

The Hanbury Brown-Twiss effect

Direct and Inverse Problems

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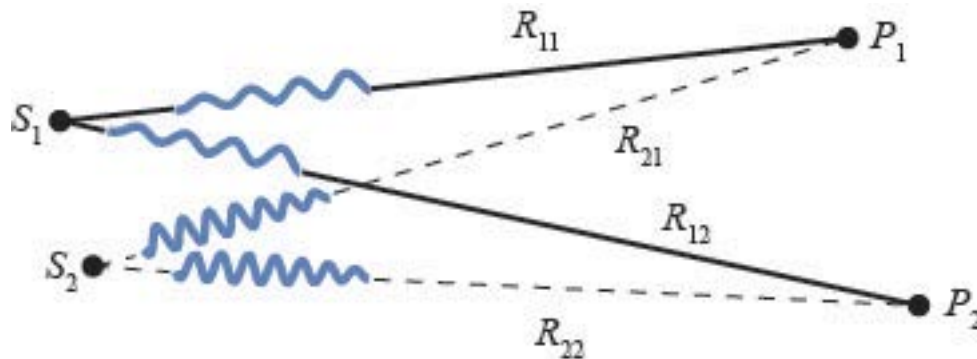
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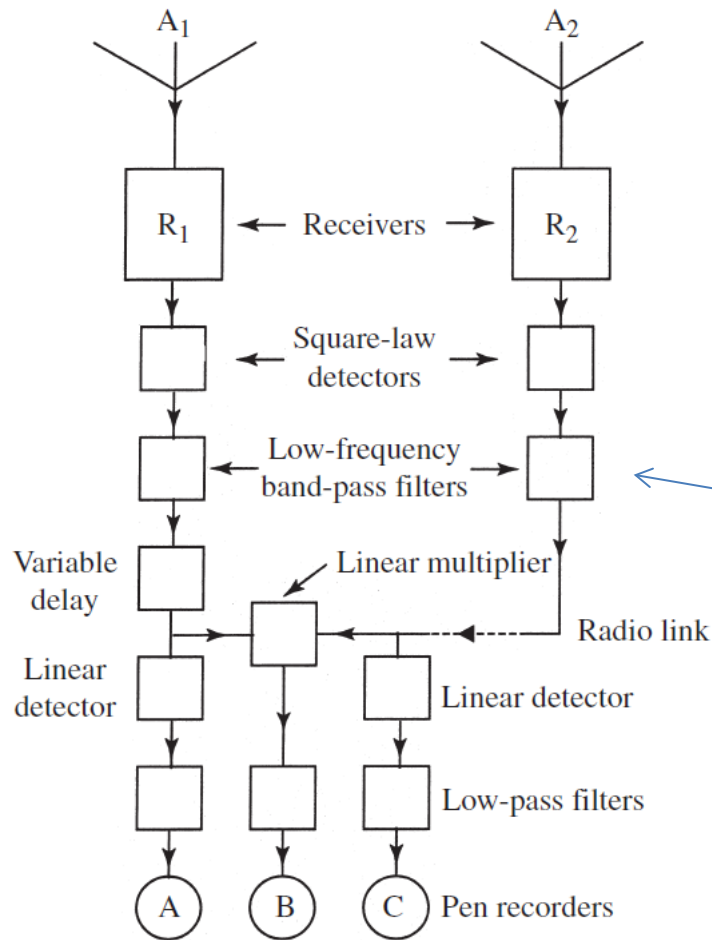
The correlation of intensity fluctuations at two detectors was first used by Hanbury Brown and Twiss to determine the diameter of radio stars.

How can the signal from a completely incoherent source like a star display any correlations?



Correlations build up on propagation.

Correlation functions obey –just like the field itself– wave equations, the so-called *Wolf equations*. From these the *van Cittert-Zernike* theorem can be derived. However, all this is restricted to *second-order (field-field) correlations*.

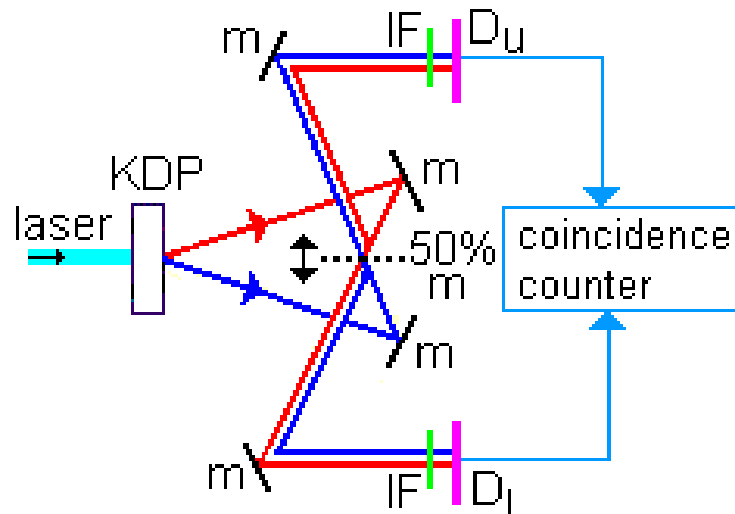


Original HBT setup

- Second-order correlations can be measured using *Michelson interferometry*.
- Intensity fluctuations deal with 4th-order correlations.

The diameter of a distant radio star is determined by measuring how rapidly the correlation decreases with an increasing delay.

The HBT effect is employed in **the** quantum optics experiment, the *Hong-Ou-Mandel interferometer*:



Both photons are *always jointly detected at either D_U or at D_L* demonstrating the non-classical nature of photon interference.

Time vs. Frequency domain

An actual HBT experiment involves observing intensity fluctuations in time.

However, the analysis is somewhat easier in the frequency domain. The results can always be transformed back to the time domain.

Mutual coherence function:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{1}{2T} \lim_{T \rightarrow \infty} \int_{-T}^T V^*(\mathbf{r}_1, t) V(\mathbf{r}_2, t + \tau) dt$$

Cross-spectral density:

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega) U(\mathbf{r}_2, \omega) \rangle$$

These two functions are a *Fourier pair*.

Time average vs. *ensemble average*.

Random Electromagnetic Beams

The coherence and polarization properties of a stochastic EM beam are described (in the frequency domain) by its *cross-spectral density matrix*:

$$\mathbf{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \begin{pmatrix} W_{xx}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) & W_{xy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) \\ W_{yx}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) & W_{yy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) \end{pmatrix}$$

with $W_{ij}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \langle E_i^*(\boldsymbol{\rho}_1, z) E_j(\boldsymbol{\rho}_2, z) \rangle$, $(i, j = x, y)$

The intensity of *a single realization* is

$$I(\boldsymbol{\rho}, z) = |E_x(\boldsymbol{\rho}, z)|^2 + |E_y(\boldsymbol{\rho}, z)|^2$$

with *expectation value*

$$\begin{aligned} \langle I(\boldsymbol{\rho}, z) \rangle &= \langle |E_x(\boldsymbol{\rho}, z)|^2 \rangle + \langle |E_y(\boldsymbol{\rho}, z)|^2 \rangle \\ &= \text{Tr } \mathbf{W}(\boldsymbol{\rho}, \boldsymbol{\rho}, z), \end{aligned}$$

The *intensity fluctuation* at a certain position is

$$\Delta I(\boldsymbol{\rho}, z) = I(\boldsymbol{\rho}, z) - \langle I(\boldsymbol{\rho}, z) \rangle$$

The Hanbury Brown-Twiss coefficient

The *correlation of intensity fluctuations* at two points in the same cross-sectional plane z (*the HBT coefficient*) is defined as

$$C(\rho_1, \rho_2, z) = \langle \Delta I(\rho_1, z) \Delta I(\rho_2, z) \rangle$$

Under the assumption that we're dealing with a source with *Gaussian statistics*, the *fourth-order* HBT coefficient can be expressed in terms of *second-order* correlations as

$$C(\rho_1, \rho_2, z) = \sum_{i,j} |W_{ij}(\rho_1, \rho_2, z)|^2$$

The *normalized* HBT coefficient is given by

$$\begin{aligned} C_N(\rho_1, \rho_2, z) &= \frac{C(\rho_1, \rho_2, z)}{\langle I(\rho_1, z) \rangle \langle I(\rho_2, z) \rangle} \\ &= \frac{\sum_{i,j} |W_{ij}(\rho_1, \rho_2, z)|^2}{\text{Tr } \mathbf{W}(\rho_1, \rho_1, z) \text{Tr } \mathbf{W}(\rho_2, \rho_2, z)}. \end{aligned}$$

and $0 \leq C_N(\rho_1, \rho_2, z) \leq 1$.

Gaussian Schell-Model Beams

GSM beams have a Gaussian intensity distribution, and a correlation function that is also Gaussian and *depends only on the separation* between the two points. In the source plane $z = 0$

$$W_{ij}(\rho_1, \rho_2, 0) = A_i A_j B_{ij} \exp \left[-\frac{\rho_1^2}{4\sigma_i^2} - \frac{\rho_2^2}{4\sigma_j^2} - \frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}^2} \right]$$

A_i = amplitude of E_i ($i, j = x, y$)

B_{ij} = correlation between E_i and E_j

σ_i = spectral density width. We assume that $\sigma_x = \sigma_y = \sigma$

The four *coherence radii* δ_{ij} have to satisfy certain constraints because \mathbf{W} is a non-negative matrix.

Because *correlation functions satisfy wave equations*, the matrix elements can be 'propagated' to a plane away from the source by using the Green's function of the *paraxial wave equation*.

Gaussian Schell-Model Beams (2)

The propagated matrix elements in a plane z have the form

$$W_{ij}(\rho_1, \rho_2, z) = \frac{A_i A_j B_{ij}}{\Delta_{ij}^2(z)} \exp \left[-\frac{(\rho_1 + \rho_2)^2}{8\sigma^2 \Delta_{ij}^2(z)} \right] \\ \times \exp \left[-\frac{(\rho_1 - \rho_2)^2}{2\Omega_{ij}^2 \Delta_{ij}^2(z)} + \frac{ik(\rho_2^2 - \rho_1^2)}{2R_{ij}(z)} \right]$$

with

$$\frac{1}{\Omega_{ij}^2} = \frac{1}{4\sigma^2} + \frac{1}{\delta_{ij}^2},$$

B_{ij} = correlation between E_i and E_j .

$$\Delta_{ij}^2(z) = 1 + (z/\sigma k \Omega_{ij})^2,$$

$$R_{ij}(z) = [1 + (\sigma k \Omega_{ij}/z)^2]z$$

We can now use the expressions for W_{ij} to calculate the HBT coefficient in any cross-section of the beam.

Determining source shapes using the HBT effect

*In the far zone of a **quasi-homogeneous (QH)** source, information about its shape is lost.*

For example, *the far-zone radiation pattern* of a QH *rectangular* source with sides a and $4a$ looks like this:

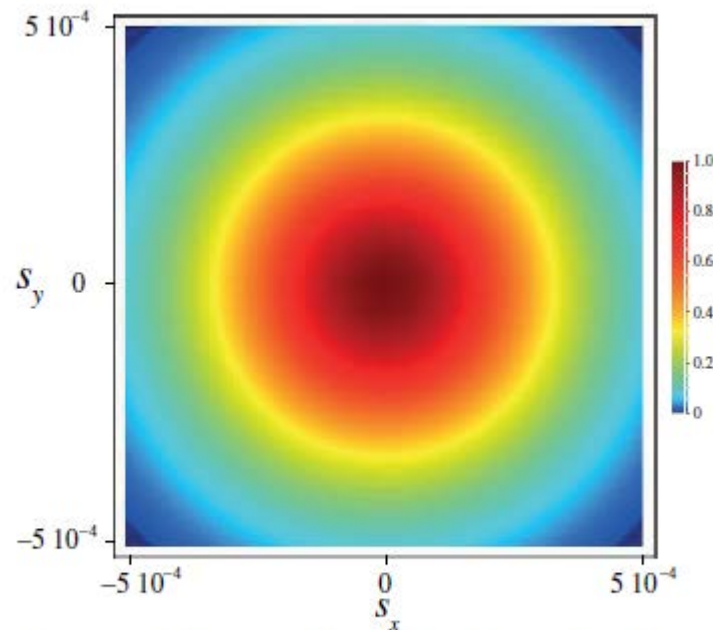
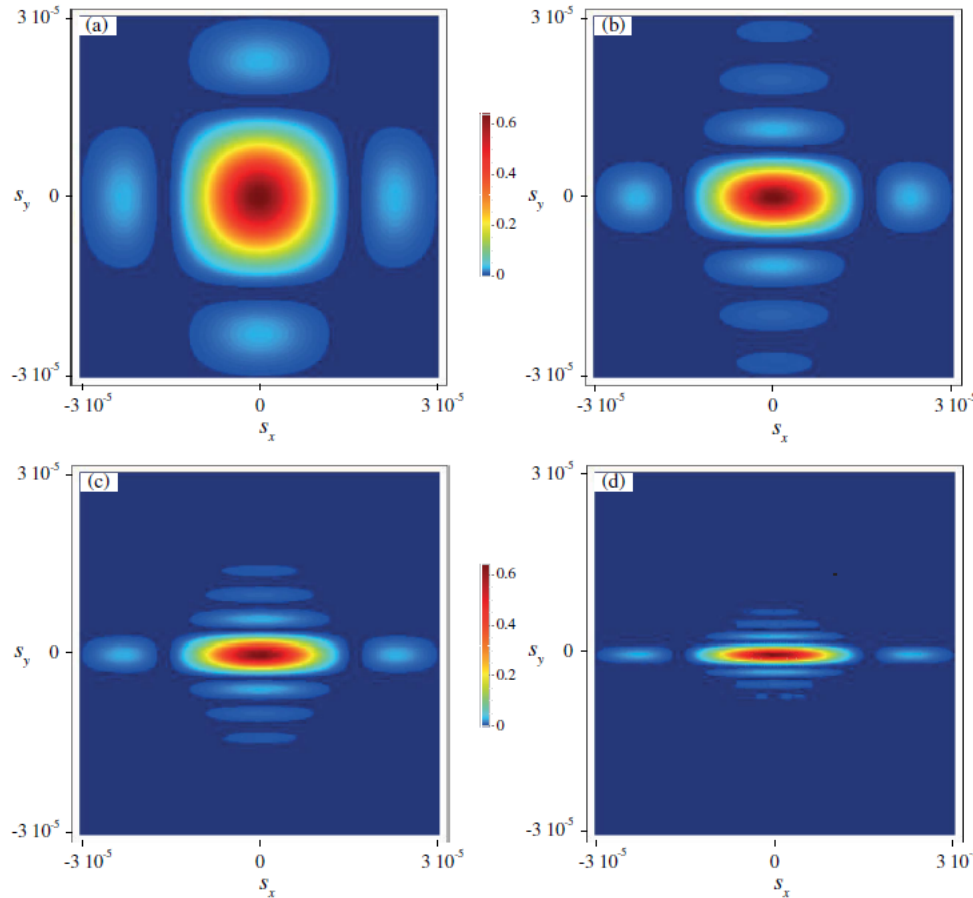


Fig. 6. Contours of the normalized radiant intensity of beams generated by rectangular sources. The parameters are the same as in Fig. [5](#).

However, the source shape *is somehow retained in the far-zone HBT effect*:



Far-zone HBT coefficient
for QH sources with
different aspect ratios.

$$C_N^{(\infty)}(r\mathbf{s}_1, r\mathbf{s}_2)$$

$$\mathbf{s}_1 = (s_x, s_y, s_z),$$

$$\mathbf{s}_2 = (-s_x, -s_y, s_z).$$

Fig. 5. Contours of the normalized correlation of intensity fluctuations of beams generated by rectangular sources with sides a and b . The sides are chosen as (a) $b = a$, (b) $b = 2a$, (c) $b = 4a$, and (d) $b = 8a$. In these examples $A_x = 2$, $A_y = 1$, $B_{xy} = 0.2$, $\delta_{xx} = 0.4$ mm, $\delta_{yy} = 0.6$ mm, $\delta_{xy} = 0.75$ mm, $a = 2$ cm, and $\lambda = 632.8$ nm.

This allows us to reconstruct the source shape from far-zone measurements of the HBT coefficient:

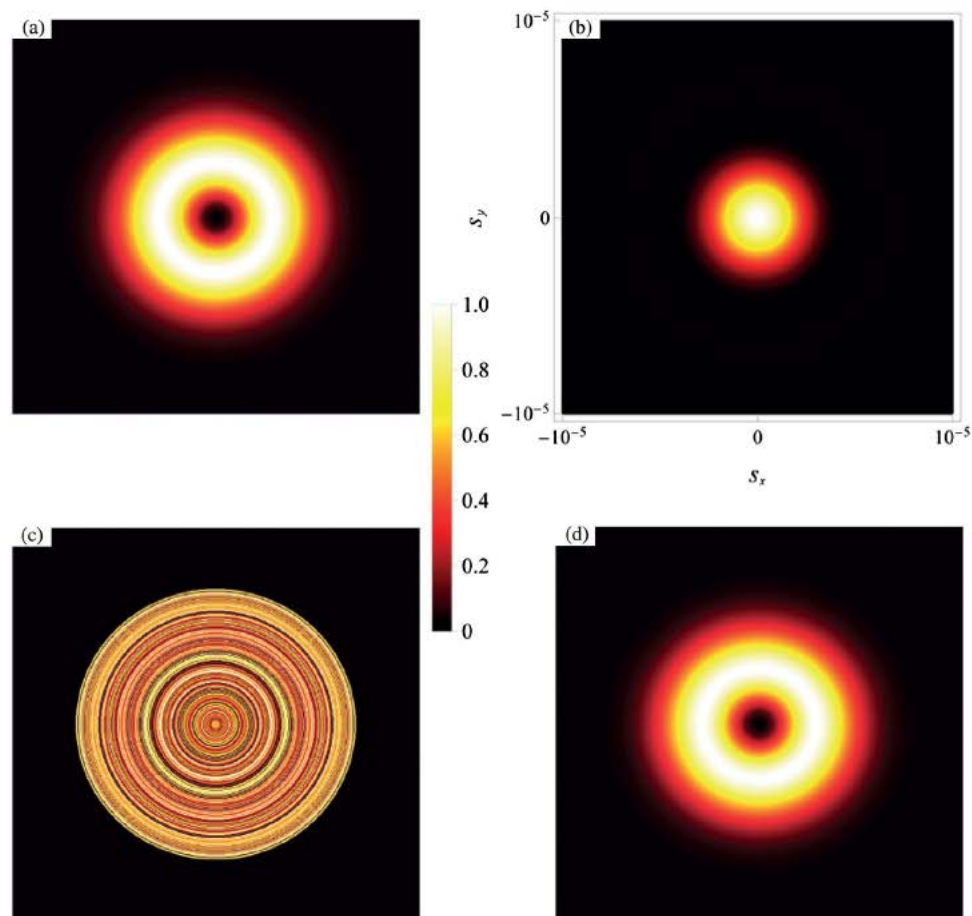
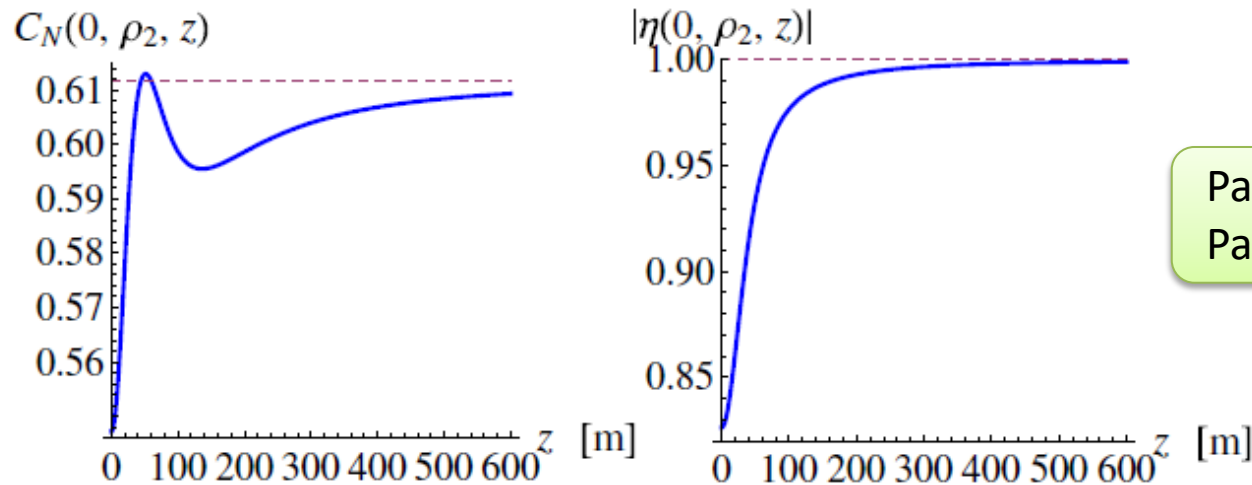


Fig. 7. Retrieval of a simulated spectral density from its normalized correlation of intensity fluctuations in the far-zone. (a) The spectral density of a partially coherent Laguerre–Gauss beam in the source plane. In this example $\lambda = 632.8$ nm and $\sigma_x = 15$ μ m. (b) The normalized correlation of intensity fluctuations in the far zone. (c) The initial guess for the source spectral density that is used to start the algorithm: a completely random pattern with rotational symmetry and with values between 0 and 1. (d) The result of the reconstructed source spectral density after 80 iterations.

The evolution of the HBT effect



Propagation distance \longrightarrow

The *EM spectral degree of coherence* η is a measure of the *fringe visibility* as observed with Young's experiment. It is a *second-order correlation*:

$$\eta(0, \rho_2, z) = \frac{\text{Tr } \mathbf{W}(0, \rho_2, z)}{\sqrt{\langle I(0, z) \rangle} \sqrt{\langle I(\rho_2, z) \rangle}}.$$

As the beam propagates, the spectral degree of coherence gradually builds up and asymptotically tends to unity.

The fourth-order HBT coefficient has a more complicated behavior*:

It does *not increase monotonically, and does not approach unity*. **[WHY?]**

* Wu and Visser, *Optics Letters* **39**, 256 (2014).

The far zone HBT effect for different types of sources

$$W_{ij}(\rho_1, \rho_2, z) = \frac{A_i A_j B_{ij}}{\Delta_{ij}^2(z)} \exp \left[-\frac{(\rho_1 + \rho_2)^2}{8\sigma^2 \Delta_{ij}^2(z)} \right] \\ \times \exp \left[-\frac{(\rho_1 - \rho_2)^2}{2\Omega_{ij}^2 \Delta_{ij}^2(z)} + \frac{ik(\rho_2^2 - \rho_1^2)}{2R_{ij}(z)} \right]$$

with

$$\frac{1}{\Omega_{ij}^2} = \frac{1}{4\sigma^2} + \frac{1}{\delta_{ij}^2},$$

$$\Delta_{ij}^2(z) = 1 + (z/\sigma k \Omega_{ij})^2, \quad \lim_{z \rightarrow \infty} \Delta_{ij}^2(z) = \frac{z^2}{(\sigma k \Omega_{ij})^2},$$

$$R_{ij}(z) = [1 + (\sigma k \Omega_{ij}/z)^2]z \quad \lim_{z \rightarrow \infty} R_{ij}(z) = z.$$

B_{ij} = correlation between E_i and E_j .

$$B_{xx} = B_{yy} = 1;$$

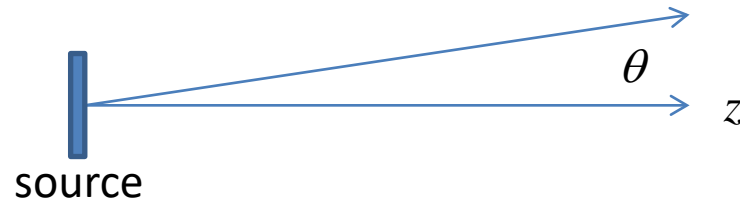
$$B_{xy} \in \mathbb{C}; \quad |B_{xy}| \leq 1; \quad B_{yx} = B_{xy}^*$$

Unlike the results for the evolution on propagation, we can now derive analytic expressions for the HBT effect.

Unpolarized source

For an *unpolarized, rotationally symmetric source* we have

$$\begin{aligned}A_x &= A_y = A, \\ \delta_{xx} &= \delta_{yy} = \delta, \\ B_{xy} &= B_{yx} = 0, \\ \delta_{xy} &= \delta_{yx} = 0.\end{aligned}$$



This leads to

$$C_N(\mathbf{0}, \theta) = \frac{1}{2} \exp \left(-\frac{4\theta^2 k^2 \sigma^4}{\delta^2 + 4\sigma^2} \right)$$

The HBT coefficient depends on the source width σ and the correlation radius δ . It's upper bound as $\theta \rightarrow 0$, equals $\frac{1}{2}$.

Linearly polarized source

For a *rotationally symmetric, linearly polarized source* (with polarization along x):

$$\begin{aligned} A_x &= A, \\ \delta_{xx} &= \delta. \end{aligned}$$

In that case

$$C_N(\mathbf{0}, \theta) = \exp \left(-\frac{4\theta^2 k^2 \sigma^4}{\delta^2 + 4\sigma^2} \right).$$

*The HBT coefficient does **not** depend on the direction of polarization. Its upper bound as $\theta \rightarrow 0$ equals 1, which is twice as large as for the unpolarized case.*

Partially polarized source

For a *rotationally symmetric source that generates a partially polarized, partially coherent beam*, we have:

$$A_x = A_y = A,$$

$$\delta_{xx} = \delta_{yy} = \delta.$$

$$\Omega_{xx} = \Omega_{yy} = \Omega \neq \Omega_{xy},$$

$$\Delta_{xx} = \Delta_{yy} = \Delta \neq \Delta_{xy}.$$

$$C_N(\mathbf{0}, \theta) = \frac{1}{2} \exp\left(-\frac{4\theta^2 k^2 \sigma^4}{\delta^2 + 4\sigma^2}\right) \times \left\{ 1 + |B_{xy}|^2 \left(\frac{\Omega_{xy}}{\Omega}\right)^4 \exp\left[-\frac{\theta^2 k^2 (\Omega_{xy}^2 - \Omega^2)}{4}\right] \right\}.$$

In this more general case the upper bound as $\theta \rightarrow 0$ exceeds $\frac{1}{2}$ because of the presence of the polarization coefficient B_{xy} :

A non-zero correlation between E_x and E_y increases the HBT effect.

NB. The realizability conditions give a limit to B_{xy}

$$\sqrt{\frac{\delta_{xx}^2 + \delta_{yy}^2}{2}} \leq \delta_{xy} \leq \sqrt{\frac{\delta_{xx}\delta_{yy}}{|B_{xy}|}}.$$

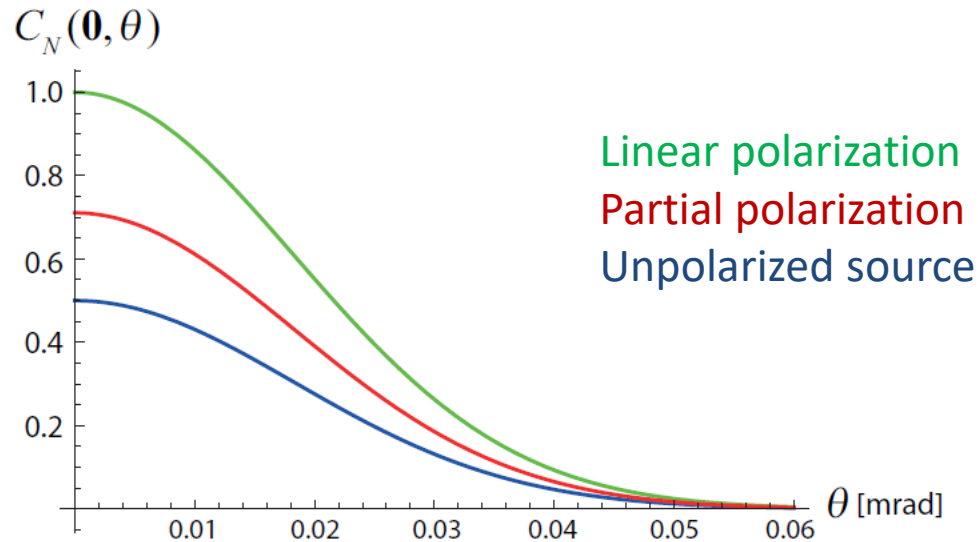


Figure 2. The normalized far-zone Hanbury Brown-Twiss coefficient for three EGSM sources with different states of polarization: linearly polarized (green curve), partially polarized (red curve) and unpolarized (blue curve). In these examples the parameters are: $\lambda = 632.8$ nm, $\sigma = 4$ mm, $\delta = 2$ mm, $\delta_{xy} = 2.3$ mm, and $B_{xy} = 0.5$.

The HBT coefficient for a linearly polarized source is the only with an upper bound that is one. Its value exceeds that for other sources at all observation angles.

An upper bound $> 1/2$ indicates a partially polarized source.

Determining the Gaussian nature of source statistics

A central equation links the *fourth-order* HBT effect to *second-order* field correlations:

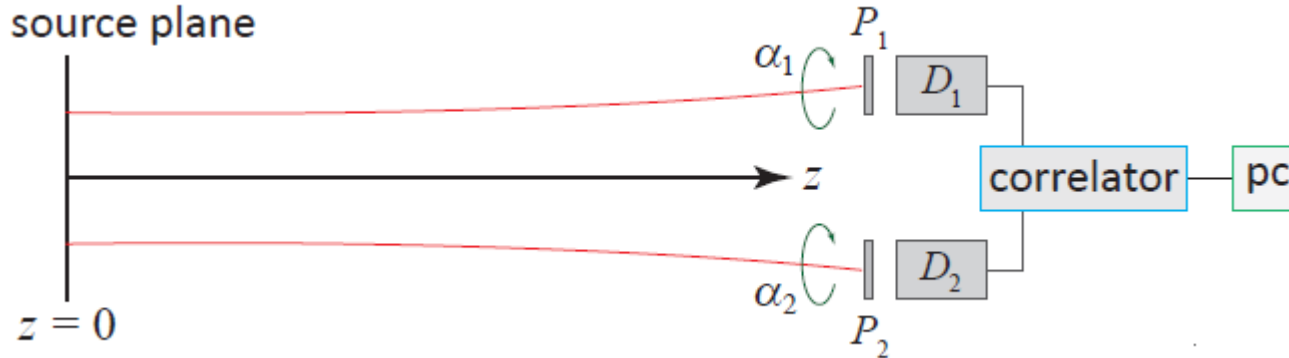
$$\begin{aligned} C_N(\rho_1, \rho_2, z) &= \frac{C_0(\rho_1, \rho_2, z)}{\langle I(\rho_1, z) \rangle \langle I(\rho_2, z) \rangle} \\ &= \frac{\sum_{i,j} |W_{ij}(\rho_1, \rho_2, z)|^2}{\text{Tr } \mathbf{W}(\rho_1, \rho_1, z) \text{Tr } \mathbf{W}(\rho_2, \rho_2, z)}. \end{aligned} \quad (34)$$

This expression is valid if the source fluctuations are governed by Gaussian statistics: *All joint probabilities are also Gaussian.*
NB. This is independent of the usual Gaussian distribution of single quantities.

The justification of this assumption under practical circumstances is unclear...[*Central limit theorem, multi-mode operation, etc.*]

The HBT effect can be applied to test the validity of the assumption underlying Eq. (34).

The polarization-resolved HBT effect



Cross-spectral density matrix:

$$W^{(\text{in})}(\rho_1, \rho_2, z) = \begin{pmatrix} W_{xx}^{(\text{in})}(\rho_1, \rho_2, z) & W_{xy}^{(\text{in})}(\rho_1, \rho_2, z) \\ W_{yx}^{(\text{in})}(\rho_1, \rho_2, z) & W_{yy}^{(\text{in})}(\rho_1, \rho_2, z) \end{pmatrix}$$

Matrix elements:

$$W_{ij}^{(\text{in})}(\rho_1, \rho_2, z) = \langle E_i^{(\text{in})*}(\rho_1, z) E_j^{(\text{in})}(\rho_2, z) \rangle \quad (i, j = x, y).$$

Define correlation coefficients:

$$\mu_{ij}^{(\text{in})}(\rho_1, \rho_2, z) = \frac{W_{ij}^{(\text{in})}(\rho_1, \rho_2, z)}{\sqrt{W_{ii}^{(\text{in})}(\rho_1, \rho_1, z) W_{jj}^{(\text{in})}(\rho_2, \rho_2, z)}}.$$

Only if the source is governed by Gaussian statistics, does the following relation hold:

$$C_{2N}(\rho_1, \rho_2, z; \alpha_1 = 0, \alpha_2 = \pi/2) = \frac{|W_{xy}^{(\text{in})}(\rho_1, \rho_2, z)|^2}{W_{xx}^{(\text{in})}(\rho_1, \rho_1, z)W_{yy}^{(\text{in})}(\rho_2, \rho_2, z)} = |\mu_{xy}^{(\text{in})}(\rho_1, \rho_2, z)|^2$$



Fourth-order HBT effect

Transmission angles



Second-order correlation coefficient

Comparing a far-zone measurement of the HBT effect with an interferometric (Michelson) measurement of the 2nd order correlation coefficient, verifies the assumption of the Gaussian nature of a distant source.

For different polarizer angles α_1 and α_2 , the other three identities can be tested.

➡ *Determine the nature of the source statistics in the far zone, by testing four identities.*

Conclusions

- The *evolution of the Hanbury Brown-Twiss effect on propagation* is more complex than that of 2nd order correlation functions.
- We examined the far-zone behavior of the HBT effect for a wide class of sources with *rotational symmetry*.
- Not just the *coherence properties* of the source, but also its state of *polarization* play a crucial role in the correlation of intensity fluctuations.
- *Analytic expression* for the HBT coefficient in the far zone were derived.
- Different types of sources (*unpolarized, partially polarized and linearly polarized*) are shown to have *different upper bounds* for their respective HBT coefficients.
- The far-zone HBT effect can be used for *two inverse problems*:
 1. Reconstruction of the source shape (QH)
 2. Characterization of the source statistics