Placement of multi-fidelity sensors and balancing transformations

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Two Goals:

1. Account for dynamics/actuation in sensor placement
2. Account for different sensor costs and performance

Control systems: Respecting Dynamics in Sensor Placement

- Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}.$$

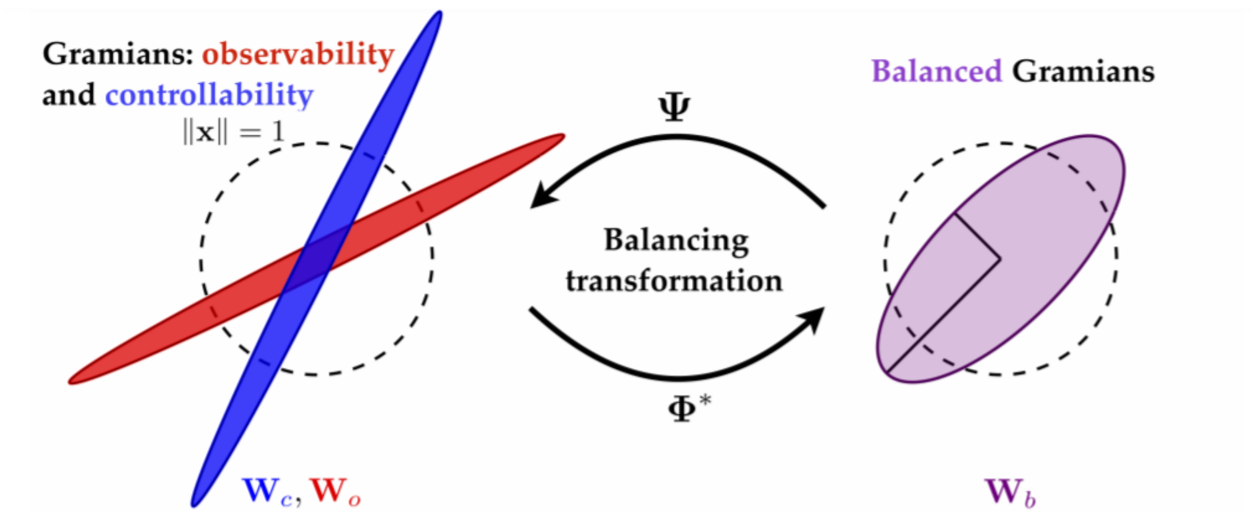
- The controllability and observability Gramians are given by:

$$\mathbf{W}_C = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^*\mathbf{e}^{\mathbf{A}^*t}$$
$$\mathbf{W}_O = \int_0^\infty e^{\mathbf{A}^*t}\mathbf{C}^*\mathbf{C}\mathbf{e}^{\mathbf{A}t}$$

- Their eigendecompositions define the directions in which the system is most controllable and observable.

- Apply a balancing transformation, changing to a basis where the system is maximally jointly controllable and observable.

$$\tilde{\mathbf{W}}_C = \mathbf{\Phi}^* \mathbf{W_C} \mathbf{\Phi}$$
$$\tilde{\mathbf{W}}_O = \mathbf{\Psi}^* \mathbf{W_O} \mathbf{\Psi}$$
$$\tilde{\mathbf{W}}_C = \tilde{\mathbf{W}}_O$$



Manohar, Kutz, and Brunton (2018). Optimal Sensor and Actuator Placement Using Balanced Model Reduction.

QR decomposition

- Perform the column-pivoted QR decomposition on $\mathbf{\Psi}^T$:

$$\mathbf{\Psi}^T \mathbf{P}^T = \mathbf{Q}\mathbf{R},$$

where $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is orthogonal, $\mathbf{R} \in \mathbb{R}^{m \times n}$ is upper triangular, and $\mathbf{P}$ is made up of rows of the identity.

- This is achieved by iteratively applying Householder transformations to make $\mathbf{\Psi}^T$ upper triangular.
- At each step, the transformation is applied to the column with the largest norm, indexed by $\mathbf{P}$.
- For *p* sensors, taking the measurement matrix to be the first *p* rows of $\mathbf{P}$ provides nearly-optimal sensor locations by approximately maximizing the volume of space spanned by the sensors.

Drmac and Gugercin (2016). A new selection operator for the discrete empirical interpolation method—improved a priori error bound and extensions.

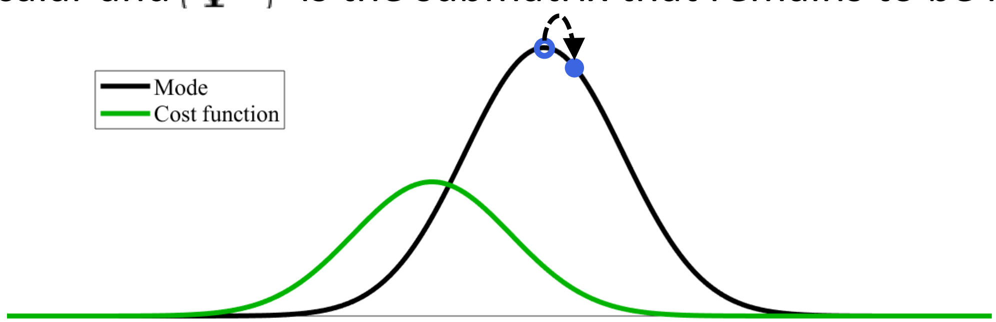
Incorporating a cost function

- Now assume there is some non-negative cost function on sensor location, $\mathbf{f} \in \mathbb{R}^n$

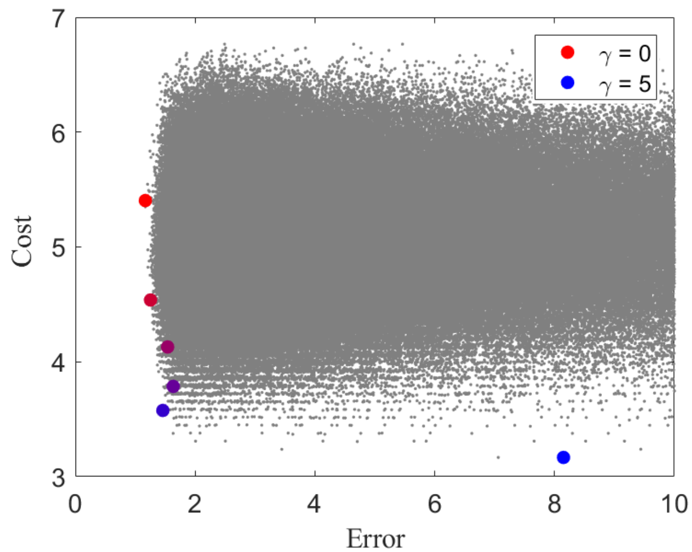
- We modify the column-pivoted QR decomposition algorithm:
- At the kth iteration, pivot about the column *i* that maximizes

$$||(\mathbf{\Psi}^T)_i^k|| - \gamma f_i,$$

where γ is a scalar and $(\mathbf{\Psi}^T)^k$ is the submatrix that remains to be made upper triangular.



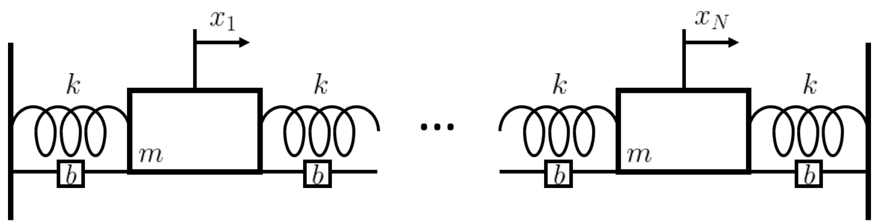
- Place sensors by performing the modified column-pivoted QR decomposition on the truncated balancing modes $\boldsymbol{\Psi}_r^T$.
- Place actuators by performing QR on the truncated adjoint modes $\boldsymbol{\Phi}_r^T$.



Reconstruction error for a 25-dimensional randomized control system with gaussian cost function. Colored points represent results from placing 7 principled sensors as the cost function is weighted more heavily. Gray points show every other permutation of sensor arrays.

Example

- Damped mass-spring system:

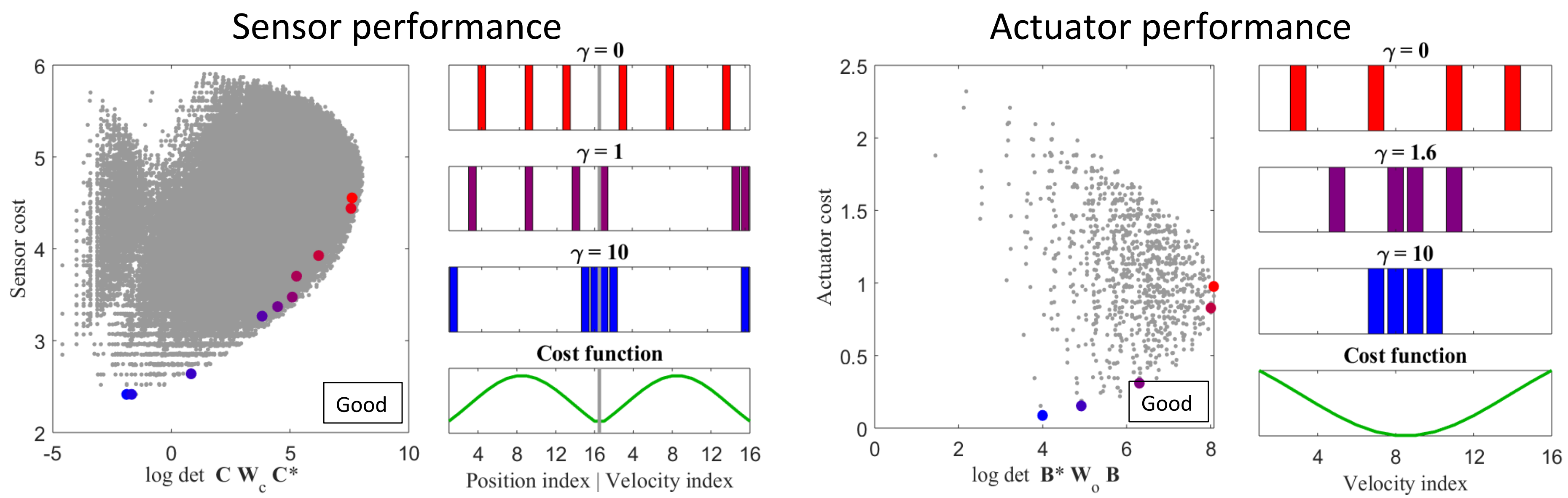


- The i^{th} mass has the equation of motion

- The full system evolves as $m\ddot{x}_i = k(x_{i-1} + x_{i+1} - 2x_i) + b(\dot{x}_{i-1} + \dot{x}_{i+1} - 2\dot{x}_i) + f_i$

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{0}_N & \mathbb{I}_N \\ \mathbf{T} & \mathbf{T} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathbf{0}_N \\ \mathbb{I}_N \end{pmatrix} \mathbf{f}, \quad \text{where } \mathbf{T} = \begin{pmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & \ddots & \\ 0 & \ddots & \ddots & \\ \vdots & & & \end{pmatrix}.$$

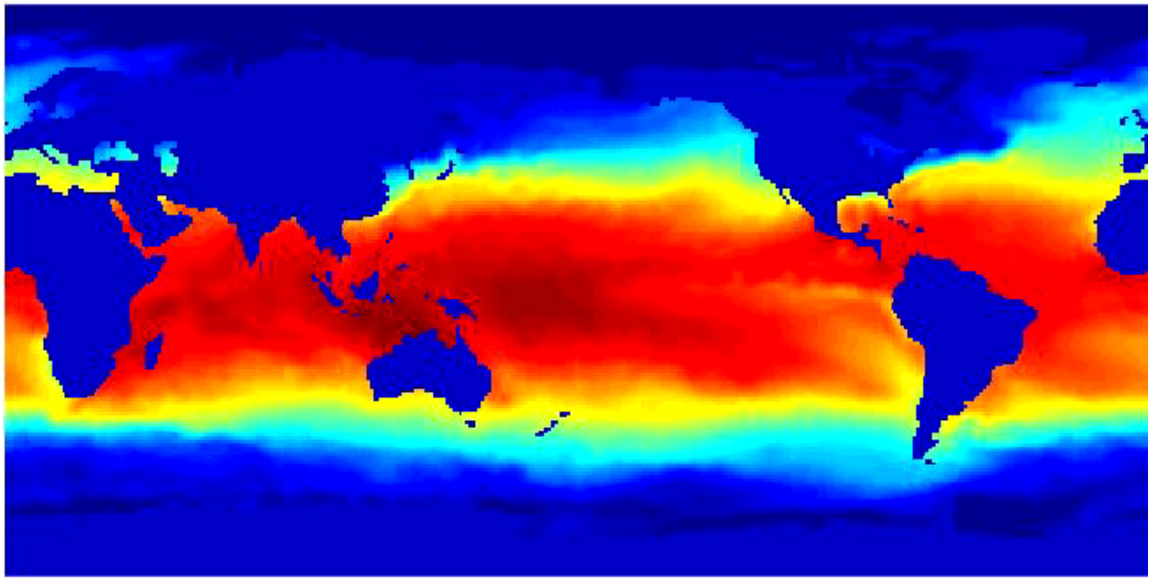
$$\mathbf{y} = \mathbb{I}_{2N}\mathbf{x}$$



Sensor performance
Actuator performance
Sensor cost
log det C W_c C*
Good
γ = 0
γ = 1
γ = 10
Cost function
Position index | Velocity index
Actuator cost
log det B* W_o B
γ = 1.6
Velocity index

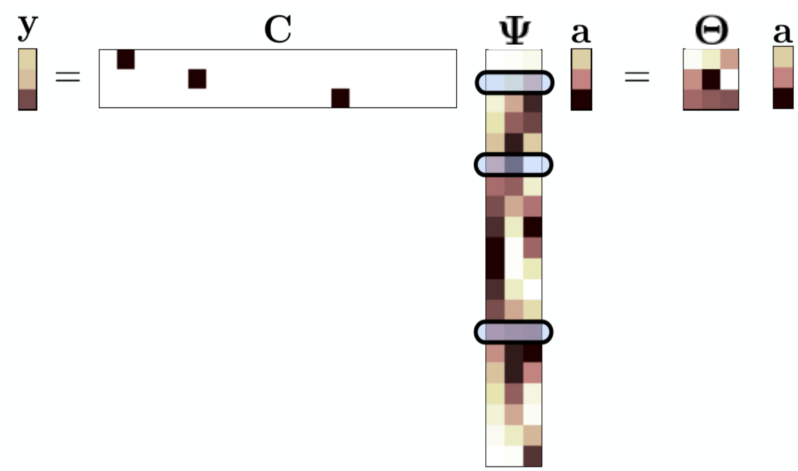
Multi-fidelity sensor placement

- Consider a large system whose full state we want to estimate from sparse measurements.
- Assume there are two types of sensors: Cheap sensors with high noise levels, and expensive ones with less noise.
- Where should we place the sensors to get the best reconstruction?



Background

- First, consider the case of sensor placement without noise.
- Rewrite the full state $\mathbf{x}$ by choosing a new basis $\mathbf{\Psi}$:
$$\mathbf{x} = \mathbf{\Psi}\mathbf{a}.$$
- Sparsely sample. Collect measurements $\mathbf{y}$ at the locations indexed by $\mathbf{C}$:
$$\mathbf{y} = \mathbf{C}\mathbf{\Psi}\mathbf{a} = \mathbf{\Theta}\mathbf{a}.$$
- Given $\mathbf{y}$, get an estimate $\hat{\mathbf{a}}$ of the coefficients: $\hat{\mathbf{a}} = \mathbf{\Theta}^{\dagger}\mathbf{y}.$
- Design $\mathbf{C}$ and $\mathbf{\Psi}$ to get the best reconstruction possible, i.e. $\min ||\mathbf{a} - \hat{\mathbf{a}}||_2.$



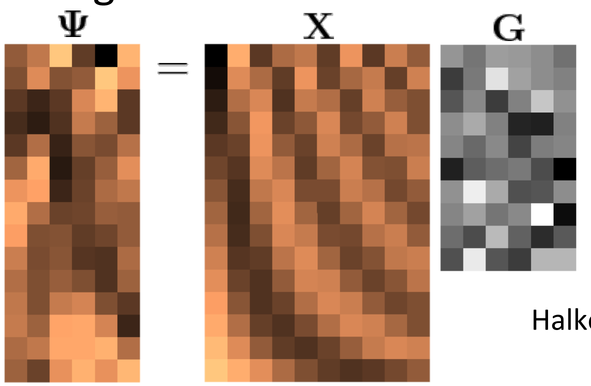
Manohar et al (2017). Data-driven sparse sensor placement for reconstruction.

Building a basis

- Assume we have *m* snapshots in time of the full state. Organize them into a snapshot matrix

$$\mathbf{X} = \left(\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \mathbf{x}_m \right).$$

- We take $\mathbf{\Psi}$ to be randomized linear combinations of the $\mathbf{x}_i$, $\mathbf{\Psi} = \mathbf{X}\mathbf{G}$, $\mathbf{G}$ where has Gaussian i.i.d. entries. We take twice the number of modes as sensors. This seems to provide a good balance between order reduction and information retained.



Halko, Martinsson, and Tropp (2011).

Choosing sensor locations

- Perform column-pivoted QR decomposition on $\mathbf{\Psi}^{\mathrm{T}}$:

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{Q}\mathbf{R},$$

  where $\mathbf{Q}$ is orthonormal, $\mathbf{R}$ is upper triangular, and $\mathbf{P}$ is made up of rows of the identity matrix.
- For *p* sensors, the first *p* rows of $\mathbf{P}$ are nearly-optimal sensor locations.
- This is because the column-pivoted QR decomposition greedily maximizes the determinant of $\mathbf{\Theta}$. This is called D-optimal design.
- This also minimizes the condition number of $\mathbf{\Theta}$, which leads to stable reconstructions that are robust to noise.

Drmac and Gugercin (2016). A new selection operator for the discrete empirical interpolation method—improved a priori error bound and extensions.

Multifidelity sensors

- Now, the measurements are given by

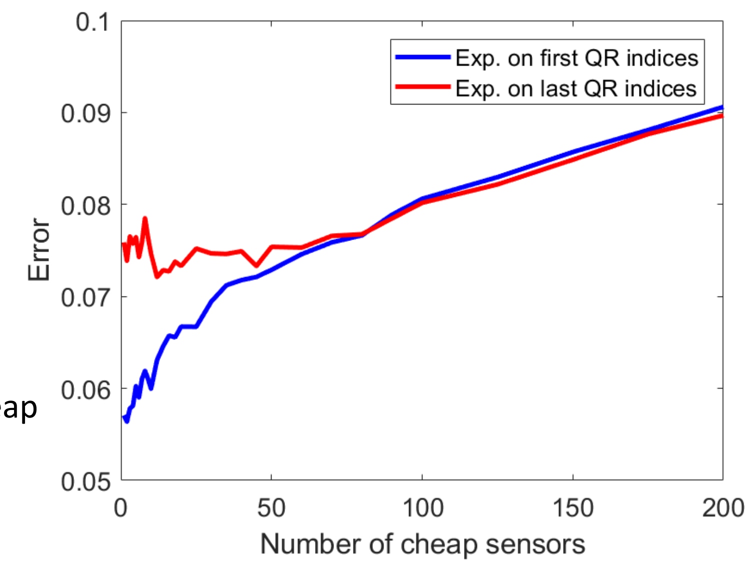
$$\mathbf{y} = \mathbf{\Theta}\mathbf{a} + \epsilon,$$

  where the noise vector ϵ can have two values, $\epsilon_i \in \{\sigma_{\mathbf{e}}, \sigma_{\mathbf{c}}\}$, with $\sigma_{\mathbf{e}} < \sigma_{\mathbf{c}}$ being the noise levels of the expensive and cheap sensors, respectively.

- We still choose the sensor locations using the column-pivoted QR decomposition.

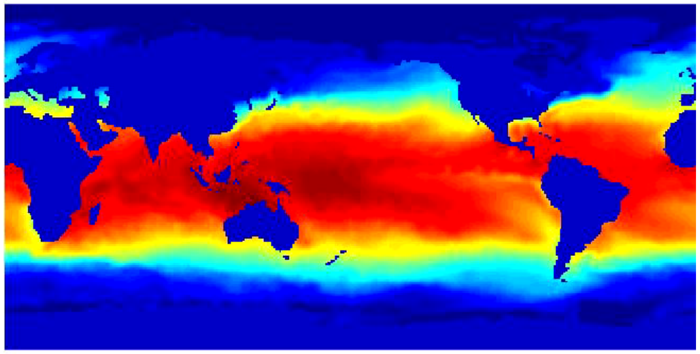
- For now, we place p_c cheap sensors at the first p_c QR indices, and p_e expensive sensors at the remaining QR indices.

- As shown on the right, this leads to better reconstructions at a large number of sensors.

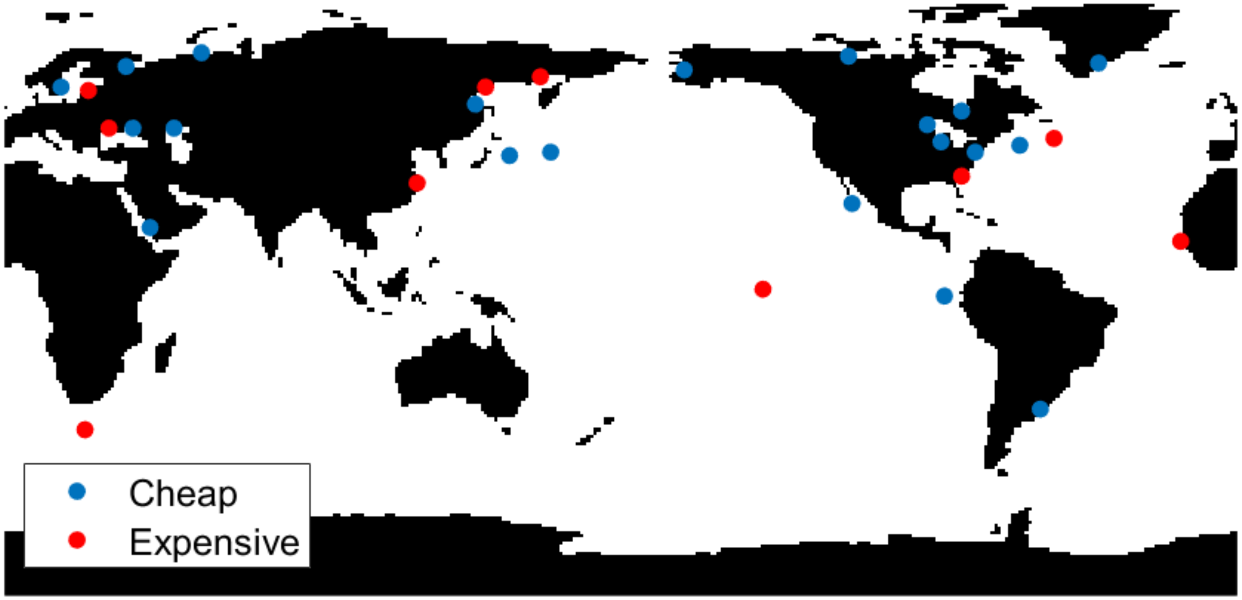


Reconstruction errors using between 1 and 200 cheap sensors ($\sigma_{\mathbf{c}} = \mathbf{5}\%$, $\text{cost}_c = 1$), with an additional 10 expensive sensors ($\sigma_{\mathbf{e}} = \mathbf{1}\%$, $\text{cost}_e = 10$), placed on either the first or last QR indices.

Example

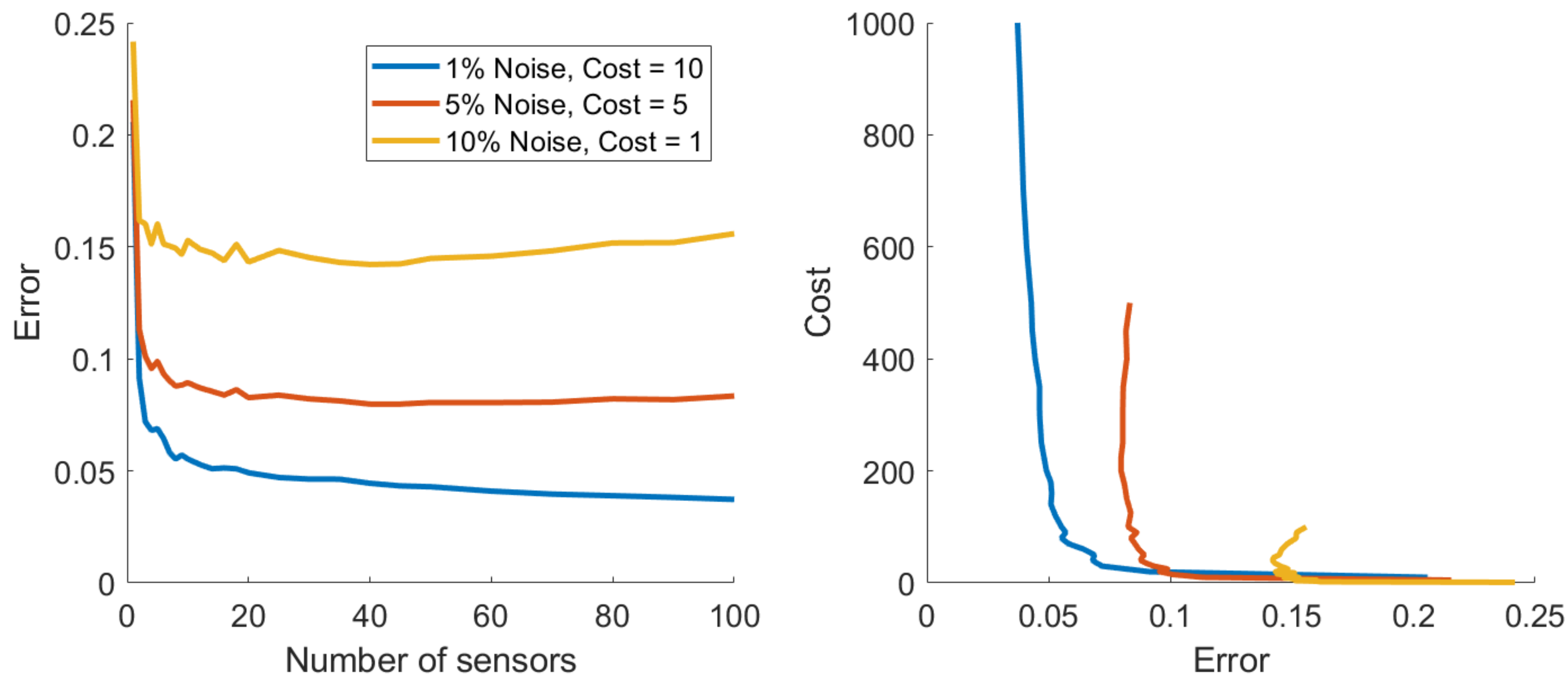


- NOAA weekly sea surface temperature data 1990-2016.
  (https://www.esrl.noaa.gov/psd/data/gridded/data.noaa.oisst.v2.html)
- Below is an example sensor array, placing 20 cheap sensors and 10 expensive sensors.

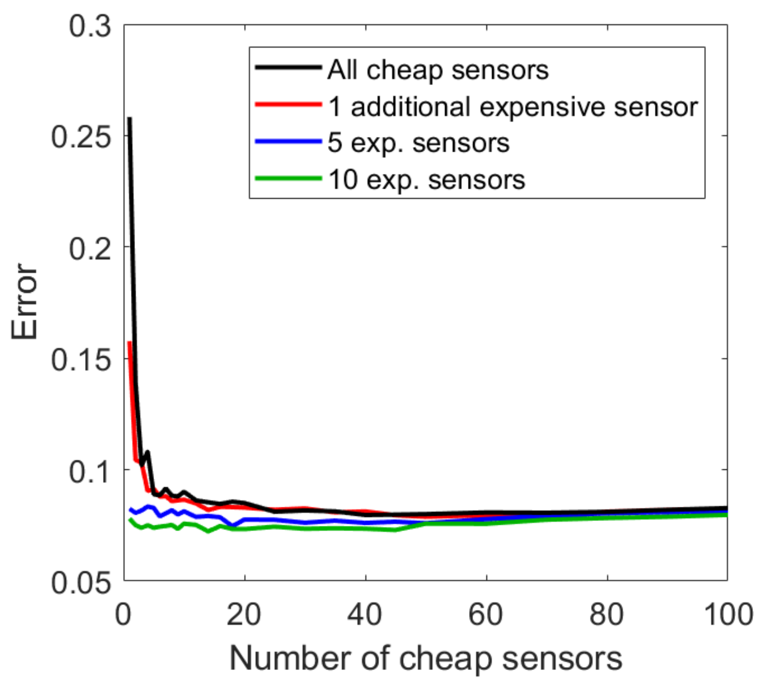


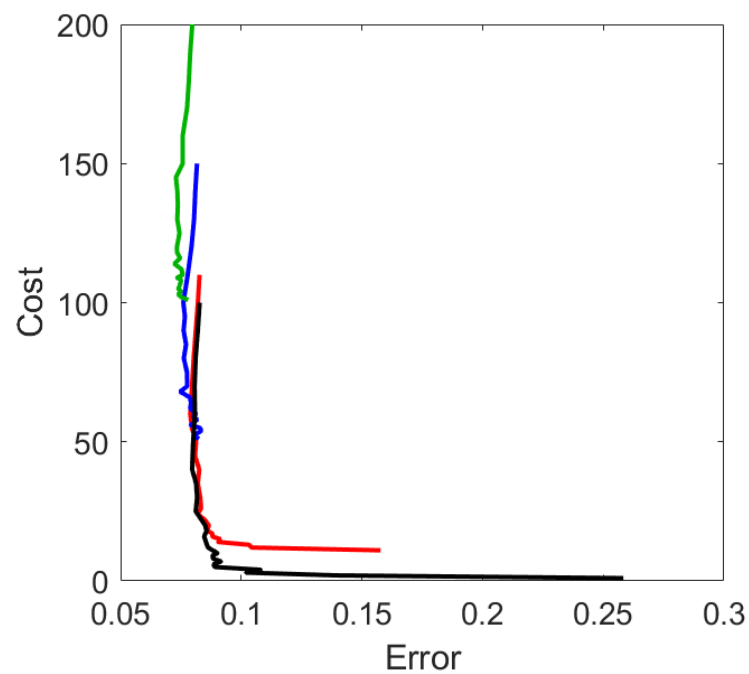
Example

The figure shows reconstruction error and cost using only one type of sensor.

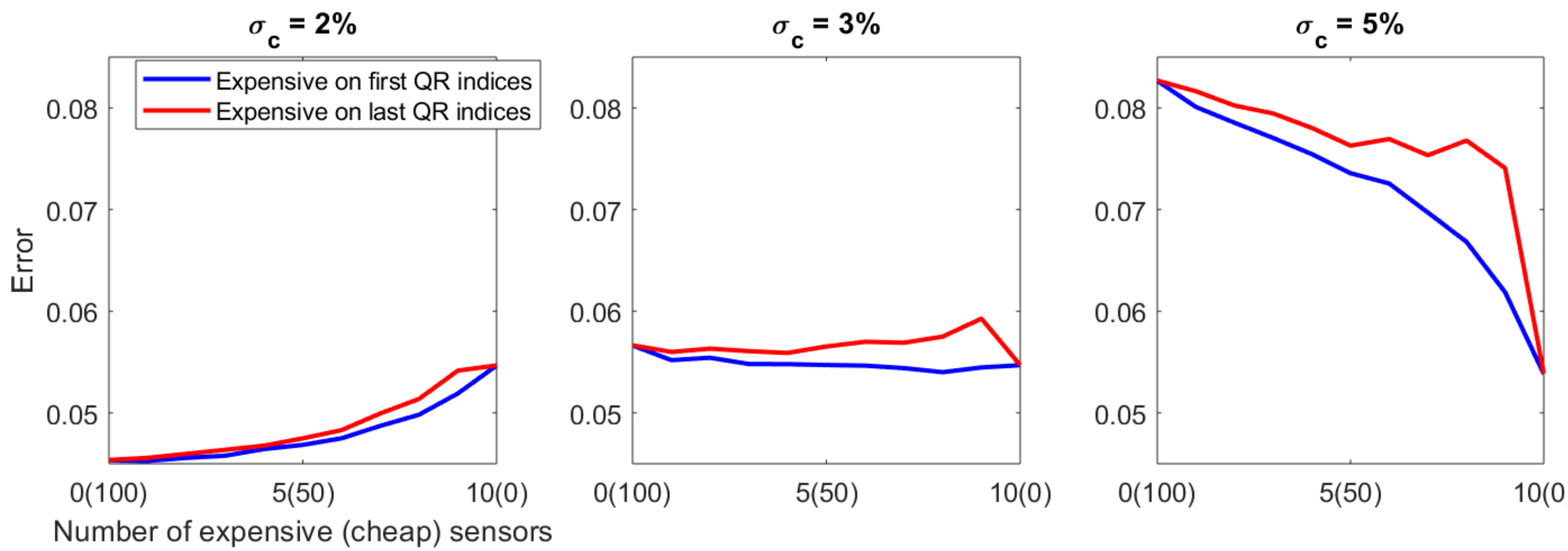


- Now add on additional low-noise sensors to an existing set of noisy sensors.
- The left plot demonstrates that this improves reconstruction performance, but the right plot shows that the price may not be worth it.
- Cheap sensors have noise level 5% and cost 1. Expensive sensors have noise level 1% and cost 10.





- Performance varies greatly with parameters. Depending on noise levels, costs, budget, and type of data, it may be best to have all cheap sensors, all expensive sensors, or a mix of both. Whether to put expensive sensors on the first or the last set of QR pivots is also parameter dependent.
- Below are SST reconstruction errors with a set budget of 100. Expensive sensors have noise level $\sigma_e = 1\%$ and cost 10, while cheap sensors have cost 1.



Conclusions & Future directions

- The column-pivoted QR decomposition can be used for sensor placement and modified to include a cost on sensor location.
- We now extend the cost-modified QR decomposition for sensor and actuator placement for control systems.
- The algorithm identifies sensor and actuator arrays with simultaneously good performance metric and low cost.

- Systems are highly dependent on the noise and cost ratios of the two types of sensors. Ultimately, we want to learn a principled method for knowing how many of each type of sensor should be placed where, given only costs, noise levels, training data, and a budget.
- Other considerations:
  - Add in a cost function on spatial location.
  - Consider more than two types of sensors.
  - Explore sensors with different time resolutions.