left[1 - \sum_{j=1}^r \frac{|\tilde{\mathbf{u}}_j^\dagger \mathbf{h}_1(\vec{\mathbf{y}})|^2}{\cos^2 \theta_j} \right]^{-1/2}$ and then estimate their reflectivity.

- Incomplete measurements \rightsquigarrow low r and worse imaging.

*Estimation of σ from data is carefully studied by Garnier, Solna.

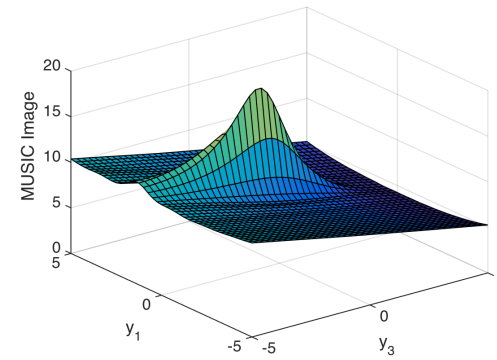
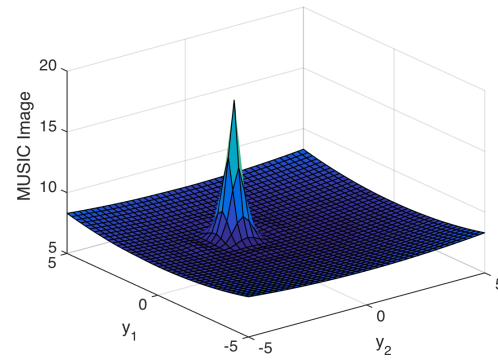
Numerical simulations



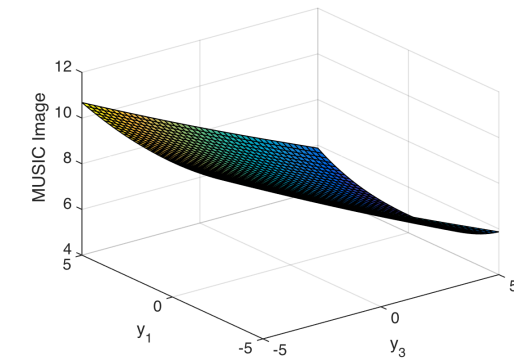
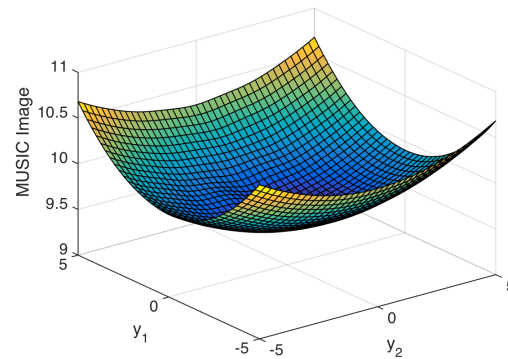
- Planar square array of aperture $a = 10\lambda$ has $N = 441$ antennas. "Large aperture" for $L = a$ and "small aperture" for $L = 10a$.
- Scatterers are small ellipsoids with various orientations of axes.
- Assume best (Hadamard) data acquisition scheme so noise standard deviation is σ/\sqrt{M} for $M = 3N$ or $|\mathcal{S}|N$.
- Entries of unperturbed data matrix $\sim \frac{\sigma_1}{M}$ meaning $SNR = \frac{\sigma_1}{\sigma\sqrt{M}}$
 $\frac{\sigma}{\sigma_1}$ as high as 75%. **This is a lot of noise:** $SNR = \frac{4}{3\sqrt{M}} = 0.03$
in complete measurement case.

Image of one scatterer

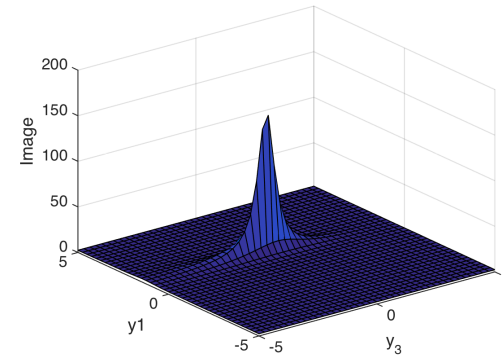
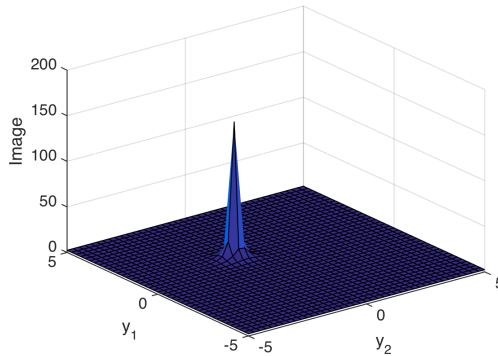
25% noise



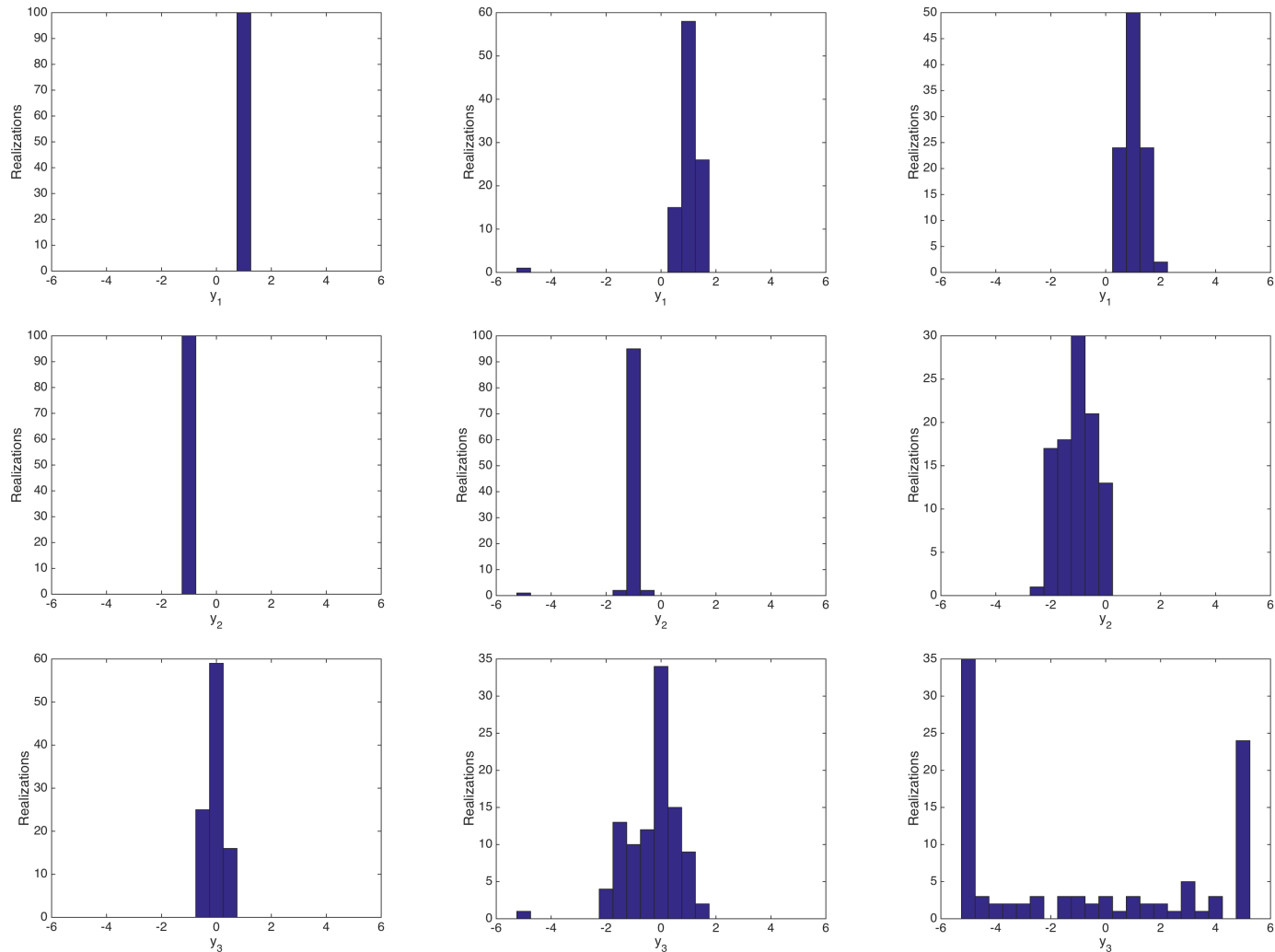
75% noise



75% noise

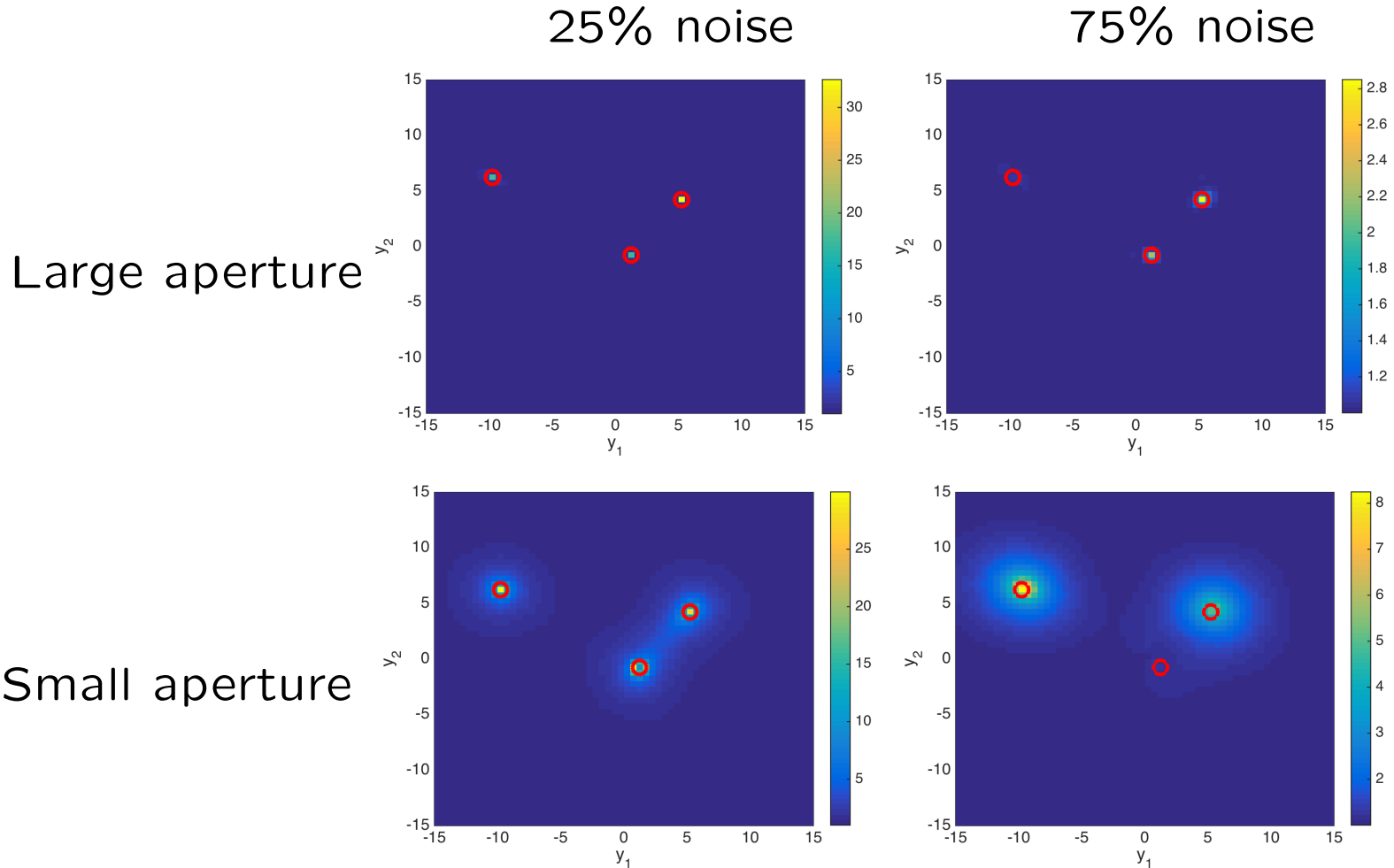


Histogram of location estimates at 50% noise



Left: large aperture, complete measurements. Middle: large aperture, measurements along \vec{e}_1 . Right: small aperture. Abscissa is in units of the wavelength.

Imaging of three inclusions with complete measurements



Results are worse for measurements only along \vec{e}_1 , specially for estimation of reflectivity. In the small aperture case, complete measurements and measurements along \vec{e}_1, \vec{e}_2 are similar.

Random media effects are not like additive noise

- We studied passive array imaging of sources in random media

$$p(t, \vec{x}) \approx \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) \sum_{p=1}^P \rho(\vec{y}_p) \hat{G}(\omega, \vec{x}, \vec{y}_p)$$

with geometrical optics model

$$\hat{G}(\omega, \vec{x}, \vec{y}_p) = \frac{\exp \left\{ i\omega \left[\tau(\vec{x}, \vec{y}_p) + \delta\tau(\vec{x}, \vec{y}_p) \right] \right\}}{4\pi |\vec{x} - \vec{y}_p|}$$

where $\delta\tau(\vec{x}, \vec{y}_p)$ give large random wave front distortions.

- CINT imaging* uses cross-correlations of measurements to mitigate distortions. Robustness comes at cost of resolution.

Question: Can we improve resolution using optimization?

*B., Garnier, Papanicolaou, Tsogka

Imaging in random media with convex optimization

- Seek $\rho(\vec{y})$ s.t. CINT image $\mathcal{I}(\vec{y}) \approx \sum_{p,q=1}^P \rho(\vec{y}_p) \overline{\rho(\vec{y}_q)} K(\vec{y}, \vec{y}_p, \vec{y}_q)$

with some penalty on $\rho(\vec{y})$ to emphasize sparse support.

- Let $\vec{y} = (\mathbf{y}, z)$ with z along range. Generically,

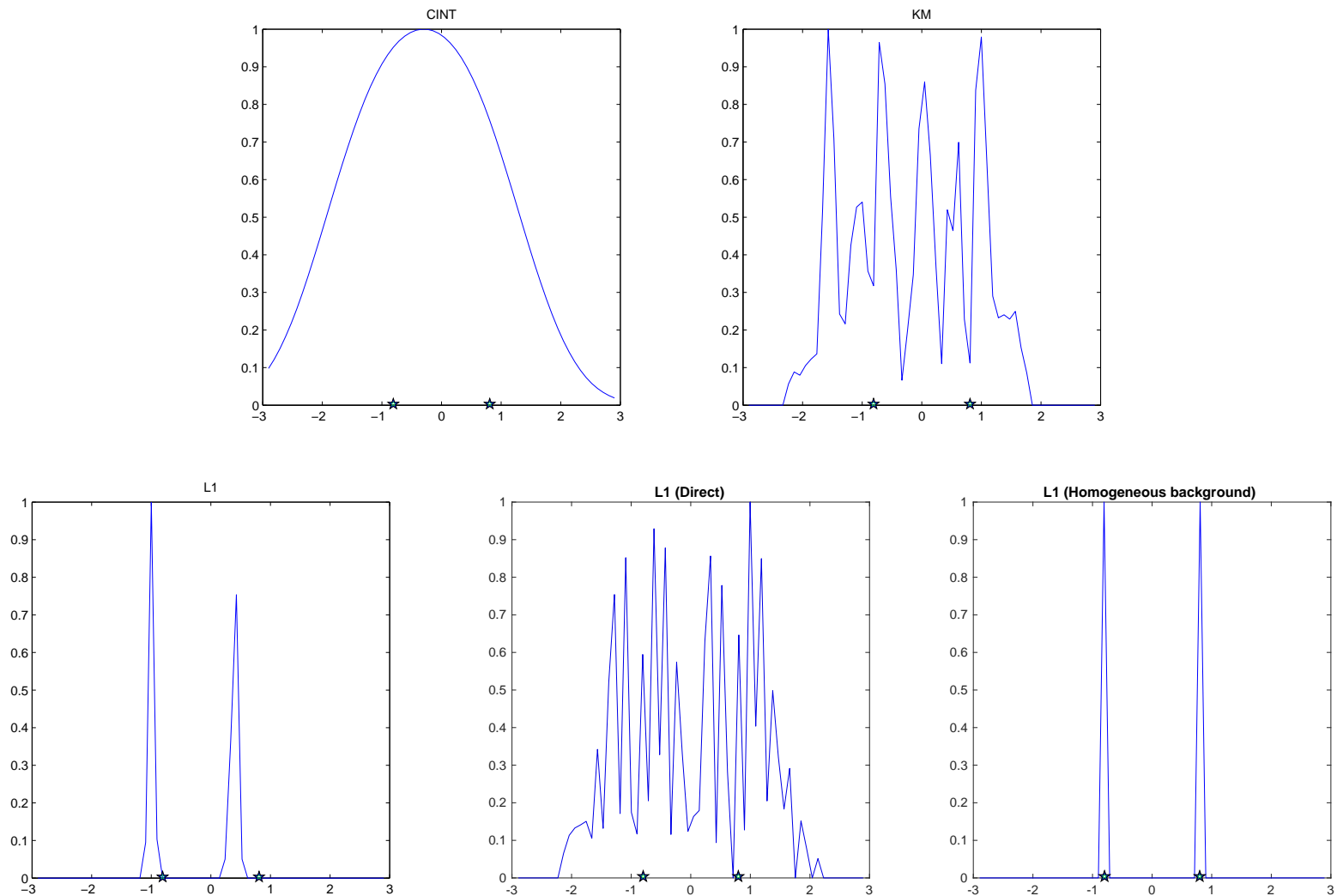
$$K(\vec{y}, \vec{y}_p, \vec{y}_q) \sim \exp \left[-\frac{\left| \mathbf{y} - \frac{\mathbf{y}_p + \mathbf{y}_q}{2} \right|^2}{2R^2} - \frac{\left| z - \frac{z_p + z_q}{2} \right|^2}{2R_z^2} \right] \mathfrak{M}(\vec{y}, \vec{y}_p, \vec{y}_q)$$

where $R = \frac{\lambda_o L}{X_d}$ and $R_z = \frac{c}{\Omega_d}$ describe CINT resolution in terms of decorrelation length X_d and frequency Ω_d in random medium.

$\mathfrak{M}(\vec{y}, \vec{y}_p, \vec{y}_q)$ decays rapidly in $\mathbf{y}_p - \mathbf{y}_q$ and $z_p - z_q \rightsquigarrow$

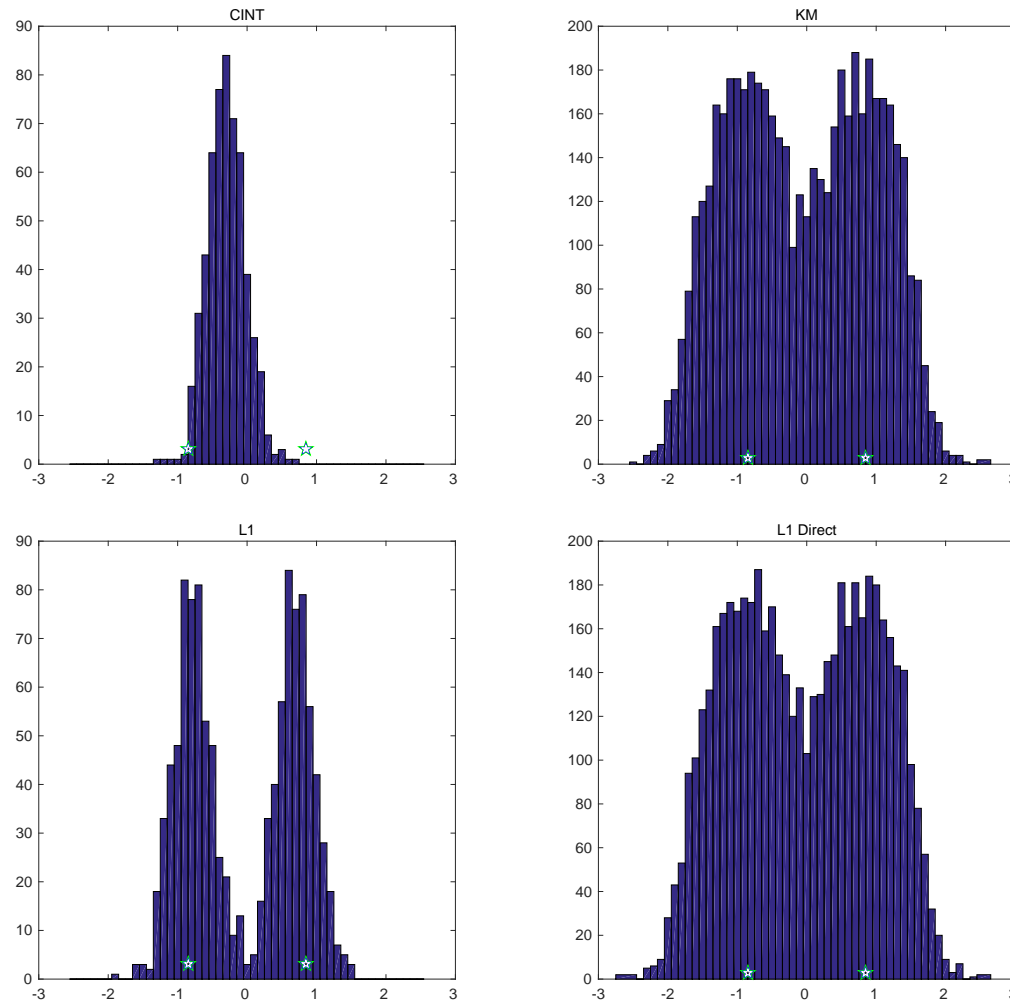
$K \approx$ diagonal (convolution) kernel. Unknown becomes $|\rho(\vec{y}_p)|^2$.

Numerical results. Typical cross-range localization.



Abscissa is in $R = \frac{\lambda_o L}{X_d}$ units. Direct ℓ_1 means minimize $\|\rho\|_1$ constrained by fitting array data within a tolerance.

Numerical results. Histograms of cross-range localization.



Histograms of number of peaks (value $> 33\%$ of maximum) vs. cross-range in units $\lambda_o L / X_d$.

Current projects

- With I. Kocyigit we are working on convex optimization approach to radar imaging with polarization.

We use the Multiple Measurement Vector (MMV) formalism* (simultaneous sparse approximation).

- We discovered connections to: B., Moscoso, Papanicolaou, Tsogka, "Synthetic Aperture Imaging of Direction-and Frequency-Dependent Reflectivities." SIAM Journal on Imaging Sciences 9.1 (2016): 52-81.

We seek a theory for understanding the data (aperture, bandwidth) segmentation in these problems and the benefits of MMV.

*Malioutov, Cetin, Willsky (2005),