

Null controllability properties for multi-D fractional heat equations

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- 1 Controllability of the classical heat equation
- 2 Control theory of nonlocal evolution equations
- 3 Recent controllability results for fractional evolution equations
- 4 Open problems

Outline

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What is controllability/observability?

- 1 Given a **dynamical system (ODE or PDE)** in a time interval $(0, T)$, **initial and final states**, **controllability** consists to determine whether there is a **control driving the initial data to the given final targets in finite time T** .
- 2 This is equivalent to the problem of **observability**. That is, the possibility of recovering full estimates on solutions of the uncontrolled adjoint system in terms of partial measurements done on the control region.
- 3 **Observability/controllability** properties depend in a very sensitive way on the class of PDEs or ODEs under consideration. In particular, heat and wave equations behave in a significantly different way, **because of their different behavior with respect to time reversal**.

Few earlier main contributions in the field

- ① **Fattorini and Russell (1971)**: They introduced the use of biorthogonal sequences to control $(1 - D)$ -heat equations.
- ② **J.-L. Lions (1988)**: Controllability can be reduced to observability of the associated adjoint system (as we mentioned above).
 - This has been used to prove several important unique continuation results.
 - This also helped to develop techniques that are nowadays the key tools in the field: **Multipliers, Microlocal Analysis, Carleman Estimates, Non-harmonic Fourier series and Numerical Methods** to study these issues.

Controllability of the classical heat equation

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set with boundary $\partial\Omega$.

1 Interior controllability:

$$\begin{cases} u_t(t, x) - \Delta u(t, x) = f\chi_\omega & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

The control f is localized in a non-empty subset $\omega \subset \Omega$.

2 Boundary controllability:

$$\begin{cases} u_t(t, x) - \Delta u(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ u = g\chi_\omega & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.2)$$

The control g is localized on a non-empty subset $\omega \subset \partial\Omega$.

The set of reachable states

- ① For the interior control the set of reachable states is given by

$$\mathcal{R}(u_0, T) = \{u(\cdot, T) : f \in L^2((0, T) \times \omega)\}.$$

- ② For the boundary control the set of reachable states is given by

$$\mathcal{R}(u_0, T) = \{u(\cdot, T) : g \in L^2((0, T) \times \omega)\}.$$

Notice that for the reachable set, we consider controls for which the heat equation has finite energy solutions. For the boundary control, it may happen that $g \in L^2((0, T) \times \omega)$ is not enough to have finite energy solutions. In that case, we may consider controls in an appropriate function space, for example, $g \in L^2((0, T); H^{1/2}(\omega))$ will work.

The three notions of controllability

- (1.1) or (1.2) is called **approximately controllable** in time T if

$$\mathcal{R}(u_0, T) \text{ is dense in } L^2(\Omega), \forall u_0 \in L^2(\Omega).$$

- (1.1) or (1.2) is said to be **exactly controllable** in time T if

$$\mathcal{R}(u_0, T) = L^2(\Omega), \forall u_0 \in L^2(\Omega).$$

- (1.1) or (1.2) is said to be **null controllable** in time T if

$$0 \in \mathcal{R}(u_0, T), \forall u_0 \in L^2(\Omega).$$

Since the system (1.1) or (1.2) is linear and reversible in time, **null and exact controllabilities** are equivalent notions and they trivially imply the **approximate controllability**, but the converse is not true in general.

Interior null/exact controllability: Characterization

Consider the interior null/exact controllability of the heat equation (1.1).
Then the following assertions are equivalent.

- ① The system is null/exact controllable in time T .
- ② The following observability inequality holds:

$$\|v(0, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega} |v|^2 dx dt, \quad (1.3)$$

where v is the unique solution of the associated adjoint system:

$$\begin{cases} -v_t(t, x) - \Delta v(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(T, \cdot) = v_T & \text{in } \Omega. \end{cases} \quad (1.4)$$

Boundary null/exact controllability: Characterization

Consider the boundary null/exact controllability of the heat equation (1.2). **Then the following assertions are equivalent.**

- ① The system is null/exact controllable in time T .
- ② The following observability inequality holds:

$$\|w(0, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega} |\partial_{\nu} w|^2 dx dt, \quad (1.5)$$

where w is the unique solution **of the associated adjoint system:**

$$\begin{cases} -w_t(t, x) - \Delta w(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{on } (0, T) \times \partial\Omega, \\ w(T, \cdot) = w_T & \text{in } \Omega. \end{cases} \quad (1.6)$$

What is known about the classical heat equation?

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set.

- ① **(Interior Control):** The heat equation (1.1) is null/exact controllable in any time $T > 0$ and $f \in L^2((0, T) \times \Omega)$.
- ② **(Boundary Control):** The heat equation (1.2) is null/exact controllable in any time $T > 0$ and $g \in L^2((0, T), H^{\frac{1}{2}}(\omega))$.

What is needed in the proofs?

- If Ω is such that the eigenvalues and eigenfunctions of Δ are known explicitly, then one usually uses **Ingham type inequalities and the theory of biorthogonal sequences**. This is the case of very special domains, such as, intervals in \mathbb{R} .
- Otherwise, one needs to use the so called **Carleman estimates**.

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What is a nonlocal PDE?

- ① A non-local PDE is a differential equation involving integral terms or pseudo-differential operators of fractional orders.
- ② What do they model?
 - They model anomalous phenomena (transport and diffusion).
 - **In elasticity**: The Peierls-Nabarro equation arises in the description of phenomena of dislocation dynamics in crystals.
 - **In material sciences**: Nonlocal models take into account that the stress in a point may depend on the strains in the neighboring points.
 - **In population dynamics**: Nonlocal reaction-diffusion equations are models that analyze the interplay between food-dependent growth and size-dependent mortality in predator-prey systems.
 - **Other examples**: Image processing, laser design, porous media flow and wave propagation in heterogeneous high contrast media.

A typical example: The fractional Laplace operator

For a measurable function u and $\varepsilon > 0$ we let

$$(-\Delta)_\varepsilon^s u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N: |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

The fractional Laplacian $(-\Delta)^s u$ of u is defined for $x \in \mathbb{R}^N$ by,

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x)$$

provided that the limit exists.

Interior control of the fractional heat equation (FHE)

Interior controllability ($0 < s < 1$): Let $f \in L^2((0, T) \times \Omega)$. The system

$$\begin{cases} u_t(t, x) + (-\Delta)^s u(t, x) = f \chi_\omega & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (2.1)$$

is null/exact controllable in time $T > 0$ if and only if

$$\|v(0, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega |v|^2 dx dt, \quad (2.2)$$

where v is the unique solution of the **associated dual system**:

$$\begin{cases} -v_t(t, x) + (-\Delta)^s v(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ v(T, \cdot) = v_T & \text{in } \Omega. \end{cases}$$

Exterior control of the fractional heat equation (FHE)

Exterior controllability ($0 < s < 1$): Let $g \in L^2((0, T), H^s(\mathcal{O}))$. Then

$$\begin{cases} u_t(t, x) + (-\Delta)^s u(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ u = g\chi_{\mathcal{O}} & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (2.3)$$

is null/exact controllable if and only if

$$\|w(0, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s w|^2 dx dt, \quad (2.4)$$

where w is the unique solution of the **associated dual system**:

$$\begin{cases} -w_t(t, x) + (-\Delta)^s w(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ w(T, \cdot) = w_T & \text{in } \Omega. \end{cases}$$

What is so far known about the fractional heat equation?

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set.

- ① (Interior or Exterior Control): $N \geq 1$: Approximately controllable.
- ② (Interior or exterior Control) $N = 1$: The system is null/exact controllable in time $T > 0 \iff \frac{1}{2} < s < 1$.

All these results have been obtained by my team in several papers.

Tools needed!!!

- ① Properties of eigenvalues and eigenfunctions, Ingham inequalities and biorthogonal sequences. Only possible if $N = 1$!!
- ② Otherwise, one needs to use Carleman estimates.
- ③ Unfortunately, there are no Carleman estimates available for $(-\Delta)^s$ on bounded open subsets of \mathbb{R}^N .

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The multi-D fractional heat equation

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set. Recall that the system

$$\begin{cases} u_t(t, x) + (-\Delta)^s u(t, x) = f \chi_\omega & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (3.1)$$

is **null/exact controllability in time $T > 0$** if and only if the solution v of the dual system satisfies

$$\|v(0, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega |v|^2 dx dt. \quad (3.2)$$

- ① To prove (3.2) we need Carleman estimates (**not available**).
- ② We do not know how to prove (3.2) without using Carleman estimates. For that reason, we will proceed differently.

Step 1: Analysis of the adjoint fractional wave equation

- ① Consider the following adjoint fractional wave equation:

$$\begin{cases} p_{tt} + (-\Delta)^s p = 0 & \text{in } \Omega \times (0, T), \\ p = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ p(\cdot, 0) = p_0, \quad p_t(\cdot, 0) = p_1 & \text{in } \Omega. \end{cases} \quad (3.3)$$

- ② The energy of solutions is conserved along time and is given by:

$$E_s(t) := \frac{1}{2} \int_{\Omega} |p_t|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla^s p|^2 dx,$$

where we have set

$$\int_{\Omega} |\nabla^s p|^2 dx := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|p(x) - p(y)|^2}{|x - y|^{N+2s}} dx dy.$$

Solutions in $\mathcal{H}_J := \text{span}\{\phi_1, \dots, \phi_J\}$

Let (ϕ_n) = normalized eigenfunctions of $(-\Delta)^s$ with eigenvalues (λ_n) .

- ① We consider solutions of the wave equation (3.3) in the form

$$p(x, t) = \sum_{j=1}^J p_j(t) \phi_j(x).$$

- ② Using the Pohozaev identity for $(-\Delta)^s$, we can establish the identity

$$\begin{aligned} sTE_s(0) + \int_{\Omega} p_t \left(x \cdot \nabla p + \frac{N-s}{2} p \right) dx \Big|_{t=0}^{t=T} \\ = \frac{\Gamma(1+s)^2}{2} \int_0^T \int_{\partial\Omega} (x \cdot \nu) \left| \frac{p}{\rho^s} \right|^2 d\sigma dt, \end{aligned} \quad (3.4)$$

where $\rho(x) = \text{dist}(x, \partial\Omega)$, $x \in \Omega$.

Why do we consider solutions in $\mathcal{H}_J := \text{span}\{\phi_1, \dots, \phi_J\}$?

- 1 In order to obtain a finite-time observability inequality for a solution (p, p_t) of the fractional wave equation, one needs to get an upper bound on the term

$$\int_{\Omega} p_t \left(x \cdot \nabla p + \frac{N-s}{2} p \right) dx \Big|_{t=0}^{t=T} \quad (3.5)$$

in terms of the energy $E_s(0)$. But this is impossible when $0 < s < 1$ due to the presence of the term involving ∇p .

- 2 For this reason, we have restricted our attention to a particular class of solutions $p \in \mathcal{H}_J$, for which boundary observability results may be obtained in finite time.

Fractional wave equation: The observation domain.

From now on, our observation domain ω will be as follows:

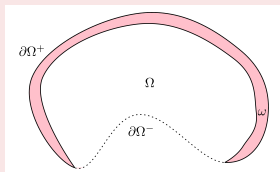


Figure: Domain Ω and control region ω .

Here,

$$\partial\Omega^+ := \left\{ x \in \partial\Omega : (x \cdot \nu) > 0 \right\} \text{ and } \partial\Omega^- = \partial\Omega \setminus \partial\Omega^+.$$

Fractional wave equation: Partial boundary observability

Let $1/4 \leq s < 1$ and $R := \max_{x \in \Omega} |x|$. Then $\exists C(s) > 0$ such that

$$E_s(0) \leq \frac{C(s)}{T - T_0} \int_0^T \int_{\partial\Omega^+} (x \cdot \nu) \left| \frac{p}{\rho^s} \right|^2 d\sigma dt, \quad \forall T > T_0 > 0. \quad (3.6)$$

❶ If $1/2 < s < 1$, then

$$T_0 := \frac{R}{s} \left(1 + C(N, s, \Omega, \varepsilon) \lambda_J^{\frac{1-s}{s-\varepsilon}} \right), \quad \forall 0 < \varepsilon < s.$$

❷ If $1/4 \leq s \leq 1/2$, then

$$T_0 := \begin{cases} \frac{R^2}{s} \lambda_J^2 + C(N, s, \Omega) \lambda_J^{\frac{2}{s}-5} + C(N, s, \Omega) & \text{if } 1/4 \leq s < 2/5, \\ \frac{R^2}{s} \lambda_J^2 + C(N, s, \Omega) & \text{if } 2/5 \leq s < 1/2, \\ 2R \lambda_J^2 + C(N, \Omega) & \text{if } s = 1/2. \end{cases}$$

Fractional wave equation: Partial interior observability/controllability

Using (3.6), we get the following. Let $1/4 \leq s < 1$ and $T > T_0$.

- ① There is a constant $C = C(N, s, \Omega, T) > 0$ such that

$$E_s(0) \leq C \int_0^T \int_{\omega} |p_t|^2 dx dt. \quad (3.7)$$

- ② $\exists f \in L^2(\omega \times (0, T))$ such that the solution (y, y_t) of

$$\begin{cases} y_{tt} + (-\Delta)^s y = f \chi_{\omega} & \text{in } \Omega \times (0, T), \\ y = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ y(\cdot, 0) = y_0, \quad y_t(\cdot, 0) = y_1 & \text{in } \Omega. \end{cases}$$

satisfies $\Pi_{\mathcal{H}_J} y(\cdot, T) = \Pi_{\mathcal{H}_J} y_t(\cdot, T) = 0$ a.e. in Ω .

- ③ Even if our result is partial, it is the first available result regarding the null controllability properties of the fractional wave equation.

Step 2: Transmutation techniques

- 1 We want to transfer the observability inequality (3.7) for low-frequency solutions of the **fractional wave equation** into an observability inequality for low-frequency solutions of the **fractional heat equation**.
- 2 We will use the so-called **transmutation techniques**. The techniques provide the inversion of the so called Kannai formula. In fact, Kannai's formula allows to transform **solutions of a wave equation** into **solutions of a heat equation**.
- 3 We have established the corresponding inversion formula. This very important result will be exploited to obtain **some partial null controllability results for the fractional heat equation**.

Fractional heat equation: Partial observability/controllability

Let $1/4 \leq s < 1$, ω as in Figure 1 and $v \in \mathcal{H}_J$ the solution of:

$$\begin{cases} -v_t + (-\Delta)^s v = 0 & \text{in } \Omega \times (0, T), \\ v = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v(\cdot, T) = v_T & \text{in } \Omega. \end{cases}$$

① $\exists C > 0$ such that the following observability inequality holds:

$$\|v(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega} |v|^2 dx dt. \quad (3.8)$$

② As a consequence of (3.8), the solution u of the fractional heat equation satisfies $\Pi_{\mathcal{K}_J} u(\cdot, T) = 0$ a.e. in Ω .

Fractional heat equation: Full null controllability

Finally, we have been able to prove the following important result.

Theorem (Biccari, W. & Zuazua (2020))

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded smooth open set, $\omega \subset \Omega$ as in Figure 1 and $1/2 < s < 1$. Then, for any time $T > 0$ and $u_0 \in L^2(\Omega)$, $\exists f \in L^2(\omega \times (0, T))$ such that the unique finite energy solution u of

$$\begin{cases} u_t(t, x) + (-\Delta)^s u(t, x) = f \chi_\omega & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (3.9)$$

satisfies $u(T, \cdot) = 0$ a.e. in Ω . That is, (3.9) is null/exactly controllable

Let us notice that the obtained result is sharp in the sense that we know that even in dimension $N = 1$, the fractional heat equation is not null controllable if $0 < s \leq 1/2$.

Main ideas of the proof

We use an iteration strategy (having in mind the above steps for the FWE and the FHE)

1. A partition of $[0, T] = \bigcup_{J \in \mathbb{N}} [a_J, a_{J+1}]$, $a_0 := 0$, $a_{J+1} := a_J + 2T_J$, $T_J := \gamma 2^{-J\varepsilon}$ and $\gamma > 0$ chosen such that $2 \sum_{J \in \mathbb{N}} T_J = T$.

2. Construction of the control function f :

- If $t \in (a_J, a_J + T_J]$, we apply a control such that $\Pi_{\mathcal{K}_J} u(\cdot, a_J + T_J) = 0$ a.e. in Ω . Then ($0 < \varepsilon < s$),

$$\|u(a_J + T_J, \cdot)\|_{L^2(\Omega)} \leq \left(1 + C_1 \exp\left(\frac{C_2}{T} 2^{J \frac{2-2s}{s-\varepsilon}}\right)\right) \|u(a_J, \cdot)\|_{L^2(\Omega)}.$$

- If $t \in (a_J + T_J, a_{J+1}]$, we apply no control. We take advantage of the exponential decay of the solution to get

$$\|u(a_J + 1, \cdot)\|_{L^2(\Omega)} \leq \exp(-2^{2J} T_J) \|u(a_J + T_J, \cdot)\|_{L^2(\Omega)}.$$

Main ideas of the proof: Continuation

3. Combining the above two estimates we get that ($0 < \varepsilon < s$)

$$\|u(a_J + 1, \cdot)\| \leq C_1 \exp \left(\sum_{\ell=0}^J \left(\frac{C_2}{T} 2^{\ell \frac{2-2s}{s-\varepsilon}} - \gamma 2^{\ell(2-\varepsilon)} \right) \right) \|u(a_J, \cdot)\|.$$

If $\frac{1+\varepsilon-\varepsilon^2}{2-\varepsilon} =: s_0 \leq s < 1$, then the above sum diverges to $-\infty$ as $J \rightarrow +\infty$. Thus $u(\cdot, T) = 0$.

4. We show that $\|f\|_{L^2(\omega \times (0, T))}^2 = \sum_{J \in \mathbb{N}} \|f\|_{L^2(\omega \times (a_J, a_J + T_J))}^2 < \infty$.
5. Finally, since

$$s_0 := \frac{1 + \varepsilon - \varepsilon^2}{2 - \varepsilon} = \frac{1}{2} + \frac{3\varepsilon - 2\varepsilon^2}{4 - 2\varepsilon} = \frac{1}{2} + \mathcal{O}(\varepsilon),$$

we have the full null-controllable in any time $T > 0$ provided that $s \in [\frac{1}{2} + \mathcal{O}(\varepsilon), 1)$. That is, for every $\frac{1}{2} < s < 1$.

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Some selected interesting open problems

- ① Carleman estimates for $(-\Delta)^s$ on bounded domains.
- ② Multi-D controllability results of the fractional heat equation when the observation domain ω is not as in Figure 1.
- ③ Multi-D exterior controllability results of the fractional heat equation.
- ④ Controllability results of the fractional heat equation when one replaces the Dirichlet exterior condition with the nonlocal Neumann or Robin exterior condition. For this problem, the difficulty is that, there is still no Pohozaev identity available for the nonlocal Neumann or Robin exterior condition.

THANKS!!!!

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