Turnpike property for fractional control problems

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Outline

Origins of the turnpike

1. Although the idea goes back to John von Neumann in 1945, Lionel W. McKenzie traces the term to Robert Dorfman, Paul Samuelson, and Robert Solow's "Linear Programming and Economics Analysis in 1958", referring to an American English word for a Highway.
2. The origin of the term turnpike is in the interpretation that Samuelson did of this phenomenon: suppose we want to travel from city A to city B by car, the best way to do it, the optimal way, is to take the highway (the turnpike) as near as we can from A, and leave it when we are close to B. So, except nearby A and B, we are expected to be on the highway: in other words, the turnpike.

The turnpike phenomenon

- There is always a fastest route between any two points.
- If the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
- But if the origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

Example: The turnpike from New York City to Los Angeles

Figure: Traveling from New York City to Los Angeles by car.

The mathematics behind the turnpike phenomenon

1. Mathematically, the turnpike property establishes that, when a general optimal control problem is settled in large time, for most of the time the optimal controls and trajectories remain exponentially close to the optimal control and state of the steady-state problem.
2. The turnpike property is very useful, since it gives us an idea of the nature of the optimal solutions of a control problem, without having to solve it analytically. In practice, the turnpike property allows performing a significant improvement of the numerical methods used to solve optimal control problems.

Turnpike phenomenon for finite-dimensional dynamical systems

- Consider the finite-dimensional dynamical system

$$x_t + Ax = Bu, \qquad x(0) = x_0 \in \mathbb{R}^N \tag{1.1}$$

where $A, B \in M(N,N)$, with control $u \in L^2(0,T;\mathbb{R}^N)$.

- Given a matrix $C \in M(N,N)$, and some $x^* \in \mathbb{R}^N$, consider the optimal control problem

$$\min_u J^T(u) = \frac{1}{2}\int_0^T \left(|u(t)|^2 + |C(x(t)-x^*)|^2\right) dt. \tag{1.2}$$

- There exists a unique optimal control $u(t)$ in $L^2(0,T;\mathbb{R}^N)$, characterized by the optimality condition

$$u = -B^*p, \quad -p_t + A^*p = C^*C(x-x^*), \quad p(T) = 0 \in \mathbb{R}^N. \tag{1.3}$$

The associated stationary problem

- Let formulate the same problem for the steady-state model

$$Ax = Bu. \tag{1.4}$$

Then, there exists a unique minimum $\bar{u}$, and a unique optimal state $\bar{x}$, of the stationary optimal control problem

$$\min_u J_s(u) = \frac{1}{2}\left(|u|^2 + |C(x - x^*)|^2\right) \quad \text{subject to} \quad Ax = Bu. \tag{1.5}$$

- Let assume that: The pair (A, B) is controllable. The pair (A, C) is observable. Then, we have the following result: there exist positive constants λ and K, independent on T, such that $\forall t \in [0, T]$

$$|u(t) - \bar{u}| + |x(t) - \bar{x}| \leq K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right).$$

Visualization of the turnpike phenomenon

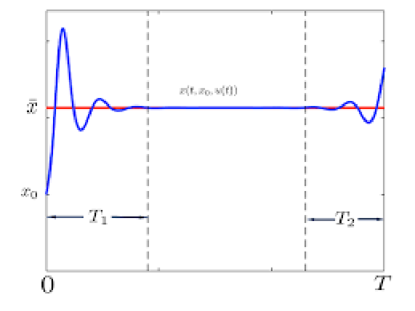

Figure: One can observe that the optimal solution remains close to certain path for most of the time, except maybe for an initial time interval T_1 and a final time interval T_2. In this Figure, $x(t, x_0, u(t))$ is the optimal solution of the time dependent control problem with a cost function, while $\bar{x}$ is the optimal solution of the stationary problem.

Relevant applications: econometric, climate science, and aeronautic.

1. The models arising in climate science are extremely complex.
2. Sustainable economic development is another area in which these issues arise and play a central role. Most often, the existing PDE and control theory, based on optimization and minimization of cost functionals, and the characterization of optimal controls through the corresponding optimality systems and adjoint methods, do not distinguish between short and long time horizons. Turnpike can maybe help to solve this gap.
3. The issue of long time versus steady state control is also relevant in shapes design. In aeronautics, most designs are computed based on steady state models and, although it is assumed, or understood that these steady optimal shapes are close to the optimal time-evolving ones. There are no results justifying such a fact rigorously. Turnpike can maybe help to justify this fact.

Outline

The time-dependent fractional control problem

Let $\mathcal{U} := L^2((0,T)\times(\mathbb{R}^N\setminus\Omega))$ and consider the optimal control problem:

$$\min_{g\in\mathcal{U}} J^T(g) := \frac{1}{2}\int_0^T \|u-u^d\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^T \|g(\cdot,t)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}dt, \quad (2.1)$$

subject to the constraints that u solves the fractional heat equation:

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } Q := \Omega\times(0,T),\\ u = g\chi_{\mathcal{O}} & \text{in } \Sigma := (\mathbb{R}^N\setminus\Omega)\times(0,T),\\ u(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \quad (2.2)$$

- Our control region $\mathcal{O}\subset(\mathbb{R}^N\setminus\Omega)$ is an arbitrary nonempty open set.
- u^d is a given fixed target and g is the control.
- $(-\Delta)^s$ $(0<s<1)$ is the fractional Laplace operator.

Theorem: Optimal solutions (Antil, Verma & W., 2020, ESAIM, COCV)

1. There exists an optimal pair (g^T, u^T) solution to the minimization problem (2.1)-(2.2).
2. There exists a function ψ^T such that $g^T = \mathcal{N}_s \psi^T\Big|_{\Sigma}$ ($\mathcal{N}_s$ is the nonlocal normal derivative operator) and we have the following optimality systems:

$$\begin{cases} u_t^T + (-\Delta)^s u^T = 0 & \text{in } Q, \\ u^T = g^T \chi_{\mathcal{O}} & \text{in } \Sigma, \\ u^T(\cdot, 0) = 0 & \text{in } \Omega, \\ & \\ -\psi_t^T + (-\Delta)^s \psi^T = u^T - u^d & \text{in } Q, \\ \psi^T = 0 & \text{in } \Sigma, \\ \psi(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

The stationary fractional control problem

Let $U = L^2(\mathbb{R}^N \setminus \Omega)$. We consider the corresponding stationary problem:

$$\min_{g \in U} J(g) := \frac{1}{2}\|u - u^d\|^2_{L^2(\Omega)} + \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}, \tag{2.4}$$

subject to the constraints that u solves the fractional elliptic equation:

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = g\chi_{\mathcal{O}} & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.5}$$

- Our control region $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$ is an arbitrary nonempty open set.
- u^d is a given fixed target.

Theoorem: Existence of minimizers (Antil, Khatri & W., 2019, Inverse Problems)

1. There exists a solution $(\overline{g}, \overline{u}) \in L^2(\mathbb{R}^N \setminus \Omega) \times L^2(\Omega)$ to the minimization problem (2.4)-(2.5).
2. In addition, there exists $\overline{\psi}$ such that $\overline{g} = \mathcal{N}_s \overline{\psi}\Big|_{\mathbb{R}^N \setminus \Omega}$, and we have the following optimality system:

$$\begin{cases} (-\Delta)^s \overline{u} = 0 & \text{in } \Omega, \\ \overline{u} = \overline{g}\chi_{\mathcal{O}} & \text{in } \mathbb{R}^N \setminus \Omega, \\ \\ (-\Delta)^s \overline{\psi} = \overline{u} - u^d & \text{in } \Omega, \\ \overline{\psi} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.6}$$

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Theorem: Behavior as $T \to +\infty$ (W. & Zamorano, 2021, COCV)

Let (u^T, g^T, ψ^T) be the solution of the optimal system (2.3) and $(\overline{u}, \overline{g}, \overline{\psi})$ be the solution of the corresponding stationary optimal system (2.6). Then, the following assertions hold:

$$\frac{1}{T}\int_0^T g^T\,dt \quad \longrightarrow \quad \overline{g} \quad \text{in } L^2(\mathbb{R}^N \setminus \Omega) \text{ as } T \to +\infty,$$

and

$$\frac{1}{T}\int_0^T u^T\,dt \quad \longrightarrow \quad \overline{u} \quad \text{in } L^2(\Omega) \text{ as } T \to +\infty.$$

Main ideas of the proof

Let $w = u^T - \overline{u}$ and $h = g^T - \overline{g}$.

- After some computations of several pages, one shows that

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T\int_\Omega |w|^2\,dxdt + \int_0^T\int_{\mathbb{R}^N\setminus\Omega} |h|^2\,dxdt\right) = 0. \tag{3.1}$$

- We have

$$\int_\Omega \left|\frac{1}{T}\int_0^T u^T\,dt - \overline{u}\right|^2 dx \le \frac{1}{T}\int_0^T\int_\Omega |w|^2\,dxdt.$$

- Similarly,

$$\int_{\mathbb{R}^N\setminus\Omega} \left|\frac{1}{T}\int_0^T g^T\,dt - \overline{g}\right|^2 dx \le \frac{1}{T}\int_0^T\int_{\mathbb{R}^N\setminus\Omega} |h|^2\,dxdt.$$

Outline

Theorem: The turnpike property (W. & Zamorano, 2021, COCV)

1. Let (u^T, g^T, ψ^T) be the optimal solution of (2.3) and $(\overline{u}, \overline{g}, \overline{\psi})$ be the optimal solution of the corresponding stationary problem (2.6). Then, the following assertions hold: Let $\gamma \geq 0$ be a real number. There is a constant $C = C(\gamma) > 0$ (independent of T) such that for every $t \in [0, T]$ we have the following estimate:

$$\begin{aligned} &\|u^T(\cdot, t) - \overline{u}\|_{L^2(\Omega)} + \|\psi^T(\cdot, t) - \overline{\psi}\|_{L^2(\Omega)} \\ &\leq C\Big(e^{-\gamma t} + e^{-\gamma(T-t)}\Big)\Big(\|\overline{u}\|_{L^2(\Omega)} + \|\overline{\psi}\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.1}$$

2. The estimate (4.1) shows that we have the turnpike phenomena.

Is it possible to improve the above turnpike estimate?

1. We do not know if the estimate (4.1) can be improved as follows:

$$\begin{aligned}&\|u^T(\cdot,t)-\overline{u}\|_{L^2(\Omega)}+\|\psi^T(\cdot,t)-\overline{\psi}\|_{L^2(\Omega)}\\ \leq &C\left(e^{-\gamma t}\|\overline{u}\|_{L^2(\Omega)}+e^{-\gamma(T-t)}\|\overline{\psi}\|_{L^2(\Omega)}\right).\end{aligned} \tag{4.2}$$

2. Such an improved estimate hods for the local case $s=1$, with the control function localized in $\omega\subset\Omega$ and zero boundary conditions, by using Riccati's theory for infinite dimensional systems.

3. It seems that our method cannot be used to obtain (4.2). To obtain such an estimate, most likely, one has to generalize the Riccati theory to the fractional setting.

Remark

1. If the problem is coercive, then adding a stabilizability assumption, it is still possible to obtain the turnpike property.
2. Stabilizability assumptions hold for example if the associated system is controllable (null or exactly controllable).
3. The only result in this direction is our works, where we have shown that the fractional heat equation in one dimension with Dirichlet exterior controls is null controllable if and only if $1/2 < s < 1$. The same result holds in dimension N with an interior control under the assumption that the control region is a certain neighborhood of the boundary.
4. Therefore, we can only ensure a stabilizability condition in the case of Dirichlet controls.

Summary

- Here, we deal with long-time horizon control problems, and the possibility that optimal trajectories and controls simplify towards those of the corresponding steady state model.
- In practice, in long time-horizons, the effective computation of the control can be very expensive, since it requires iterative methods to solve the coupled optimality system combining the forward controlled state equation and the backward adjoint one.
- It is then natural to look for some shortcuts. That is one of the main objectives of the turnpike property.

Summary

- The question of whether the control process commutes with some qualitative aspect of PDE models is very subtle when dealing with numerical approximation methods.
- More precisely, convergent numerical algorithms for free dynamics do not necessarily lead to convergent numerical methods for control problems, especially, when one is dealing with the more demanding problem of controllability.
- This is so, in particular, when the numerical scheme is not stable enough to avoid the emergence of spurious numerical high frequency solutions.
- A certain amount of dissipativity of the numerical schemes is required, and the same can be said when dealing with long-time horizon control problems.

THANK YOU VERY MUCH AND STAY SAFE!

Some references

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