Tracking Signal Subspaces in Multistatic Radar Systems

AFOSR Electromagnetics Program Review

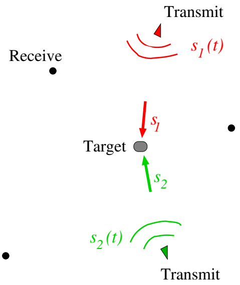
Lauren Crider and Doug Cochran
Arizona State University

Collaborators: Stephen Howard, Songsri Sirianunpiboon

6 January 2021

Signal Rank

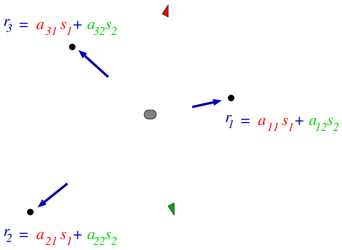
Active scenario with $K = 2$ transmitters and $M = 3$ receivers



- The illuminating signals s_1 and s_2 are linearly independent
- Each is scattered non-isotropically by the target
- Alternatively, the target may be a non-isotropic active emitter (e.g., in passive sonar or electronic surveillance)

Signal Subspace in Multi-receiver Processing

Scenario with $K = 2$ linearly independent transmitters and $M = 3$ receivers



- In the absence of noise, each receiver collects a different linear combination of the illuminating signals
- The matrix of collected data thus has the form

$$R = [a_{11}s_1 + a_{12}s_2 \;\; a_{21}s_1 + a_{22}s_2 \;\; a_{31}s_1 + a_{32}s_2]$$

- Its rank ($K = 2$) is reflected in its singular values; with noisy channels, standard estimators of signal rank are based on the spectrum of $R^\dagger R$

Grassmannians and Projective Space

- With $K \leq N$, the collection of all K-dimensional subspaces of an N-dimensional vector space V forms the Grassmannian $G(K, N)$
  - $G(K, N)$ is a Riemannian manifold of dimension $K(N - K)$
  - It is covered, except for a set of zero Haar measure, by one coordinate chart
  - An integral over $G(K, N)$ can be calculated by integrating over a single chart
- There is a one-to-one correspondence between K-dimensional subspaces of V and points on $G(K, N)$
  - Choosing a K-dimensional subspace of V "at random" supposes a probability law on $G(K, N)$
  - The probability of a collection of K-dimensional subspaces V can be obtained by an integral on $G(K, N)$
- In some of what follows, we focus on the important special case of rank-one signals
- $G(1, N)$ is called projective space, denoted $\mathbb{P}^{N-1}$

Exploiting Signal Subspaces

- Recall the "matched filter" detection statistic for a known signal S in additive ZMWGN projects the data vector X into the one-dimensional subspace spanned by S
- Higher-dimensional signal subspaces play similar roles in the solution of multi-channel detection problems
  - If it is known *a priori*, it may be exploited directly
  - If only its dimension is known, the subspace may be estimated from collected data
  - The dimension, if unknown, may also be estimated from data
- Bayesian subspace and rank estimators use prior distributions on $G(K, N)$

Signal subspaces are of fundamental importance in multi-sensor processing, and prior distributions on the Grassmannian are valuable in estimating them from sensor data. We propose iterative subspace estimator for dynamic scenarios – a Kalman-filter-like estimator on the Grassmannian for tracking temporally evolving signal subspaces.

Iterative estimation

Recall the Kalman filter on $\mathbb{R}^d$:

1. The (post-measurement) state at time $t-1$ is a normal distribution $\mathcal{N}[M_{t-1}, \Sigma_{t-1}]$
2. This state is propagated through a linear dynamical system with additive Gaussian noise to obtain a pre-measurement state at time T; due to linearity, this state is also Gaussian
3. A linear measurement with additive Gaussian noise is taken at time t
4. Bayes' rule is used to produce a post-measurement (posterior) distribution from the pre-measurement state and the measurement; the linear and Gaussian assumptions meant this state is also Gaussian, $\mathcal{N}[M_t, \Sigma_t]$

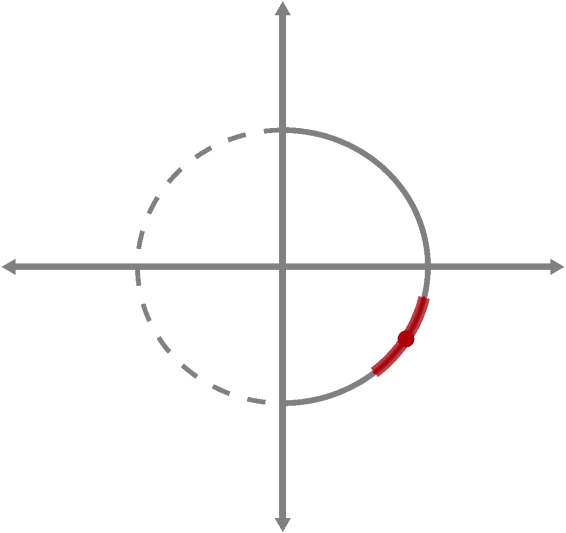
- From a Bayesian perspective, the Gaussian model is the maximum-entropy distribution on $\mathbb{R}^d$ with given covariance
- The linear-Gaussian dynamical and measurement models ensure everything remains Gaussian
- Thus propagation of the state can be reduced to equations in the mean and covariance (i.e., Riccati equations)

Iterative estimation on $G(K, N)$

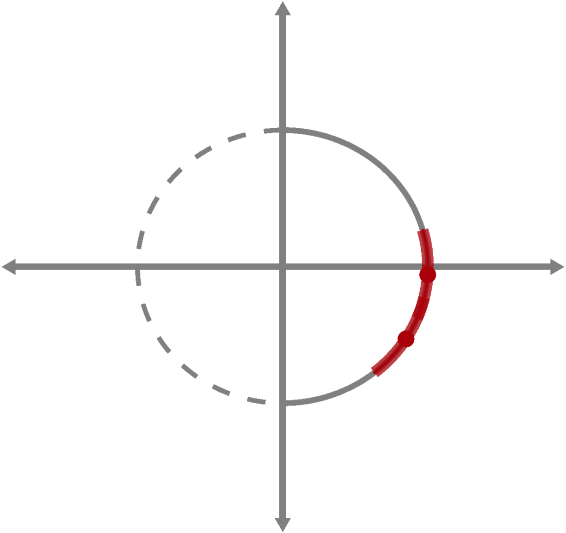
Elements of an iterative estimation algorithm for tracking:

- An invariant measure and integration method $G(K, N)$
  - ✓ Borrowed classical work by A. T. James and recent formulations by S. Howard
- A maximum-entropy family of probability distributions on $G(K, N)$
  - ✓ Completed for $G(1, N) = \mathbb{P}^{N-1}$
- Suitable dynamical models
  - ✓ Initially using constant-speed propagation on geodesics
  - ✓ Seeking to extend to models inspired by sensing scenarios
- Measurement model
  - ✓ Using standard multi-channel measurement model with Euclidean measurements mapped to $\mathbb{P}^N$
- Bayesian update of state from measurements
  - ✓ Achieved for $\mathbb{P}^N$; requires numerical integration

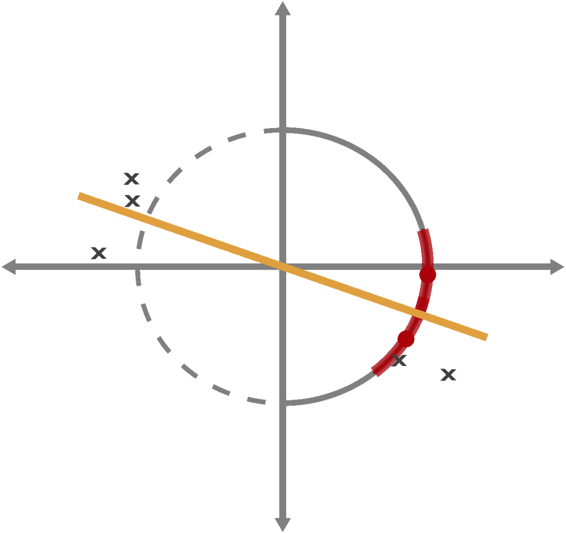
Iterative estimation on $\mathbb{RP}^1$



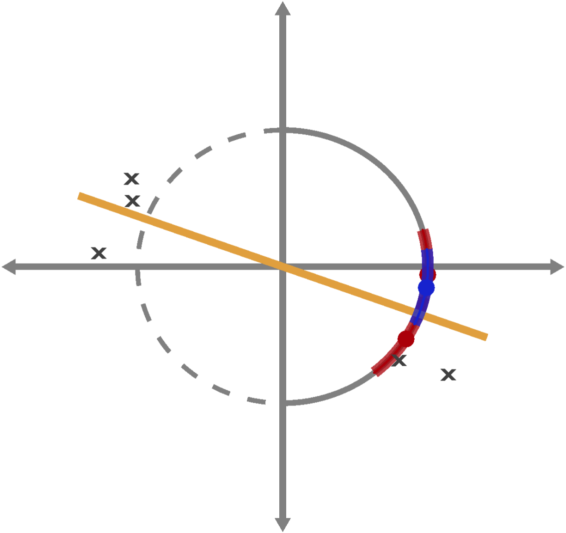
Iterative estimation on $\mathbb{RP}^1$



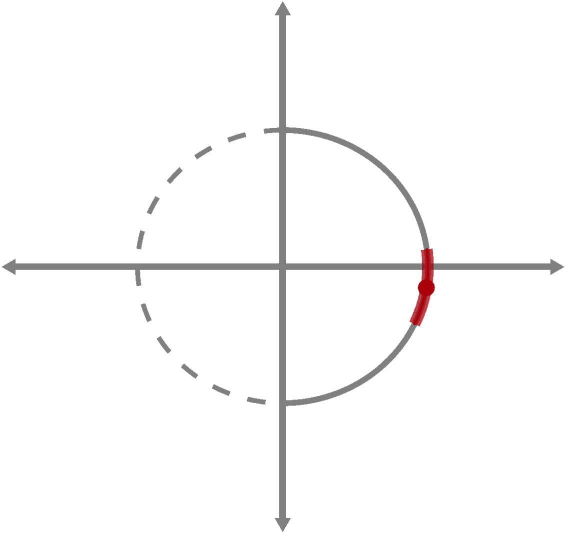
Iterative estimation on $\mathbb{RP}^1$



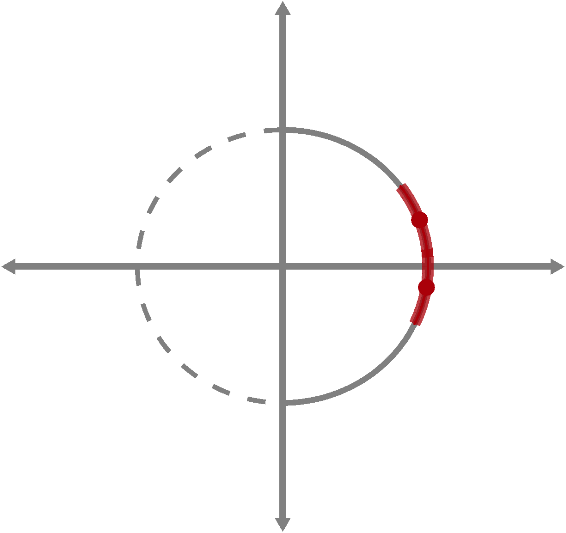
Iterative estimation on $\mathbb{RP}^1$



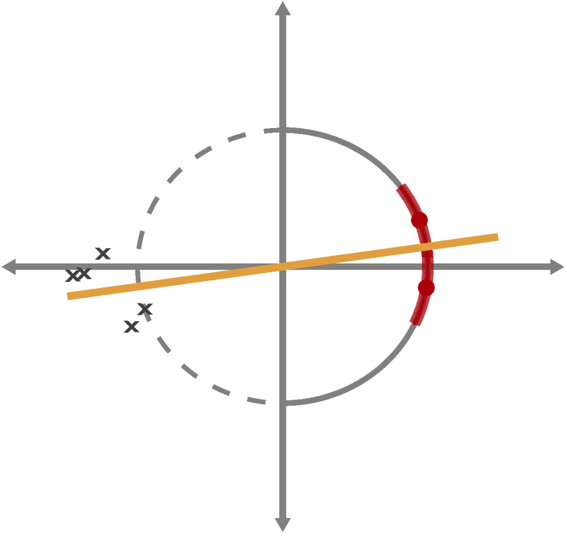
Iterative estimation on $\mathbb{RP}^1$



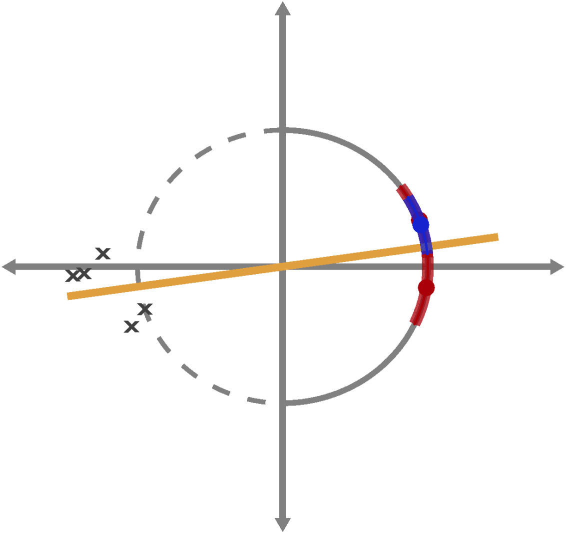
Iterative estimation on $\mathbb{RP}^1$



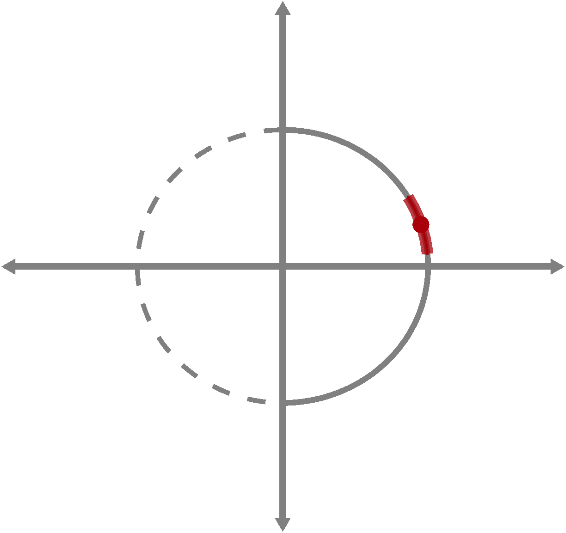
Iterative estimation on $\mathbb{RP}^1$



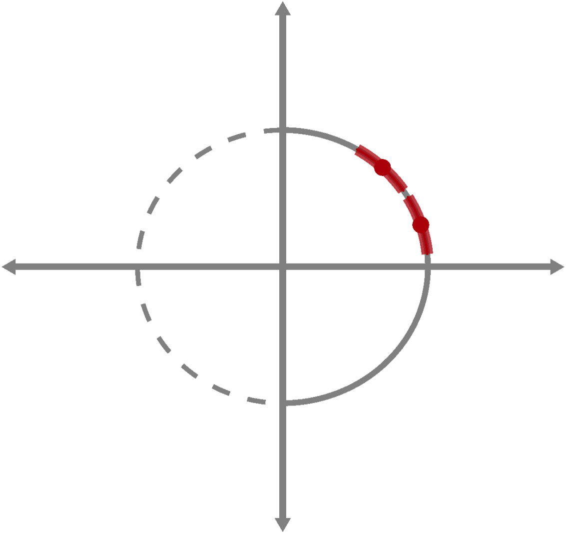
Iterative estimation on $\mathbb{RP}^1$



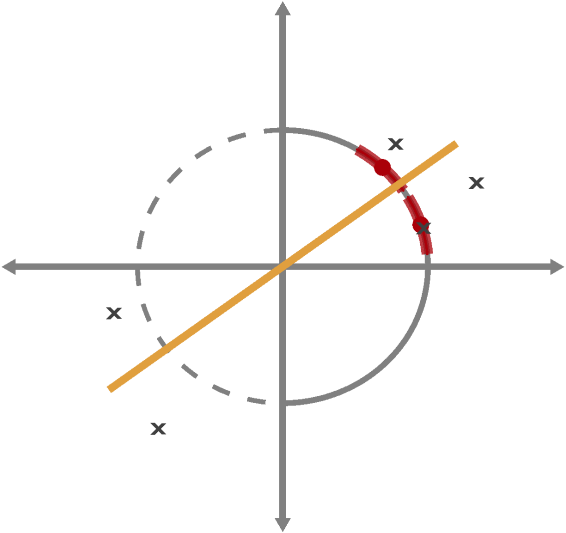
Iterative estimation on $\mathbb{RP}^1$



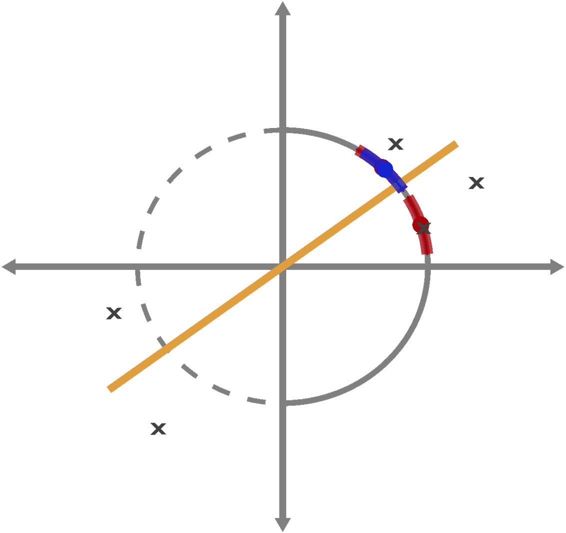
Iterative estimation on $\mathbb{RP}^1$



Iterative estimation on $\mathbb{RP}^1$



Iterative estimation on $\mathbb{RP}^1$



Iterative Estimation on $\mathbb{RP}^1$

- The maximum entropy distribution on $\mathbb{RP}^1$ (with respect to a fixed first moment) a bi-parametric distribution

$$p(x; \kappa, \mu) = \frac{1}{\pi \left[I_0(\kappa) + L_0(\kappa) \right]} e^{\kappa \mu^\top x}$$

- Our dynamical model assumes the state at time $t-1$ changes via an action of the special orthogonal group

$$\mathrm{s}_t = \mathrm{Q}_t \mathrm{s}_{t-1}, \;\; Q \in SO(2)$$

which ensures the prior distribution at time t remains of maximum entropy

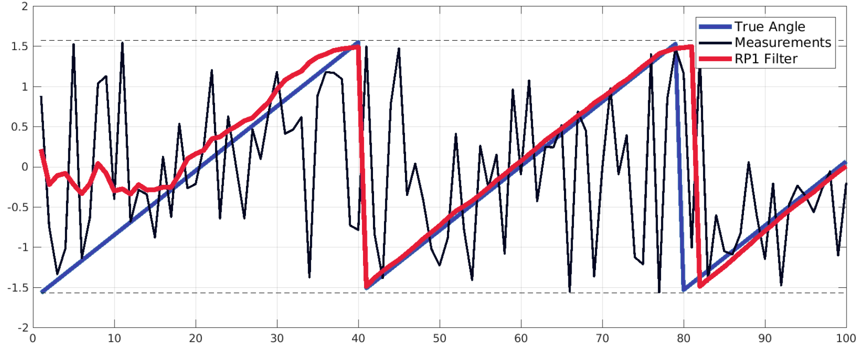
Iterative Estimation on $\mathbb{RP}^1$

- Every time epoch, we collect m measurements in $\mathbb{R}^2$ of a noisy signal whose subspace is defined by the state $\mathsf{s}_t \in \mathbb{RP}^1$

$$R_t = s_t a_t + \nu_t,$$

  where a_t is an m-dimensional (row) vector and ν_t is the additive ZMWGN whose variance is assumed known
- An update step requires
  1. Transformation of measurements from $\mathbb{R}^2$ to $\mathbb{RP}^1$
  2. Likelihood function of transformed measurements
  3. Marginalization over $\mathbb{RP}^1$ (Bayes' rule)

Iterative estimation on $\mathbb{RP}^1$: Results



Tracking on $\mathbb{RP}^1$ with measurements in $\mathbb{R}^2$, which are samples of a noisy signal whose subspace is defined by an element of $\mathbb{RP}^1$. Each measurement has its SNR set to ~ 0 dB.

Wrap-up

Motivated by the importance of subspace information in multi-sensor signal processing, we are developing an iterative estimator for tracking a subspace in a dynamic scenario

- Several mathematical results have been obtained, both for $G(K, N)$ and for the special case of $\mathbb{P}^N$ (i.e., for a one-dimensional subspace)
- These underpin a prototype tracker on $\mathbb{P}^N$
- Several challenges remain to devise an iterative estimator for the full Grassmannian