

The Linear Sampling Method in the Time Domain

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Inverse Scattering

There are three popular approaches to the inverse scattering problem for acoustic and electromagnetic waves in the frequency domain:

1 Linearization

Problem: Ignores multiple scattering and hence model may be incorrect.

2 Nonlinear Optimization

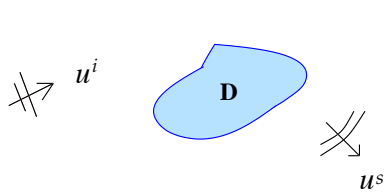
Problem: Convergence of Newton's Method for inverse scattering problem is unknown and hence method is mathematically incomplete.

3 Qualitative Methods

Problem: Only determines support of scattering object (but is mathematically rigorous with correct model).

- Linear sampling method (Colton-Kirsch 1996)
- Factorization method (Kirsch 1998)
- Generalized linear sampling method (Audibert-Haddad 2015)

Scattering by an Inhomogeneous Medium



$$\begin{aligned}
 \Delta u^s + k^2 u^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
 \Delta u + k^2 n u &= 0 && \text{in } D \\
 u &= u^s + u^i && \text{on } \partial D \\
 \frac{\partial u}{\partial \nu} &= \frac{\partial u^s}{\partial \nu} + \frac{\partial u^i}{\partial \nu} && \text{on } \partial D \\
 \lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - i k u^s \right) &= 0
 \end{aligned}$$

The function $n \in L^\infty(D)$, $n(x) > 0$, represents the **square of the refractive index** of the inhomogeneous media.

$k > 0$ is the **wave number** and is proportional to the frequency ω .

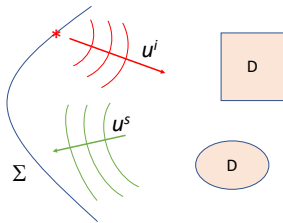
u^i is the **incident (interrogating) wave**.

Scattering by an Inhomogeneous Medium

Now consider a **point source incident wave** $u^i(x; y) := \frac{e^{ikx \cdot y}}{|x - y|}$.

Let $u^s(x; y)$ be the corresponding **scattered field**.

The source/measurements surface Σ is a portion of an analytic surface lying outside D .



Assume we know the scattered field $u^s(x; y)$ for all $x \in \Sigma$, corresponding to the incident field $u^i(x; y)$, for $y \in \Sigma$, at a fixed frequency $k > 0$. The goal is to determine D .

Scattering by an Inhomogeneous Medium

The **near field operator** $N : L^2(\Sigma) \rightarrow L^2(\Sigma)$ is defined by

$$(Ng)(x) := \int_{\Sigma} u^s(x; y) g(y) ds_y.$$

- Ng is the evaluation on Σ of the scattered field corresponding to the single layer potential

$$(Sg)(x) := \int_{\Sigma} \frac{e^{ikx \cdot y}}{|x - y|} g(y) ds_y$$

as incident field.

The Linear Sampling Method

The linear sampling method is based on "solving" the **near field equation**

$$(Ng)(x) = \Phi(x, z), \quad x \in \Sigma, \quad z \in \mathbb{R}^3, \quad \text{where } \Phi(x, z) := \frac{e^{ikx \cdot z}}{|x - z|}.$$

Theorem

Assume that k is not a **transmission eigenvalue**. Then

- If $z \in D$, there is a sequence $g_z^\alpha \in L^2(\Sigma)$ such that

$$\lim_{\alpha \rightarrow 0} \|Ng_z^\alpha - \Phi(x, z)\|_{L^2(\Sigma)} = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \|Sg_z^\alpha\|_{L^2(D)} < \infty.$$

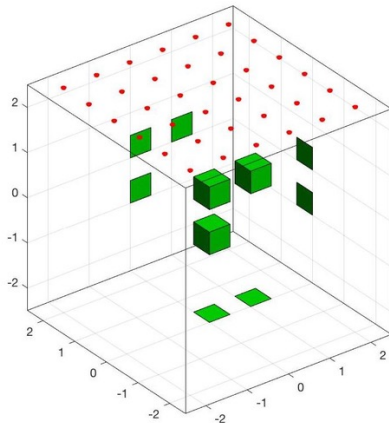
- If $z \notin D$, for every sequence $g_z^\alpha \in L^2(\Sigma)$ such that

$$\lim_{\alpha \rightarrow 0} \|Ng_z^\alpha - \Phi(x, z)\|_{L^2(\Sigma)} = 0 \quad \text{we have} \quad \lim_{\alpha \rightarrow 0} \|Sg_z^\alpha\|_{L^2(D)} = \infty.$$

The Linear Sampling Method

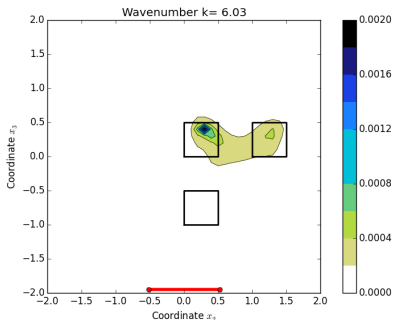
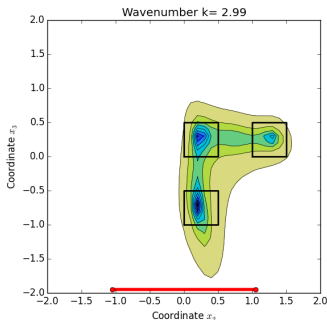
The linear sampling method uses a single frequency and for limited aperture data the method may provide a poor reconstruction of D .

For example, consider the following scatterers and measurement array where the point sources and receivers are at the point in the grid above the scatterers.



The Linear Sampling Method

Below are the cross sections in the plane $x_1 = 0.25$ corresponding to $k = 2.99$ and $k = 6.03$. Note that the cross section for $k = 6.03$ misses the lower scatterer.



The "good" frequency is not known a priori. How can this be remedied?

Scattering in the Time Domain

Let X be a Hilbert space and $f(t)$ is such that $e^{-\sigma t}f(t) \in L^1(\mathbb{R}, X)$ for some $\sigma > 0$. Define the Fourier-Laplace transform by

$$\mathcal{L}[f](s) := \int_{-\infty}^{\infty} e^{ist} f(t) dt, \quad s \in \mathbb{C}_\sigma$$

where $\mathbb{C}_\sigma := \{s \in \mathbb{C}, \Im(s) > \sigma\}$. For $p \in \mathbb{R}$ define the Hilbert space

$$H_\sigma^p(\mathbb{R}, X) := \left\{ f : \int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \|\mathcal{L}[f](s)\|_X ds < \infty \right\}$$

endowed with the norm

$$\|f\|_{H_\sigma^p(\mathbb{R}, X)} = \left(\int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \|\mathcal{L}[f](s)\|_X ds \right)^{1/2}.$$

Scattering in the Time Domain

Suppose that the support of inhomogeneity $D \subset \mathbb{R}^3$ is such that ∂D is smooth and $\mathbb{R}^3 \setminus \overline{D}$ is connected.

The square of the refractive index

$$n(x) = c_0^2/c^2(x) \geq n_0 > 0 \quad x \in \overline{D}$$

is such that $n|_{\overline{D}} \in C^\infty(\overline{D})$, $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus \overline{D}$, and $n(x) \neq 1$ for $x \in \partial D$.

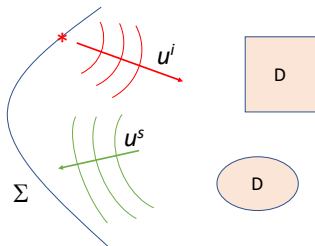
Let χ denote a smooth function of compact support on $(0, \infty)$. Then the incident field u^i is defined by

$$u^i(t, x; y) = \frac{\chi(t - |x - y|)}{4\pi|x - y|}.$$

We call u^s the corresponding scattered field and u the total field

$$u = u^s + u^i.$$

Scattering in the Time Domain



Find $u \in H^p_\sigma(\mathbb{R}, H^1(\mathbb{R}^3))$ such that

$$n(x)u_{tt} - \Delta u = \chi(t)\delta_y \quad \text{in } \mathbb{R}^3 \quad \text{for } t > 0$$

$$u = u^s + u^i$$

$$u = 0 \quad \text{in } \mathbb{R}^3 \quad \text{for } t \leq 0.$$

Theorem

There exists a unique solution to the above scattering problem.

Examples of reconstruction with time domain data

Let Σ be a portion of an analytic surface lying outside D .
Assume we know the scattered field $u^s(t, x; y)$ for all $t > 0$, $x \in \Sigma$,
corresponding to the incident field $u^i(t, x; y)$, for $y \in \Sigma$.

The aim is to determine ∂D .



Y. GUO, P. MONK, D. COLTON (2013), Toward a time domain approach to the linear sampling method, *Inverse Problems*.



F. CAKONI, P. MONK, V. SELGAS, Analysis of the linear sampling method for imaging penetrable obstacles in the time domain, (to appear).

The Near Field Operator

Define the **near field operator** $\mathcal{N} : H_\sigma^p(\mathbb{R}, L^2(\Sigma)) \rightarrow H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))$ by

$$(\mathcal{N}\varphi)(t, x) = \int_{\Sigma} \int_{-\infty}^t u^s(\tau, x; y) \varphi(t - \tau, y) d\tau ds_y$$

We also need the **retarded single layer potential** defined by $\mathcal{S} : H_\sigma^p(\mathbb{R}, L^2(\Sigma)) \rightarrow H_\sigma^{p+2}(\mathbb{R}, H^{1/2}(\Sigma))$ defined by

$$(\mathcal{S}\varphi)(t, x) = \int_{\Sigma} \int_{-\infty}^t u^i(\tau, x; y) \varphi(t - \tau, y) d\tau ds_y.$$

Theorem

\mathcal{N} is injective with dense range.

Theorem

\mathcal{S} is injective and bounded.

The Near Field Equation

The **near field equation** is

$$(\mathcal{N}\varphi_z)(t, x) = u^i(t, x; z), \quad z \in \mathbb{R}^3.$$

There exists a unique solution to the near field equation if and only if there exists a $w := w_z \in H^p_\sigma(\mathbb{R}, L^2(D))$ such that w and $v := \mathcal{S}\varphi_z$ satisfy the **interior transmission problem in the time domain**

$$\begin{array}{ll} \partial_{tt}^2 v - \Delta v = 0 & \text{in } \mathbb{R} \times D \\ n(x) \partial_{tt}^2 w - \Delta w = 0 & \text{in } \mathbb{R} \times D \\ w - v = u^i & \text{on } \mathbb{R} \times \partial D \\ \partial_\nu w - \partial_\nu v = \partial_\nu u^i & \text{on } \mathbb{R} \times \partial D \\ w = v = 0 & \text{in } D \text{ for } t \leq 0. \end{array}$$

The Near Field Equation

- The existence of a solution to the interior transmission problem in the time domain was an open problem until now.
- The difficulty in establishing such a result was to determine the location of transmission eigenvalues in the frequency domain.
- The relationship between transmission eigenvalues in the frequency domain and the interior transmission problem in the time-domain is arrived at through the Fourier-Laplace transform.

Transmission Eigenvalues

The **transmission eigenvalue problem in the frequency domain** is to find s such that there exists a nontrivial solution $\hat{w}, \hat{v} \in L^2(D)$, $\hat{w} - \hat{v} \in H_0^2(D)$ such that

$$\begin{aligned}\Delta \hat{v} + s^2 \hat{v} &= 0 && \text{in } D \\ \Delta \hat{w} + s^2 n(x) \hat{w} &= 0 && \text{in } D \\ \hat{w} &= \hat{v} && \text{on } \partial D \\ \partial_\nu \hat{w} &= \partial_\nu \hat{v} && \text{on } \partial D\end{aligned}$$

Such values of s are called **transmission eigenvalues**.



F. CAKONI, D. COLTON AND H. HADDAR (2016), Transmission Eigenvalues and Inverse Scattering Theory, CBMS-NSF, SIAM Publications, **88**.

Transmission Eigenvalues

Theorem

There exists $\sigma_* > 0$ sufficiently large such that there exist no transmission eigenvalues in $\mathbb{C}_{\sigma_*} = \{s \in \mathbb{C}, \Im(s) > \sigma_*\}$.



G. VODEV (2018), High-frequency approximation of interior Dirichlet-to-Neumann map and applications to transmission eigenvalues, *Analysis and PDEs*.

Example

Let $n(x) = 4/9$ and $D := \{x : |x| < 1\}$. Then s is a transmission eigenvalue if and only if

$$\sin^3\left(\frac{s}{3}\right) \left[3 + 2 \cos \frac{2s}{3}\right] = 0$$

i.e. complex transmission eigenvalues exist in this case.

The Interior Transmission Problem

For $p \in \mathbb{R}$, let $h \in H^p_\sigma(\mathbb{R}, H^1(\partial D))$ and $g \in H^{p+5/2}_\sigma(\mathbb{R}, H^2(\partial D))$ and consider the interior transmission problem in the time domain

$$\begin{aligned}\partial_{tt}^2 v - \Delta v &= 0 && \text{in } \mathbb{R} \times D \\ n(x) \partial_{tt}^2 w - \Delta w &= 0 && \text{in } \mathbb{R} \times D \\ w - v &= g && \text{on } \mathbb{R} \times \partial D \\ \partial_\nu w - \partial_\nu v &= h && \text{on } \mathbb{R} \times \partial D \\ w = v &= 0 && \text{in } D \text{ for } t \leq 0.\end{aligned}$$

Corollary

There exists a unique solution $w, v \in H^p_\sigma(\mathbb{R}, L^2(D))$, for $\sigma > \sigma_*$ to the interior transmission problem in the time domain.

The Linear Sampling Method in the Time Domain

Theorem

Let $\sigma > \sigma_*$ and $p \in \mathbb{R}$.

- 1 For $z \in D$ for every $\varepsilon > 0$, there exists some $\varphi_z^\varepsilon \in H_\sigma^p(\mathbb{R}, L^2(\Sigma))$ such that

$$\|\mathcal{N}\varphi_z^\varepsilon - u^i(\cdot, \cdot; z)\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon$$

and

$$\|\mathcal{S}\varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(D))} < C \quad \text{as } \varepsilon \rightarrow 0$$

where C is independent of ε .

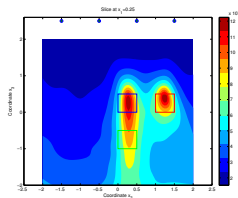
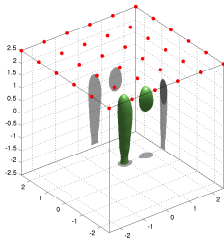
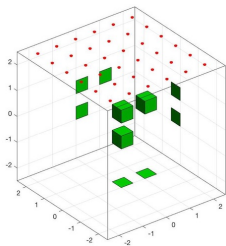
- 2 For $z \in \mathbb{R}^3 \setminus \overline{D}$, every sequence $\{\varphi_z^\varepsilon\}_{\varepsilon>0} \subset H_\sigma^p(\mathbb{R}, L^2(\Sigma))$ satisfying

$$\|\mathcal{N}\varphi_z^\varepsilon - u^i(\cdot, \cdot; z)\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon,$$

is such that

$$\|\mathcal{S}\varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(D))} \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

The Linear Sampling Method in the Time Domain



Note that the reconstruction separates the upper two boxes but fails to recognize that the lower scatterer is disconnected.

The Case When $n(x) = 1$ on ∂D

The mathematical justification of the linear sampling method in the time domain is based on the assumption that $n(x) \neq 1$ on ∂D . In particular this assumption is needed in order to show that the transmission eigenvalues all lie in a strip.

What happens when $n(x) = 1$ on ∂D ?

Consider the transmission eigenvalue problem for the case when $n(x) = n(r)$ and D is a ball of radius one. This problem reduces to

$$y'' + s^2 n(r) y = 0 \quad 0 < r < 1$$

$$y_0'' + s^2 y_0 = 0 \quad 0 < r < 1$$

$$y(1) = y_0(1)$$

$$y'(1) = y_0'(1)$$

where $y(0) = y_0(0) = 0$.

The Case When $n(x) = 1$ on ∂D

In particular s is a transmission eigenvalue if and only if

$$y(1) \cos s - y'(1) \frac{\sin s}{s} = 0.$$

The zeros of $d(s)$ can be investigated through the use of the theory of entire functions of a complex variable. The proof of the following theorem can be found in



D.COLTON, Y.J.LEUNG, S. MENG (2015), Distribution of complex transmission eigenvalues for spherically stratified media, *Inverse Problems*.

The Case When $n(x) = 1$ on ∂D

Theorem

Assume that

- 1 $n(1) = 1$.
- 2 $\int_0^1 \sqrt{n(t)} dt \neq 1$.
- 3 $n''(1) \neq 0$.

Then the transmission eigenvalues do not lie inside a fixed strip parallel to the real axis.

Conclusion

The linear sampling method in the time domain is not mathematically justified when $n(x) = 1$ on ∂D