ROOT LOCUS ANALYSIS OF THE GROUND-TO-SPACE LOCALIZATION PROBLEM, WITH ERROR ANALYSIS

PRESENTER:

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Integrity ★ Service ★ Excellence

BACKGROUND

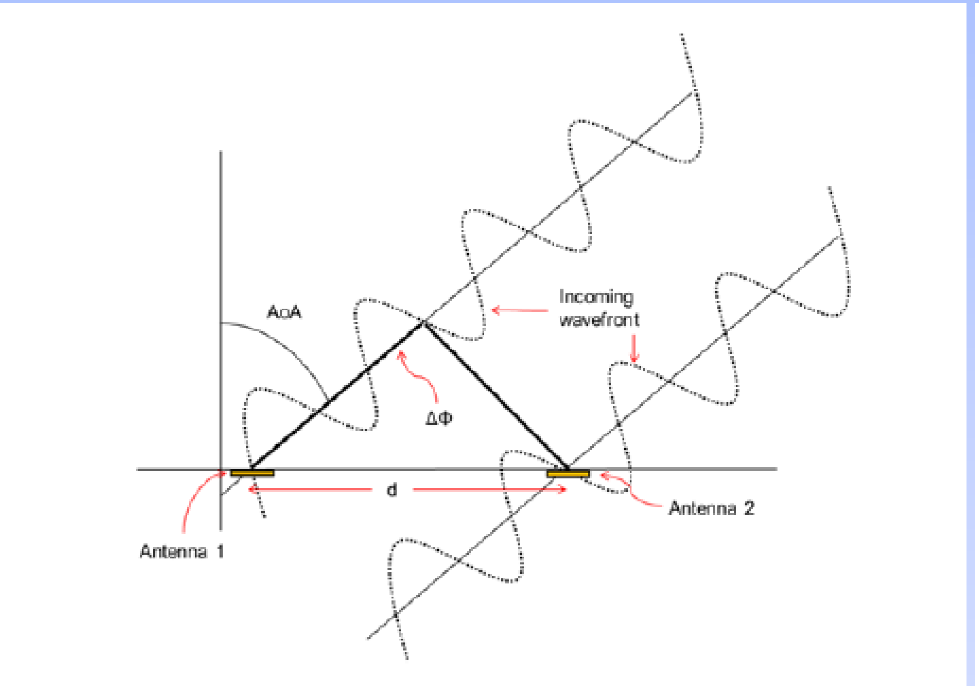
- **RESEARCH GOAL: to improve the state of the art in precise localization of electromagnetic transmitting sources for many different scenarios of practical interest**

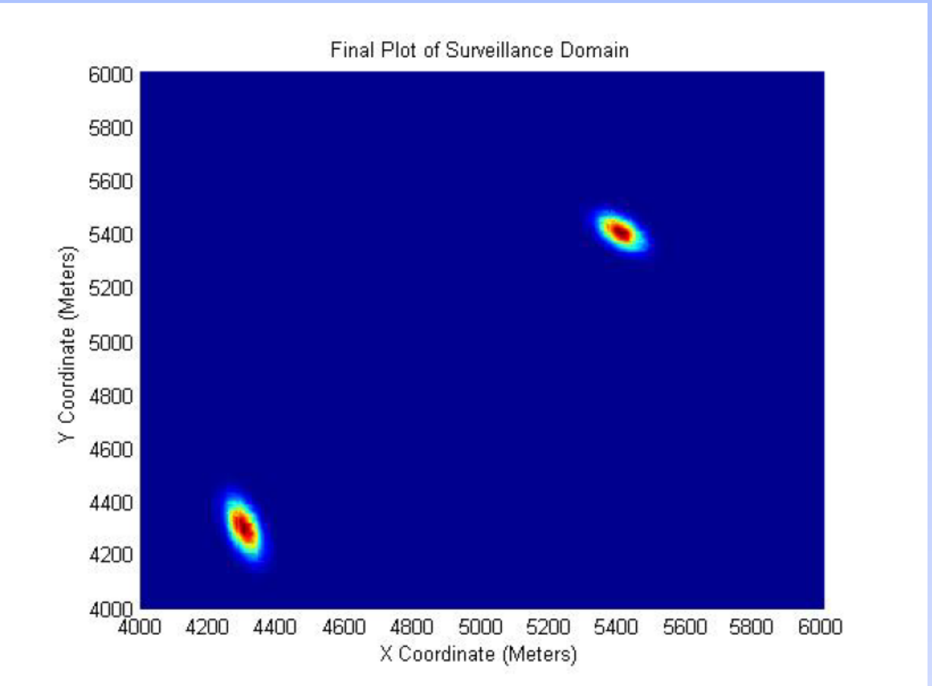
- **Generally involves the use of radio frequencies to determine the position of an emitter that may be in the air, land, sea, or space**

- **The focus of this work could best be described as passive or uncooperative localization, whereby there is no coordination between the transmitter(s) and receiver(s)**
  - **Localization is performed based only on knowledge of the received signal itself (e.g. amplitude, phase, and frequency)**

BACKGROUND

- **Many different ways to localize an emitter:**



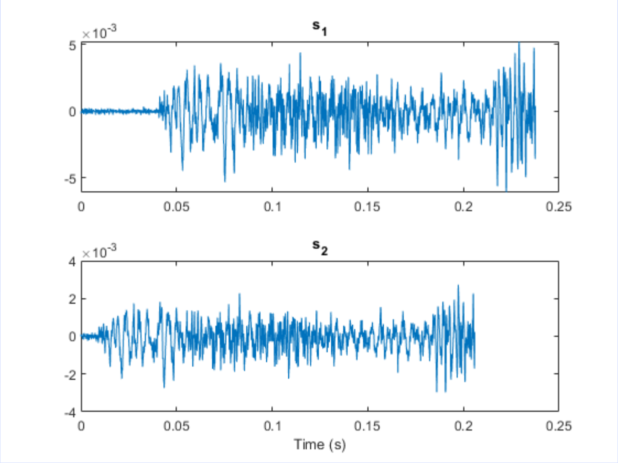


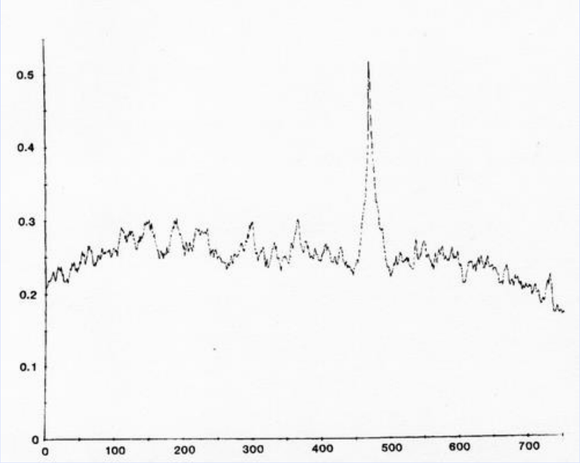
Angle of arrival/interferometry (phased array antenna)

Time-frequency SAR (Cheney)

BACKGROUND

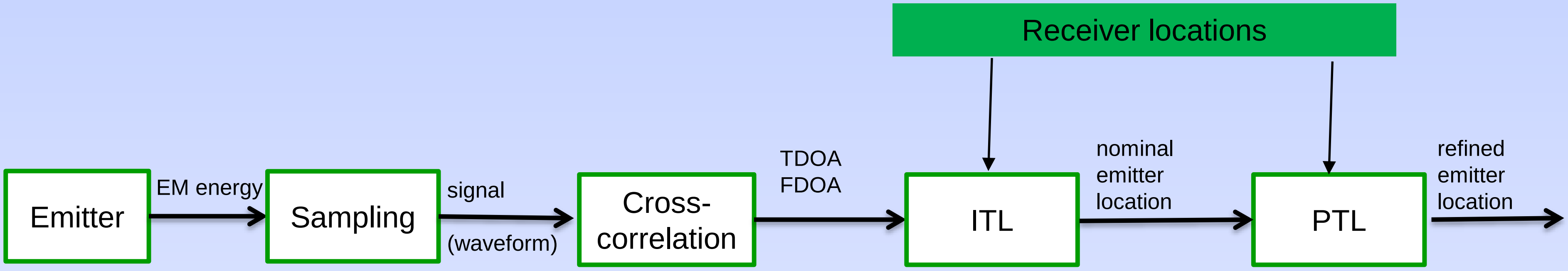
- **When two or more receivers are available, multiple data types or observables can be derived by comparing the signals obtained by the receivers (e.g. via cross-correlation)**
  - **Include time difference of arrival (TDOA) and frequency difference on arrival (FDOA)**





END-TO-END PROCESS

- **The common "2-step" localization process is depicted here:**
  - **SIGNAL PROCESSING: cross-correlation to obtain TDOA or FDOA**
  - **ESTIMATION: solving measurement equations for unknown emitter position coordinates**
- **Estimation step consists of:**
  - **INITIAL TRANSMITTER LOCALIZATION (ITL): obtaining a quick/reasonable solution from minimal measurements → 3 equations in 3 unknowns**
  - **PRECISE TRANSMITTER LOCALIZATION (PTL): refining the ITL solution with subsequent measurements → statistical process e.g. Kalman filter**



LOCALIZATION SCENARIOS

- **How do we define scenario(s) of interest?**
- **Many facets to a scenario which need to be specified → think of them as "knobs" each with different "settings"**

 # OF RECEIVERS:
1
2
3
etc

MEASUREMENT TYPES:
TDOA
FDOA/FROA
SNR/phase (AOA)

INITIAL VS PRECISE LOCALIZATION:
ITL
PTL

RECEIVER REGIME:
Terrestrial
LEO
Beyond LEO (e.g. MEO/GEO)
Hybrid (terrestrial & spaceborne)

EMITTER REGIME:
Terrestrial ("geolocation")
LEO
Beyond LEO (e.g. MEO/GEO)

EXPRESSION OF MEASUREMENT ERROR:
None
Gaussian/unmodeled
Detailed models

LOCALIZATION SCENARIOS

- **First endeavor of FY19 involved the choice of "settings" highlighted below:**
  - **TDOA geolocation from spaceborne receivers ("ground-to-space")**
  - **ITL from minimal info (3 TDOAs)**
  - **TDOA error is Gaussian/additive**

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etc

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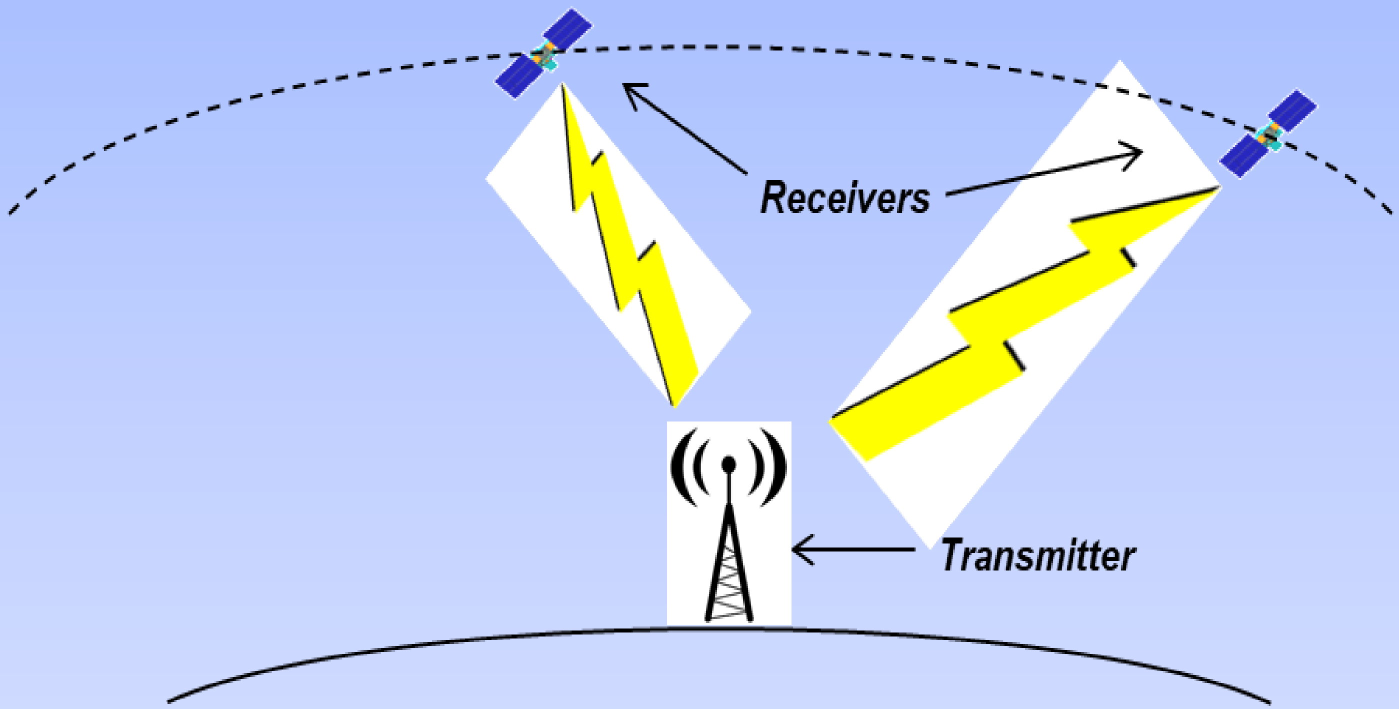
EXPRESSION OF MEASUREMENT ERROR:
None
Gaussian/unmodeled
Detailed models

LOCALIZATION SCENARIOS

- **This scenario can be visualized as follows:**



- **Goal is to obtain TDOA at three times while receivers pass overhead → solve the coordinates (x_T, y_T, z_T) of the transmitter in an Earth-centered, Earth-fixed frame**

TDOA EQUATION

TDOA * speed of light yields range difference, which is related to x_T, y_T, z_T by simple Euclidean distance formula:

*k*th TDOA measurement (known): $\Delta t_k = t_{2,k} - t_{1,k}$

Receiver locations (known): $x_{1,k},\ y_{1,k},\ z_{1,k}$ / $x_{2,k},\ y_{2,k},\ z_{2,k}$

Transmitter location (unknown): $x_T,\ y_T,\ z_T$

$$\begin{aligned}\Delta\rho_k = c\Delta t_k &= \rho_{2,k} - \rho_{1,k} \\ &= \left[(x_T - x_{2,k})^2 + (y_T - y_{2,k})^2 + (z_T - z_{2,k})^2\right]^{\frac{1}{2}} \\ &\quad - \left[(x_T - x_{1,k})^2 + (y_T - y_{1,k})^2 + (z_T - z_{1,k})^2\right]^{\frac{1}{2}}\end{aligned}$$

TDOA EQUATION

Squaring both sides & expanding yields:

$$\rho_{2,k}^2 = (\Delta\rho_k + \rho_{1,k})^2 = \Delta\rho_k^2 + 2\Delta\rho_k\rho_{1,k} + \rho_{1,k}^2$$

Rearrangement yields:

$$\rho_{2,k}^2 - \rho_{1,k}^2 - \Delta\rho_k^2 = 2\Delta\rho_k\rho_{1,k}$$

Substituting x_T, y_T, z_T, etc back in yields:

$$\begin{aligned}
&(x_T - x_{2,k})^2 + (y_T - y_{2,k})^2 + (z_T - z_{2,k})^2 \\
&- (x_T - x_{1,k})^2 - (y_T - y_{1,k})^2 - (z_T - z_{1,k})^2 - \Delta\rho_k^2 \\
&= 2\Delta\rho_k\sqrt{(x_T - x_{1,k})^2 + (y_T - y_{1,k})^2 + (z_T - z_{1,k})^2}
\end{aligned}$$

TDOA EQUATION

Expanding yields:

$$
\begin{aligned}
&-2x_T x_{2,k} + x_{2,k}^2 - 2y_T y_{2,k} + y_{2,k}^2 - 2z_T z_{2,k} \\
&+ z_{2,k}^2 + 2x_T x_{1,k} - x_{1,k}^2 + 2y_T y_{1,k} - y_{1,k}^2 \\
&+ 2z_T z_{1,k} - z_{1,k}^2 - \Delta\rho_k^2 \\
&= 2\Delta\rho_k \sqrt{\left(x_T - x_{1,k}\right)^2 + \left(y_T - y_{1,k}\right)^2 + \left(z_T - z_{1,k}\right)^2}
\end{aligned}
$$

Regrouping yields:

$$
\begin{aligned}
&x_T\left(x_{1,k} - x_{2,k}\right) + y_T\left(y_{1,k} - y_{2,k}\right) \\
&+ z_T\left(z_{1,k} - z_{2,k}\right) + \frac{1}{2}\left(K_{2,k} - K_{1,k} - \Delta\rho_k^2\right) \\
&= \Delta\rho_k \sqrt{\left(x_T - x_{1,k}\right)^2 + \left(y_T - y_{1,k}\right)^2 + \left(z_T - z_{1,k}\right)^2}
\end{aligned}
$$

$$K_{i,k} = x_{i,k}^2 + y_{i,k}^2 + z_{i,k}^2$$

TDOA EQUATION

Squaring both sides yields:

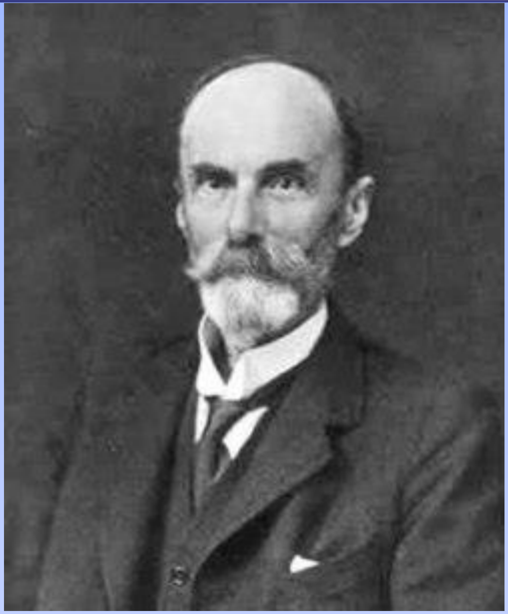
$$
\begin{aligned}
& x_T^2\left(x_{1,k}-x_{2,k}\right)^2+2x_Ty_T\left(x_{1,k}-x_{2,k}\right)\left(y_{1,k}-y_{2,k}\right) \\
& +2x_Tz_T\left(x_{1,k}-x_{2,k}\right)\left(z_{1,k}-z_{2,k}\right)+y_T^2\left(y_{1,k}-y_{2,k}\right)^2 \\
& +2y_Tz_T\left(y_{1,k}-y_{2,k}\right)\left(z_{1,k}-z_{2,k}\right)+z_T^2\left(z_{1,k}-z_{2,k}\right)^2 \\
& +\left(x_T\left(x_{1,k}-x_{2,k}\right)+y_T\left(y_{1,k}-y_{2,k}\right)+z_T\left(z_{1,k}-z_{2,k}\right)\right) \\
& \times\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)+\frac{1}{4}\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)^2 \\
& \quad=\Delta\rho_k^2\left(\left(x_T-x_{1,k}\right)^2+\left(y_T-y_{1,k}\right)^2+\left(z_T-z_{1,k}\right)^2\right)
\end{aligned}
$$

TDOA EQUATION

- Finally, rearrangement yields a 2nd-order polynomial in x_T, y_T, z_T
- Three TDOA's then yield three 2nd-order polynomials in 3 unknowns
- How to solve these eqn's??

$$
\begin{aligned}
& x_T^2\left(\left(x_{1,k}-x_{2,k}\right)^2-\Delta\rho_k^2\right) \\
& +2x_T y_T\left(x_{1,k}-x_{2,k}\right)\left(y_{1,k}-y_{2,k}\right) \\
& +2x_T z_T\left(x_{1,k}-x_{2,k}\right)\left(z_{1,k}-z_{2,k}\right) \\
& +y_T^2\left(\left(y_{1,k}-y_{2,k}\right)^2-\Delta\rho_k^2\right) \\
& +2y_T z_T\left(y_{1,k}-y_{2,k}\right)\left(z_{1,k}-z_{2,k}\right) \\
& +z_T^2\left(\left(z_{1,k}-z_{2,k}\right)^2-\Delta\rho_k^2\right) \\
& +x_T\left(\left(x_{1,k}-x_{2,k}\right)\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)+2\Delta\rho_k^2 x_{1,k}\right) \\
& +y_T\left(\left(y_{1,k}-y_{2,k}\right)\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)+2\Delta\rho_k^2 y_{1,k}\right) \\
& +z_T\left(\left(z_{1,k}-z_{2,k}\right)\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)+2\Delta\rho_k^2 z_{1,k}\right) \\
& +\frac{1}{4}\left(K_{2,k}-K_{1,k}-\Delta\rho_k^2\right)^2-\Delta\rho_k^2 K_{1,k}=0
\end{aligned}
$$

SOLUTION METHODS



- Analytical method based on early 20th-century work by F.S. Macaulay
- Numerical method developed by Bates, Hauenstein, & students

- **Goal of each method is to yield all solutions of the polynomial eqn's → Bezout number = 2^3 = 8**
- **It then remains to select which is the "correct" solution → disambiguation**
  - **For geolocation scenario, most practical choice of solution is one that is (1) real & (2) at or near Earth's surface → $||<x,y,z>|| \approx 6378$km**

ROOT LOCUS METHOD

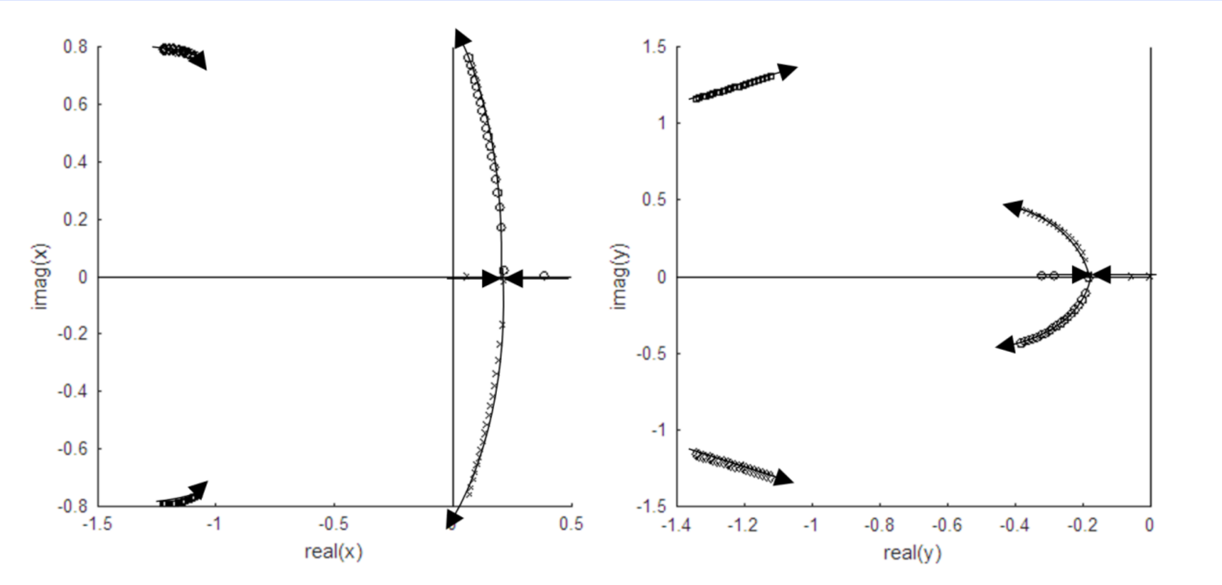
- **Given this "polynomial system" construct of the problem, how might we explore the solution space in the presence of TDOA measurement error?**
- **Consider the root locus concept → from stability & control theory**
- **Conventional root locus method illustrates the effect of changing one parameter on the solutions to a univariate polynomial**
- **To extend root locus for use in understanding the solutions to a system of polynomials, we can examine plots of each variable separately**

ROOT LOCUS METHOD

- An example of the root locus method is applied to the following system of two 2nd-order polynomials:

$$x^2 + y^2 + y + f_1 = 0$$
$$xy + x + y = 0$$

- f_1 is varied from 0 to 1 at an interval of 0.05 per step



Roots of x

Roots of y

ROOT LOCUS METHOD

For error analysis of a TDOA scenario, an application of root loci might proceed as follows:

- **Generate the governing polynomial system for the scenario (three 2nd-order polynomials), as well as a Gaussian error on TDOA**
- **Generate multiple trials of the polynomial system in Monte-Carlo fashion; for each trial, sample from the error source & add this error to TDOA in the equations**
- **Plot the "point cloud" of roots for each variable resulting from these trials**

ROOT LOCUS RESULTS

- Let's examine several scenarios, each involving a transmitter & spaceborne receivers

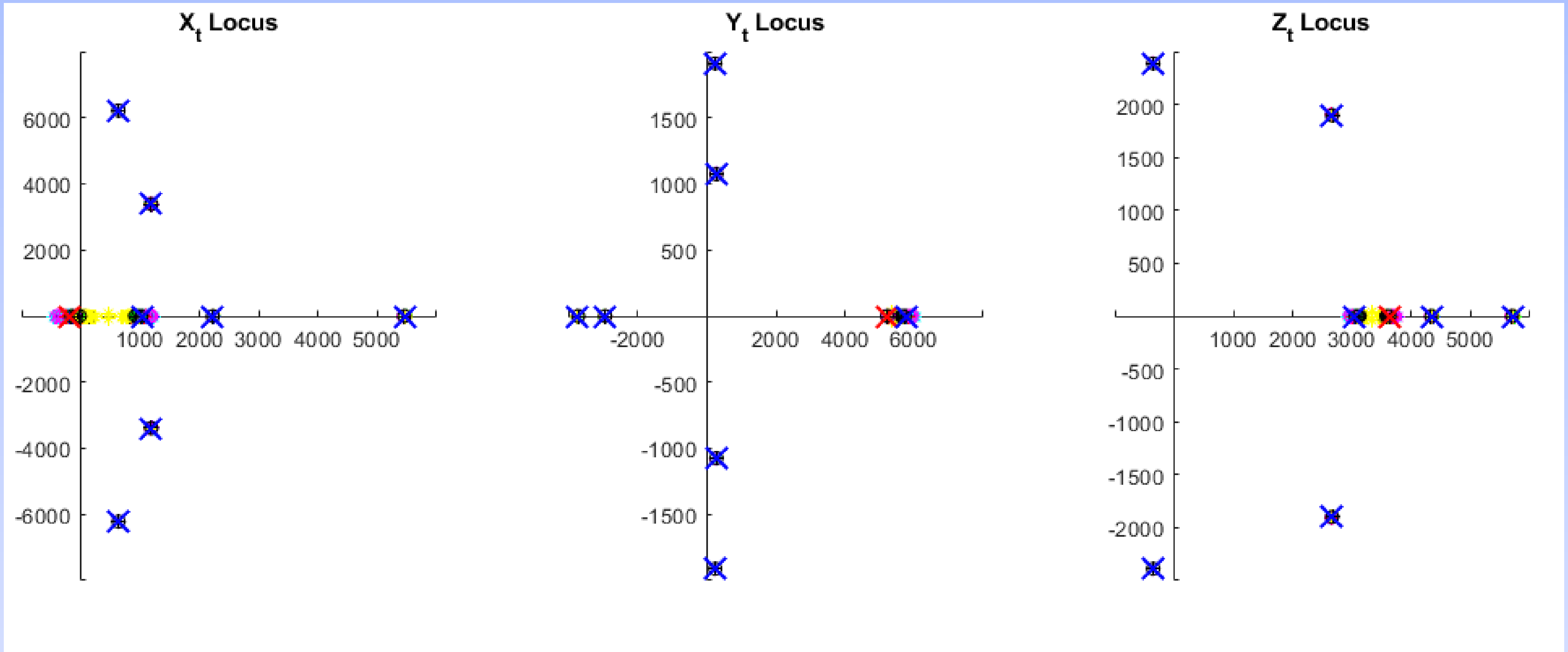
- For each scenario, we must define the following:
  - Emitter location (lat/long)
  - Receiver #1 orbit
  - Receiver #2 orbit

- For each scenario, the 3 TDOA measurement times chosen are the rise time, setting time, & the midpoint of these 2 times

- For each scenario, the TDOA Gaussian error level of σ = 3E-6 sec (~1km range difference)

ROOT LOCUS RESULTS

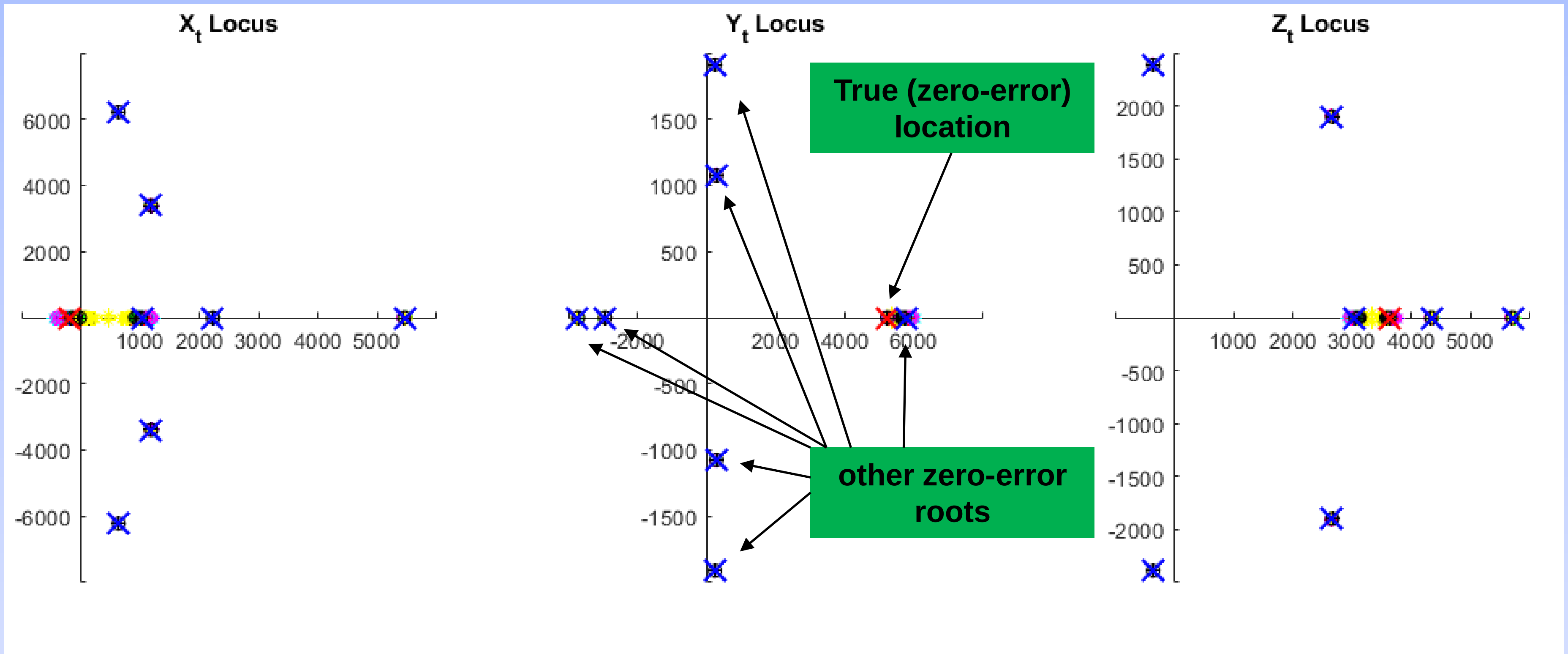
- Scenario #1
  - Receivers co-orbital at 500km altitude, 63.5° inclination, 15° apart
  - Emitter at 35°N latitude, 92°E longitude



- The larger X's denote the "zero-error" solution, with red X's marking the true transmitter coordinates, and blue X's marking all other solutions to the polynomials

ROOT LOCUS RESULTS

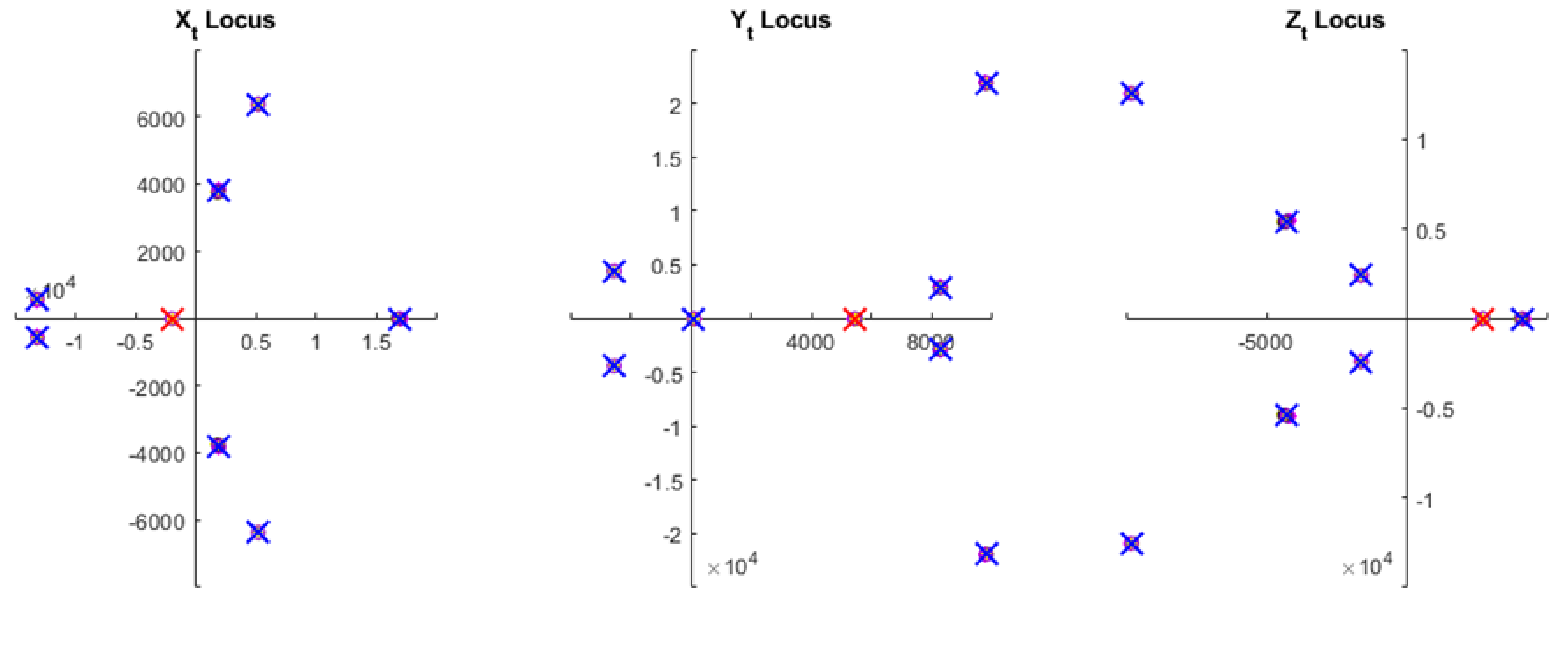
- **Scenario #1**
  - **Receivers co-orbital at 500km altitude, 63.5° inclination, 15° apart**
  - **Emitter at 35°N latitude, 92°E longitude**



- **Effect of TDOA error is for 2 "point clouds" to migrate toward each other → possible difficulty of disambiguation**
- **Remaining roots very stable/robust to this amount of error**

ROOT LOCUS RESULTS

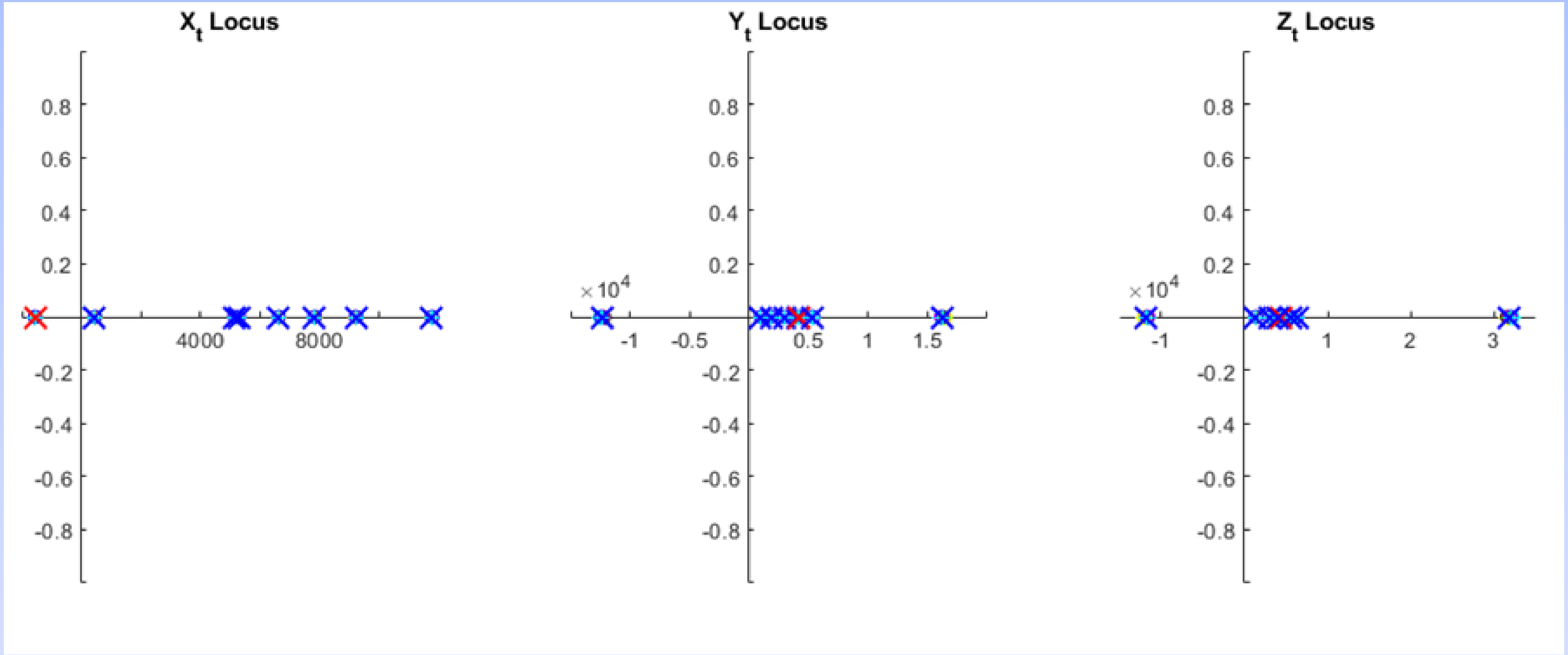
- Scenario #2
  - Receiver #1: a = 7178 km, e = 0, i = 90°, θ = 65°
  - Receiver #2: a = 10645 km, e = 0, i = 74.3°, θ = 325°
  - Emitter at 25°N latitude, 110°E longitude



- All roots very stable/robust
- Likely straightforward disambiguation

ROOT LOCUS RESULTS

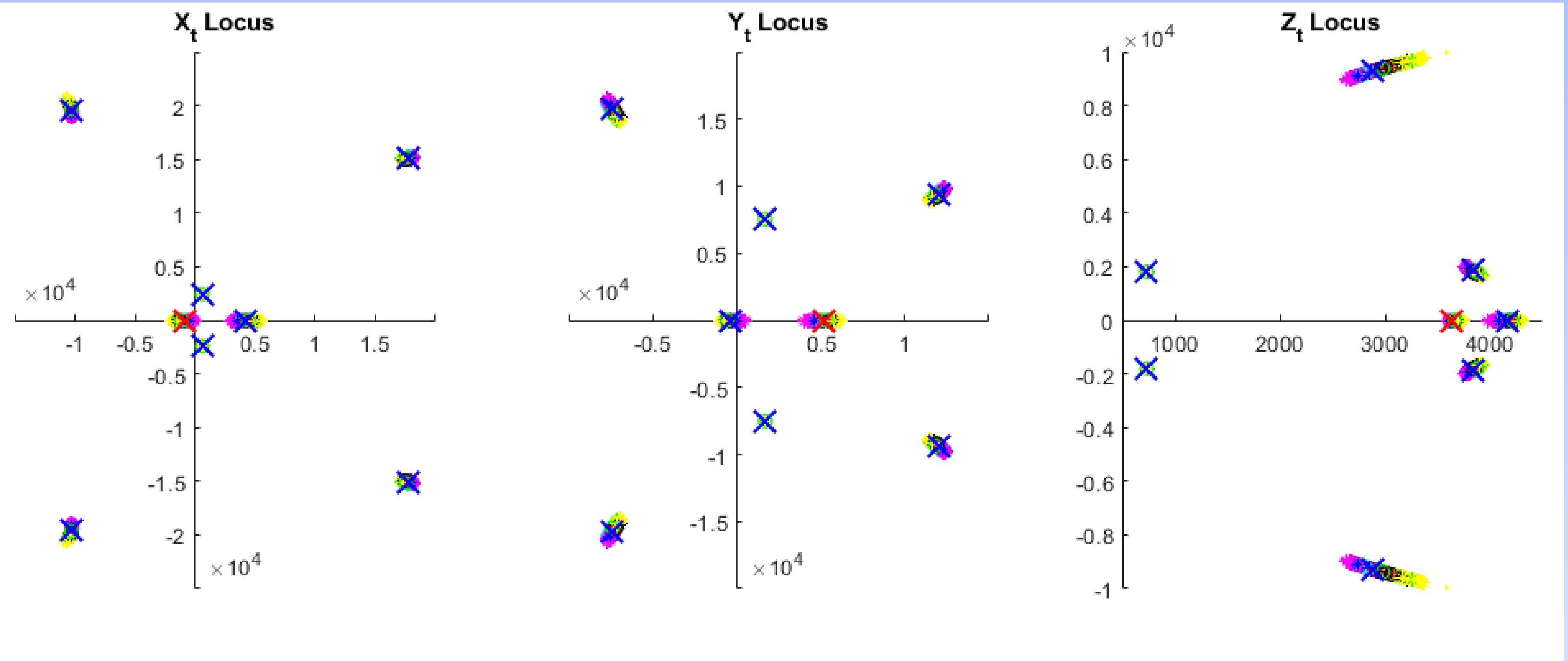
- **Scenario #3**
  - **Receiver #1: a = 7178 km, e = 0, i = 90°, θ = 45°**
  - **Receiver #2: a = 10645 km, e = 0, i = 74.3°, θ = 345°**
  - **Emitter at 45°N latitude, 110°E longitude**



- **All roots very stable/robust, but (interestingly) all are real!**
- **Disambiguation possibly NOT as straightforward as in previous case**

ROOT LOCUS RESULTS

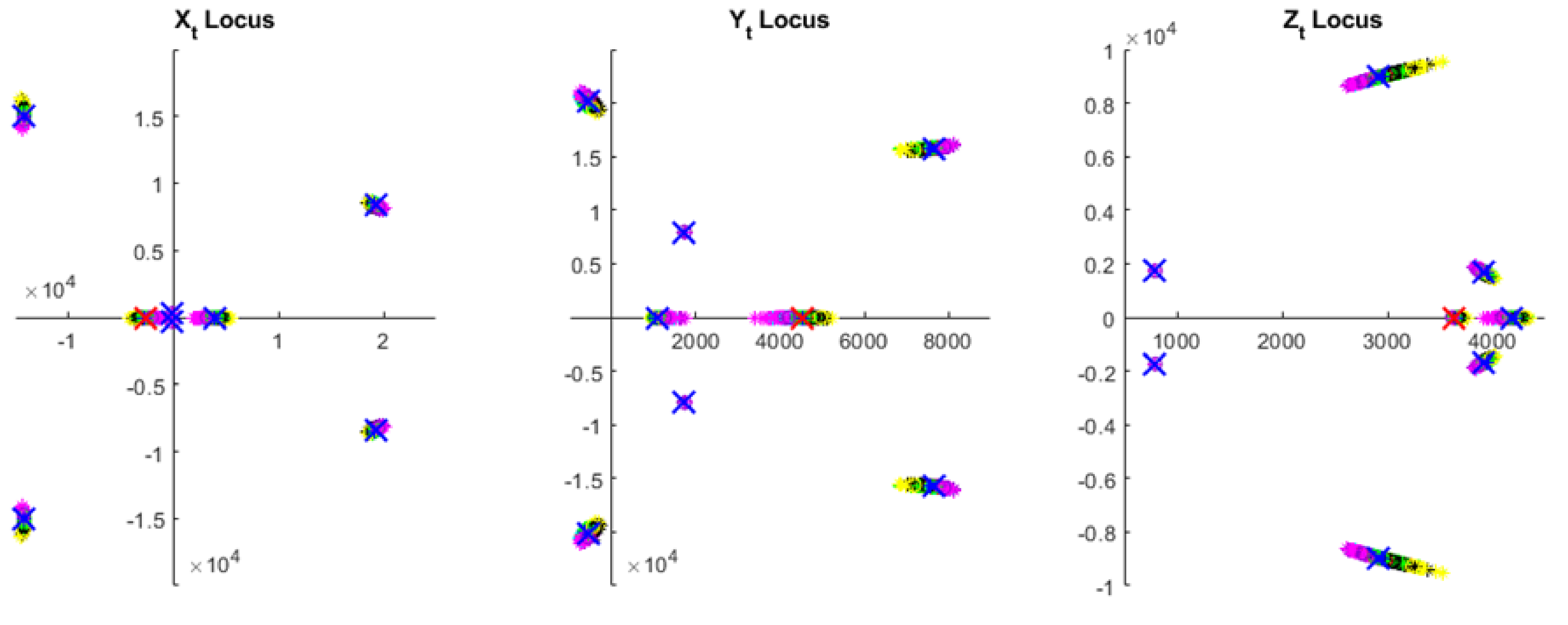
- Scenario #4
  - Receiver #1: a = 7178 km, e = 0, i = 90°, θ = 55°
  - Receiver #2: a = 10645 km, e = 0, i = 74.3°, θ = 320°
  - Emitter at 35°N latitude, 100°E longitude



- Roots display **most susceptibility** to error of any scenario so far

ROOT LOCUS RESULTS

- Scenario #5
  - Receiver #1: a = 7178 km, e = 0, i = 90°, θ = 55°
  - Receiver #2: a = 10645 km, e = 0, i = 74.3°, θ = 320°
  - Emitter at 35°N latitude, 120°E longitude



- Roots vary more with TDOA error than in previous scenario (large "point clouds")

ROOT LOCUS RESULTS

- Recall the most straightforward criteria for validity of a solution are that it be real & at or near Earth's surface

- For a given scenario, it is useful to plot (in lat/long space) all solutions meeting these criteria

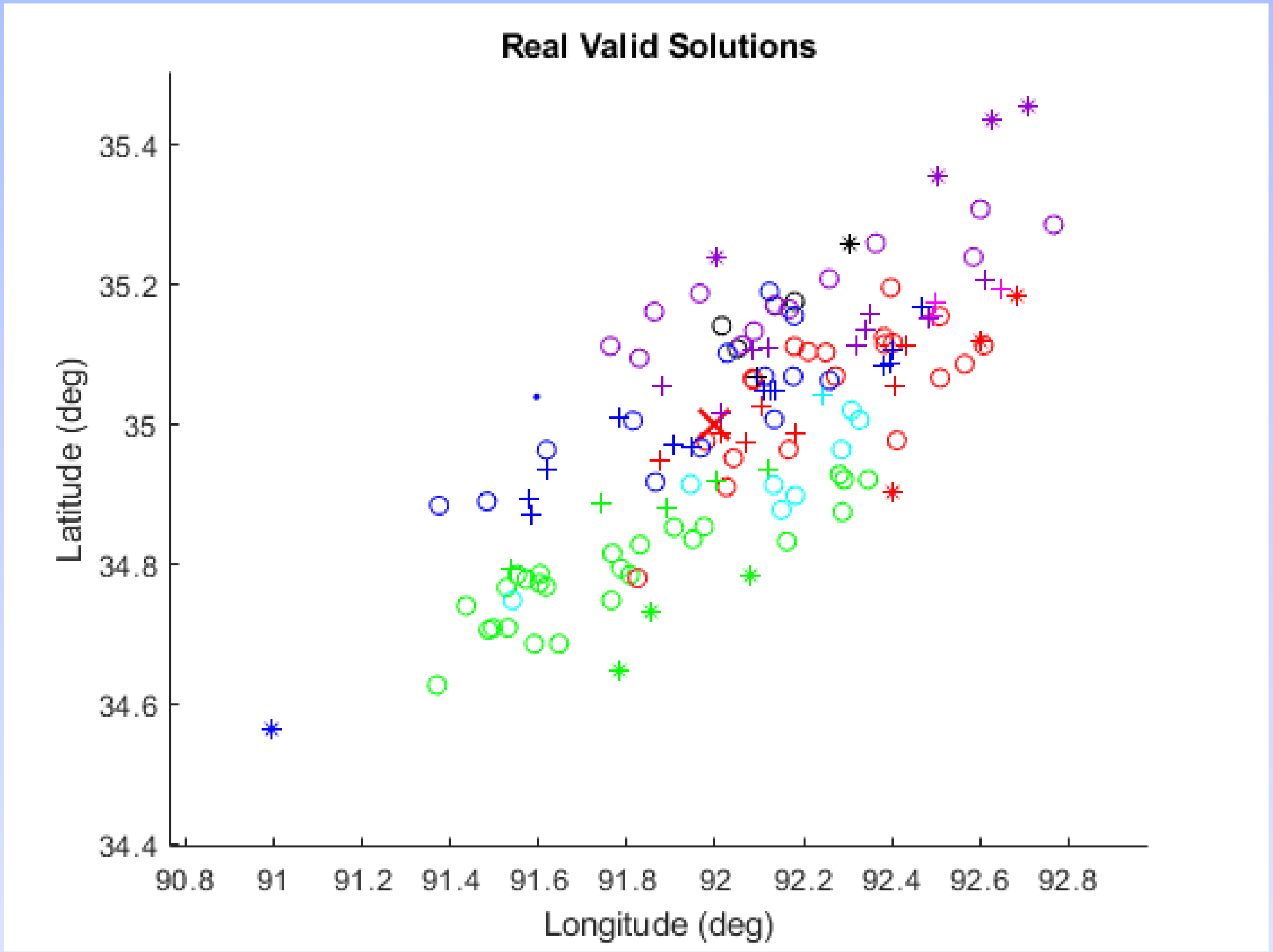
- The following plots include only solutions possessing no (or very little) imaginary component & corresponding to emitter location within 10km of Earth's surface

ROOT LOCUS RESULTS

- **Scenario #1**
  - **Receivers co-orbital at 500km altitude, 63.5° inclination, 15° apart**
  - **Emitter at 35°N latitude, 92°E longitude**

1° latitude error ≈ 110km error

1° longitude error ≈ 90km error

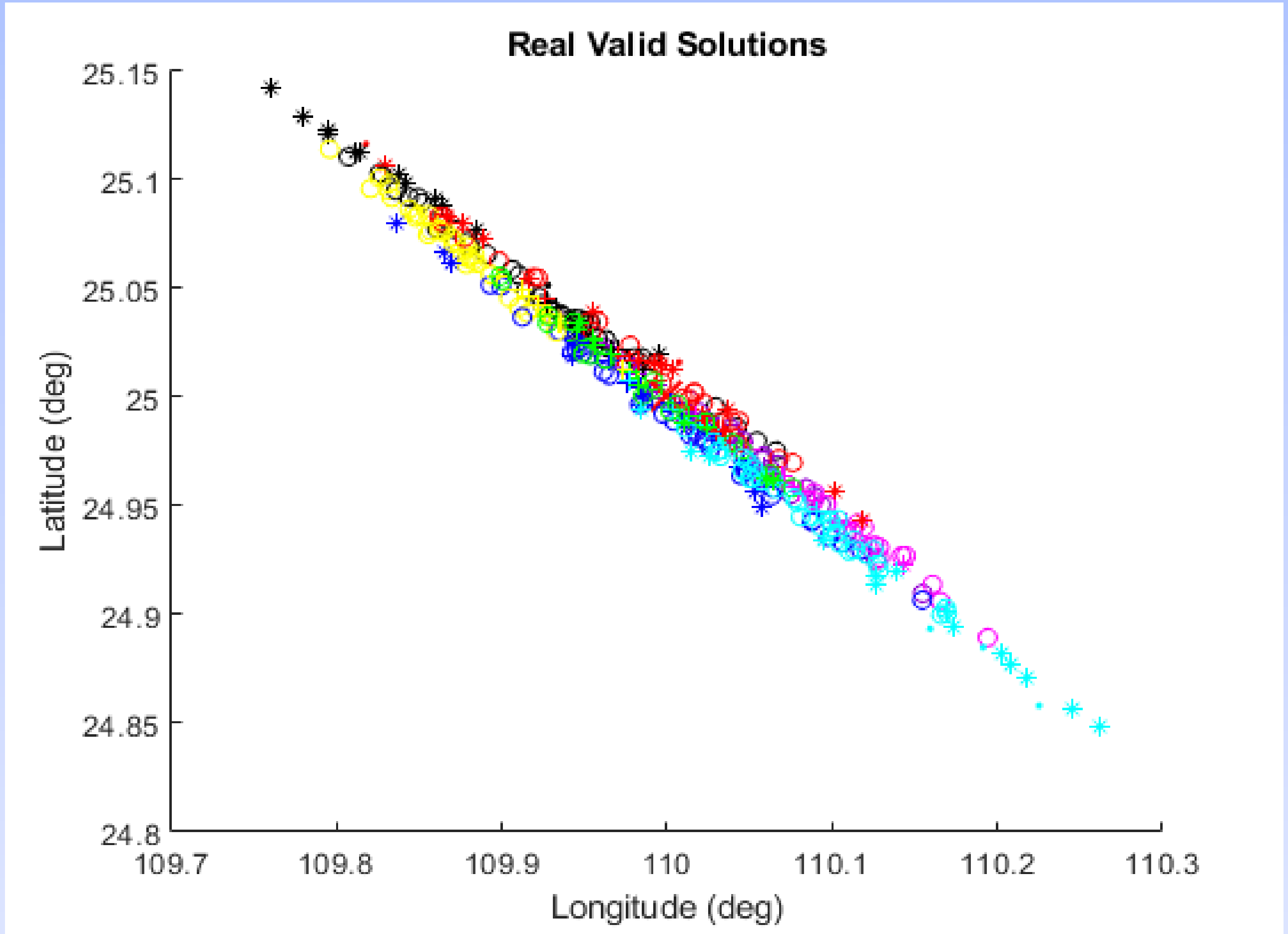


ROOT LOCUS RESULTS

- **Scenario #2**
  - **Receiver #1: a = 7178 km, e = 0, i = 90°, θ = 65°**
  - **Receiver #2: a = 10645 km, e = 0, i = 74.3°, θ = 325°**
  - **Emitter at 25°N latitude, 110°E longitude**

1° latitude error ≈ 110km error

1° longitude error ≈ 100km error



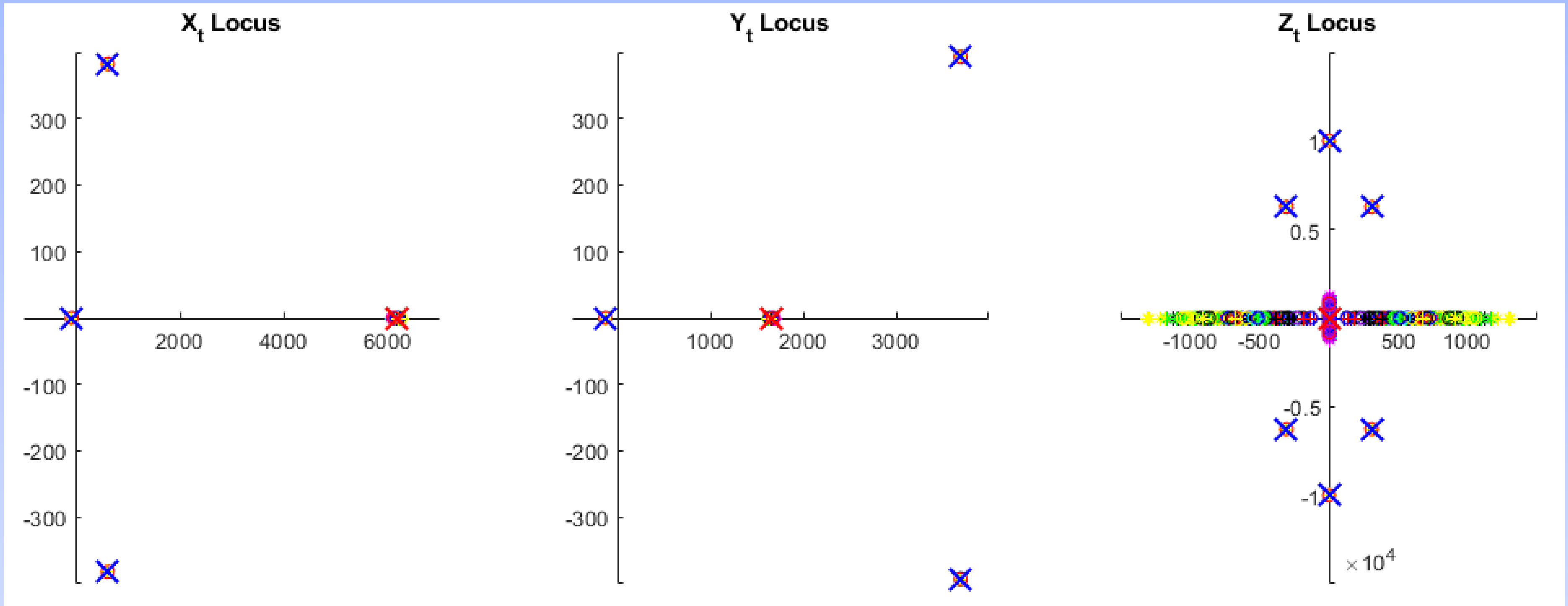
ROOT LOCUS RESULTS

- **Scenarios displayed so far were arbitrarily chosen**
  - **No basis to expect any particular solution error or ambiguity a priori**

- **We now investigate a suite of scenarios where we anticipate solution ambiguity:**
  - **Receivers co-orbital on an equatorial orbit (4000km altitude), 30° apart**
  - **Emitter first placed at the equator, then at increasing N latitude**
  - **TDOA Gaussian error level of σ = 3E-7 sec (~100m range difference)**

ROOT LOCUS RESULTS

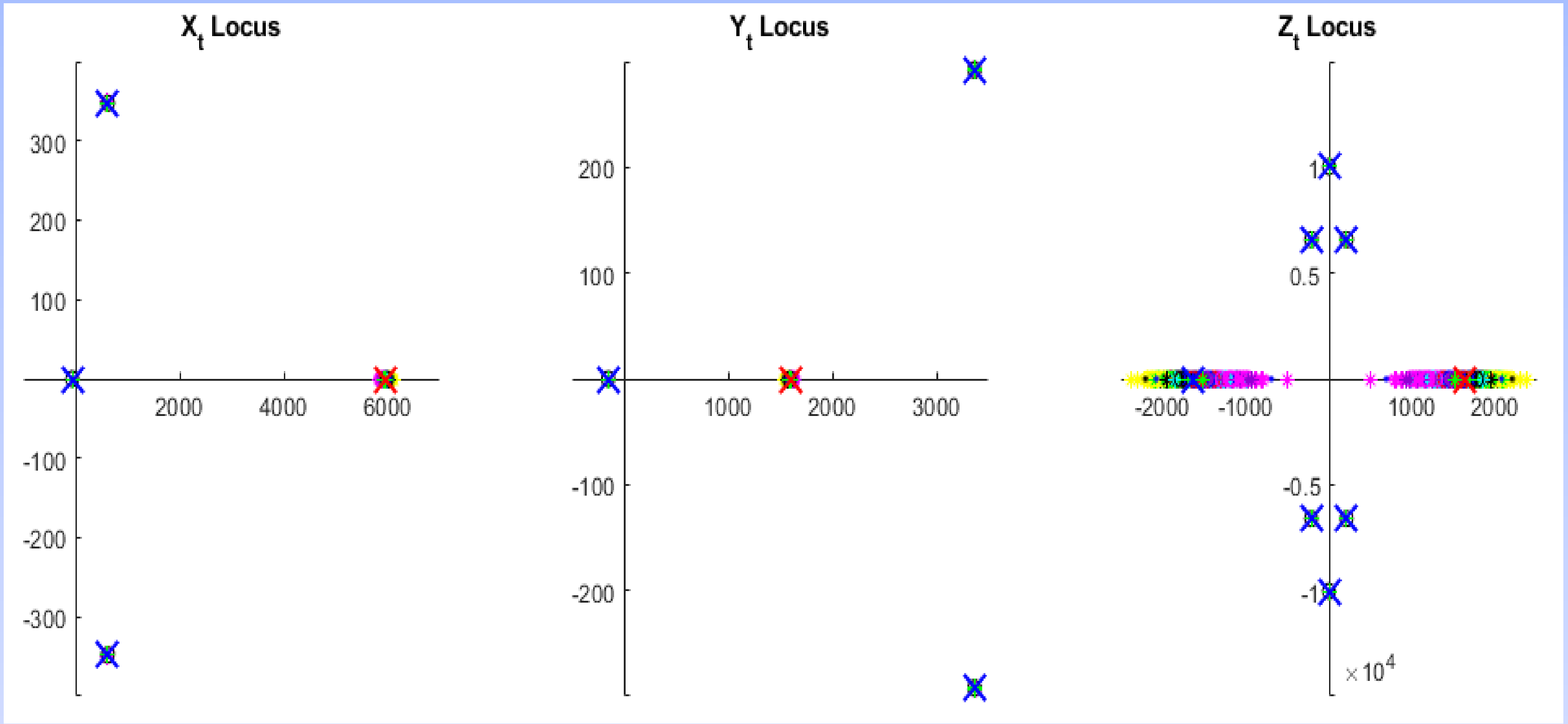
- **Scenario #1a**
  - **Emitter at 0°N latitude, 15°E longitude**



- **Roots display most susceptibility to error in the z (North/South) component**
- **Double roots at z = 0 migrate both left & right along real axis → symmetric ambiguity**

ROOT LOCUS RESULTS

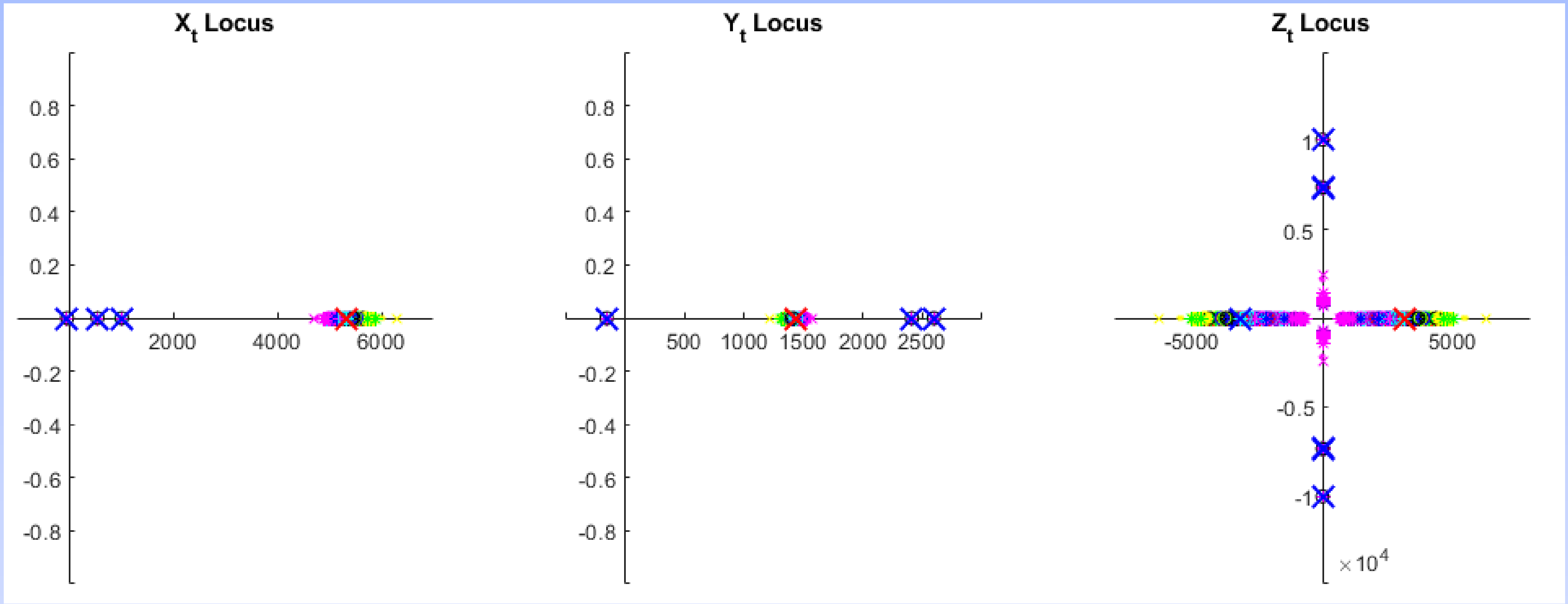
- Scenario #2a
  - Emitter at 20°N latitude, 15°E longitude



- Symmetric roots at real nonzero z values
- “Point clouds” develop symmetrically on both sides of real axis → symmetric ambiguity

ROOT LOCUS RESULTS

- **Scenario #3a**
  - **Emitter at 40°N latitude, 15°E longitude**



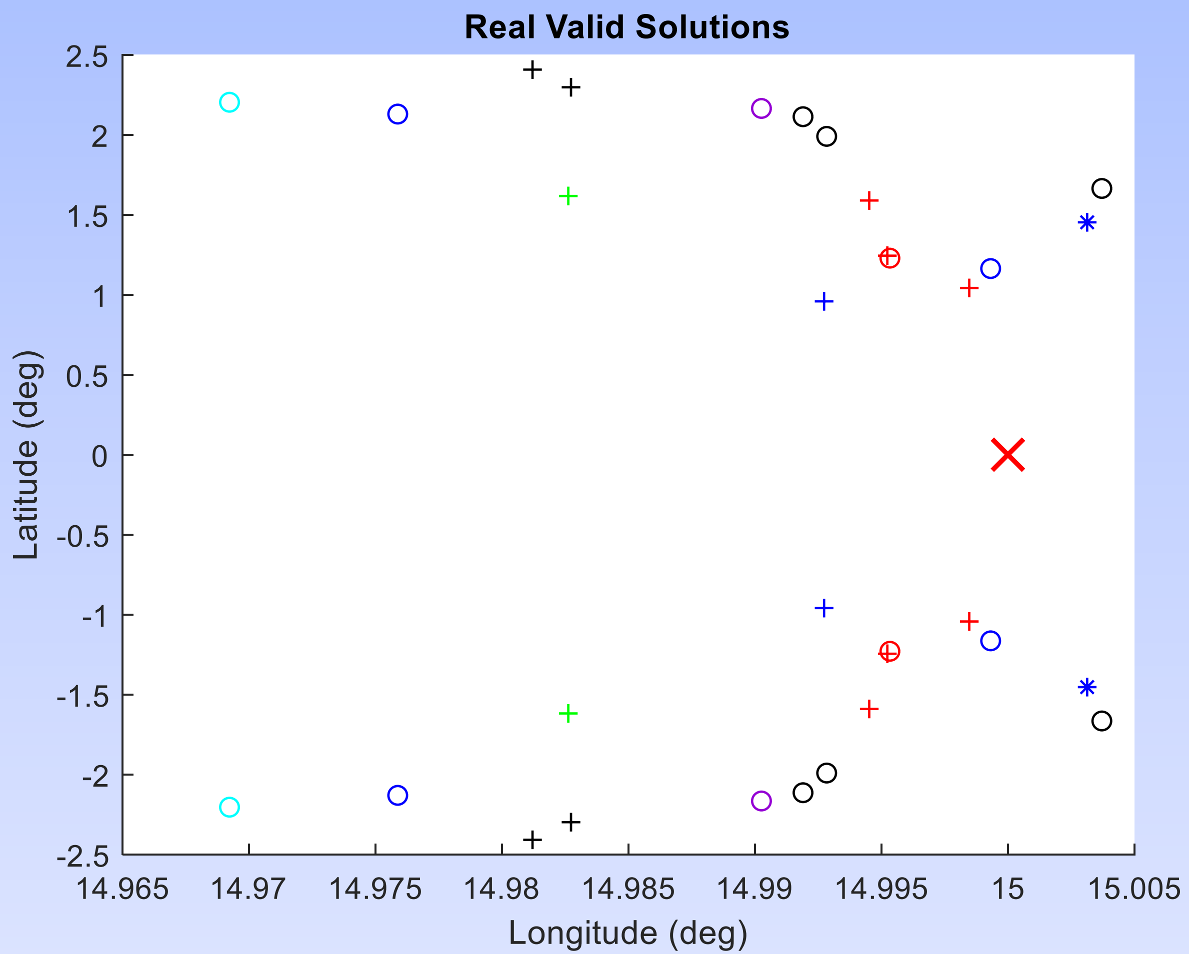
- **Significant variation with TDOA error in all 3 components**
- **All x & y roots are real**
- **"Point clouds" develop symmetrically on both sides of real axis & (to some extent) along imaginary axis → NO feasible solution**

ROOT LOCUS RESULTS

- Scenario #1a
  - Emitter at 0°N latitude, 15°E longitude

1° latitude error ≈ 110km error

1° longitude error ≈ 110km error



- Significant (symmetric) latitude error
- Very small longitude error (~ a few km)

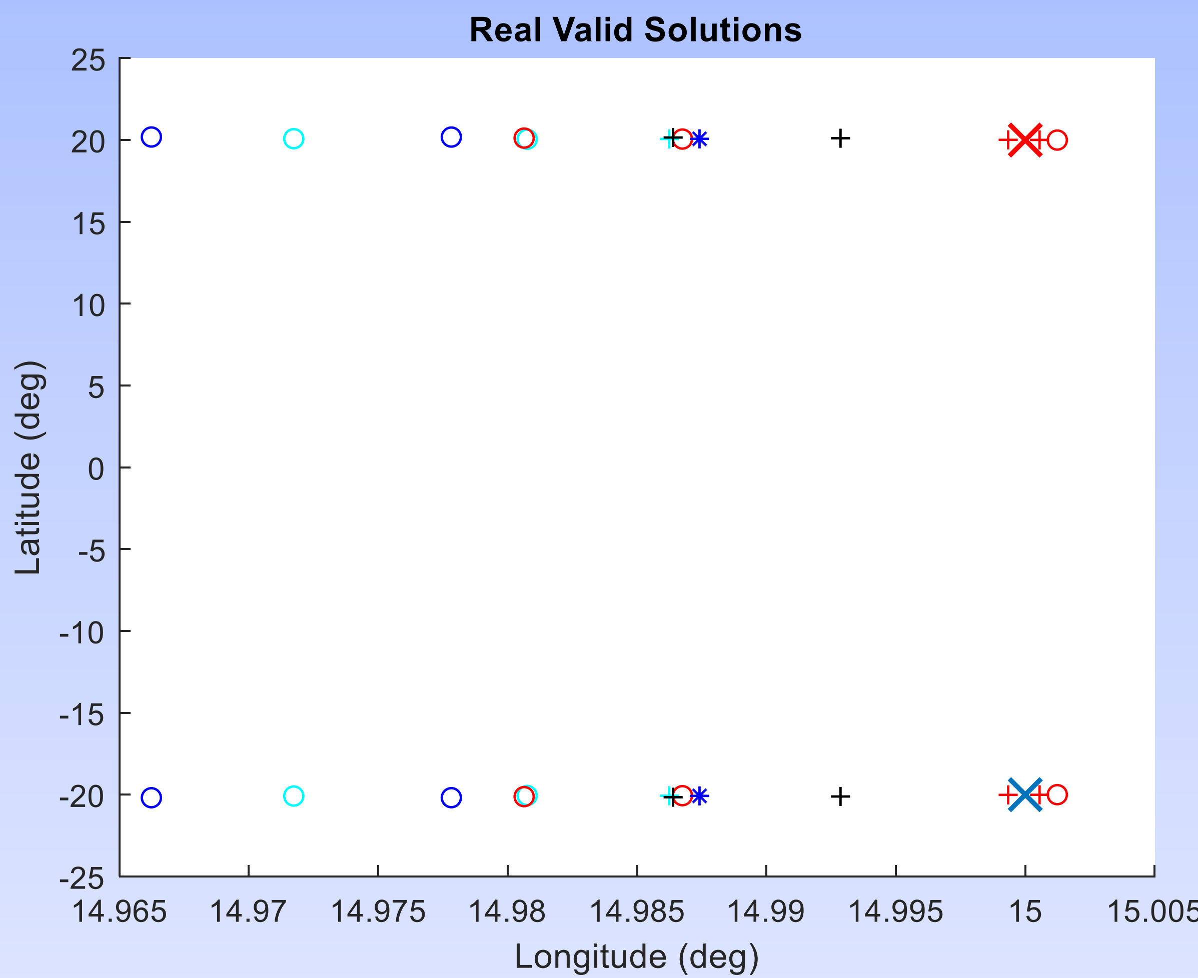
ROOT LOCUS RESULTS

- Scenario #2a
  - Emitter at 20°N latitude, 15°E longitude

1° latitude error ≈ 110km error

1° longitude error ≈ 110km error

- Very small latitude & longitude error (~ a few km)
- Symmetric latitude error

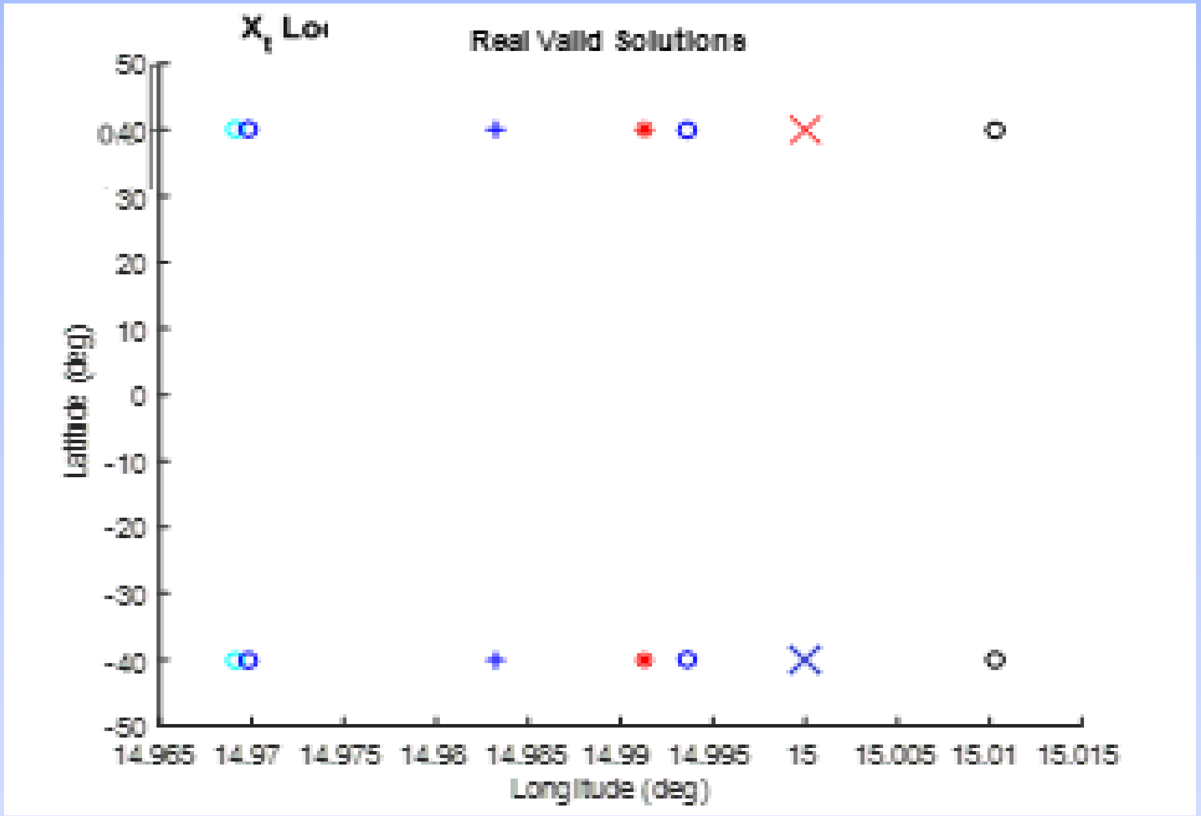


ROOT LOCUS RESULTS

- **Scenario #3a**
  - **Emitter at 40°N latitude, 15°E longitude**

1° latitude error ≈ 110km error

1° longitude error ≈ 110km error



- **Very small latitude & longitude error (~ a few km)**
- **Symmetric latitude error**

CONCLUSIONS

- **Introduced the root locus concept as a way to interpret behavior of solutions for TDOA geolocation from space, with Gaussian error → "system of polynomials" formulation**

- **Effective means of mapping solution error to TDOA error**

- **In many scenarios, point cloud pertaining to an "incorrect" root may interact with the point cloud pertaining to the "correct" root, resulting in solution ambiguity (or NO solution)**

- **For "intentionally" ambiguous scenarios with both receivers on equatorial orbits, insight gained into the nature of the ambiguity**

ADDITIONAL FY19 EFFORT

- Derived (or re-derived) the TDOA & FDOA equations accounting for varous error sources
  - Clock bias
  - Ionospheric & tropospheric path delay (or frequency shift)
  - Relativistic doppler shift

- Results in more unwieldy polynomials → 4^{th}- to 8^{th}-order → solvable with sufficient accuracy?

FUTURE WORK

- **Explore further scenarios, with more incremental changes to various attributes**
  - **Emitter latitude**
  - **Receiver orbit conditions**
  - **TDOA σ error level**

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1
2
3
etc

MEASUREMENT TYPES:
TDOA
FDOA/FROA
SNR/phase (AOA)

INITIAL VS PRECISE LOCALIZATION:
ITL
PTL

RECEIVER REGIME:
Terrestrial
LEO
Beyond LEO (e.g. MEO/GEO)
Hybrid (terrestrial & spaceborne)

EMITTER REGIME:
Terrestrial (“geolocation”)
LEO
Beyond LEO (e.g. MEO/GEO)

EXPRESSION OF MEASUREMENT ERROR:
None
Gaussian/unmodeled
Detailed models