

Analytic Bezout Equations, Deconvolution, and Sampling in Rectangular and Radial Coordinates

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C-W-W-K-S-R-K... Sampling

- “The Poisson summation formula and Cauchy’s integral and residue formulas are two different aspects of a broad gauged duality formula which lies athwart most of analysis.” *S. Bochner*



C-W-W-K-S-R-K... Sampling

- “The Poisson summation formula and Cauchy’s integral and residue formulas are two different aspects of a broad gauged duality formula which lies athwart most of analysis.” *S. Bochner*
- $\mathbb{PW}(\Omega) = \{f : f, \hat{f} \in L^2, \text{supp}(\hat{f}) \subset [-\Omega, \Omega]\}.$

Theorem (C-W-W-K-S-R-K ... Sampling Theorem)

Let $f \in \mathbb{PW}(\Omega)$, $\delta_{n\sigma}(t) = \delta(t - n\sigma)$ and $\text{sinc}_\sigma(t) = \frac{\sin(\frac{2\pi}{\sigma}t)}{\pi t}.$

a.) If $\sigma \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) \frac{\sin(\frac{2\pi}{\sigma}(t - n\sigma))}{\pi(t - n\sigma)} = \sigma \left(\left[\sum_{n=-\infty}^{\infty} \delta_{n\sigma} \right] \cdot f \right) * \text{sinc}_\sigma.$$

b.) If $\sigma \leq 1/2\Omega$ and $f(n\sigma) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

Beurling-Landau Densities

- Λ is a *set of sampling and interpolation* if there exists constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\lambda_k \in \Lambda} |f(\lambda_k)|^2 \leq B\|f\|_2^2.$$

- In particular, sampling sets are uniformly discrete.
- Let $D^-(\Lambda)$, $D^+(\Lambda)$ denote the lower and upper Beurling-Landau Densities, respectively. For exact and stable reconstruction – $D^-(\Lambda) \geq 1$. Fails – $D^-(\Lambda) < 1$. (Note – There are sets of uniqueness with arbitrarily small density. We can also conditionally construct from these sets.) If $D^+(\Lambda) \leq 1$, then Λ is a *set of interpolation*.



Proof of W-K-S Sampling via Poisson

- Poisson Summation Formula (PSF)

$$\sigma \widehat{\sum_{n \in \mathbb{Z}} \delta_{n\sigma}} = \sum_{n \in \mathbb{Z}} \delta_{n/\sigma}.$$

- Proof of Shannon:** If $f \in \mathbb{PW}_\Omega$ and $\sigma \leq 1/2\Omega$,

$$\widehat{f}(\omega) = \left(\sum_{n \in \mathbb{Z}} \widehat{f}\left(\omega - \frac{n}{\sigma}\right) \right) \cdot \chi_{\left[\frac{-1}{\sigma}, \frac{1}{\sigma}\right)}(\omega).$$

$$\widehat{f}(\omega) = \left(\sum_{n \in \mathbb{Z}} \widehat{f}\left(\omega - \frac{n}{\sigma}\right) \right) \cdot \chi_{\left[\frac{-1}{\sigma}, \frac{1}{\sigma}\right)}(\omega) = \left(\sum_{n \in \mathbb{Z}} \left[\delta_{n/\sigma} \right] * \widehat{f} \right) \cdot \chi_{\left[\frac{-1}{\sigma}, \frac{1}{\sigma}\right)}(\omega)$$

$$\stackrel{(PSF)}{\iff} f(t) = \sigma \left(\left[\sum_{n \in \mathbb{Z}} \delta_{n\sigma} \right] \cdot f \right) * \operatorname{sinc}_\sigma(t).$$

Some Number Theory

- Chinese Remainder Theorem – From *Master Sun's Mathematical Manual*

-

$$x \equiv 1 \pmod{3},$$

$$x \equiv 2 \pmod{5}.$$

- Bezout Equation: $\gcd(m_1, m_2) = 1$

-

$$m_2 \cdot [m_2^{-1}]_{m_1} + m_1 \cdot [m_1^{-1}]_{m_2} = 1$$

-

$$a_1 m_2 \cdot [m_2^{-1}]_{m_1} + a_1 m_1 \cdot [m_1^{-1}]_{m_2} \equiv a_1 \pmod{m_1}$$

-

$$a_2 m_2 \cdot [m_2^{-1}]_{m_1} + a_2 m_1 \cdot [m_1^{-1}]_{m_2} \equiv a_2 \pmod{m_2}$$



Some Number Theory, Cont'd



$$x \equiv 1 \pmod{3},$$

$$x \equiv 2 \pmod{5}.$$

Some Number Theory, Cont'd



$$x \equiv 1 \pmod{3},$$

$$x \equiv 2 \pmod{5}.$$



$$2 \cdot 3 \equiv 1 \pmod{5}$$

$$2 \cdot 5 \equiv 1 \pmod{3}$$

- Plugging into Bezout :

$$x \equiv 1 \cdot 5 \cdot 2 + 2 \cdot 3 \cdot 2 = 22 \pmod{15}$$

$$x \equiv 7 \pmod{15}$$



Strongly Coprime

- Let φ satisfy

$$\varphi : 1 :: (\varphi + 1) : \varphi.$$

- Then

$$\varphi = 1 + \frac{1}{\varphi}, \varphi = \frac{1 + \sqrt{5}}{2}.$$

- Definition :** A real number α is poorly approximated by rationals ($\alpha \in \mathbb{P}$) provided that there exist a constant $C > 0$ and $n \in \mathbb{N}$ such that for all integers p, q ,

$$|\alpha - p/q| \geq C|q|^{-n}.$$

In fact, the Golden Mean φ ($\varphi = 1 + \frac{1}{\varphi}$) satisfies $|\varphi - p/q| \geq \frac{1}{\sqrt{5}|q|^2}.$



Deconvolution

- The Fourier transform –

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \omega} dt.$$

If f is compactly supported, \widehat{f} may be analytically continued to the Fourier-Laplace transform – for $\zeta = \omega + i\eta$,

$$\widehat{f}(\zeta) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \zeta} dt$$

- Consider

$$s(t) = f * \mu(t) = \int f(\tau) \mu(t - \tau) d\tau,$$

where μ is a compactly supported finite Borel measure which is absolutely continuous w.r.t. Lebesgue measure on \mathbb{R} .



Deconvolution

- Then the set

$$\mathcal{Z}_\mu = \{\zeta \in \mathbb{C} : \hat{\mu}(\zeta) = 0\}$$

is NOT empty. Let $\eta + i\omega \in \mathcal{Z}_\mu$.

- The functions

$$g(t) = e^{-2\pi\eta t} \cos(2\pi k\omega t), \quad h(t) = e^{-2\pi\eta t} \sin(2\pi k\omega t)$$

$k \in \mathbb{N}$, are “erased” by convolution with μ . In fact, **all linear combinations** of all such signals are eliminated.

Circumventing Ill-Posedness

- Circumvented ill-posedness by creating a multichannel system.
- Choose components (convolvers) so that any information lost by one convolver is retained by another. Convolver array must also be invertible.
- **Theorem (Hörmander) (1967)** : For the compactly supported distributions $\{\mu_i\}_{i=1}^n$ on \mathbb{R} , there exists compactly supported distributions $\{\nu_i\}_{i=1}^n$ such that

$$\mu_1 * \nu_1 + \dots + \mu_n * \nu_n = \delta$$

if and only if there exist positive constants A and B and a positive integer N such that

$$\left(\sum_{i=1}^n |\widehat{\mu_i}(\zeta)|^2 \right)^{\frac{1}{2}} \geq A e^{-2\pi B |\Im \zeta|} (1 + |\zeta|)^{-N}, \quad \zeta \in \mathbb{C}.$$



Circumventing Ill-Posedness, Cont'd

- Hörmander's Theorem is an existence theorem, giving the criteria by which to create an invertible system.
- **Definitions** : A set of convolvers $\{\mu_i\}_{i=1}^n$ that satisfy the inequality in the theorem is defined to be **strongly coprime**. The ν_i are called **deconvolvers**.
- Overdetermine f by a system of convolution equations,

$$s_i = f * \mu_i, \quad i = 1, \dots, n.$$

- Then the problem of solving for f is **well-posed** if the set of convolvers $\{\mu_i\}$ is strongly coprime. This condition gives a lower bound on $\sum_i |\hat{\mu}_i(\zeta)|^2$, which guarantees, among other things, that the $\hat{\mu}_i(\zeta)$ have no common zeros. Consequently, this guarantees that in the system $\{s_i = f * \mu_i\}$ NO information about f is lost.



Multichannel Deconvolution

- **Definitions :** The δ **equation**

$$\mu_1 * \nu_1 + \dots + \mu_n * \nu_n = \delta,$$

has Fourier-Laplace transform

$$\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \dots + \widehat{\mu}_n \cdot \widehat{\nu}_n = 1.$$

This is called the **analytic Bezout equation**.

- We solve for a set of deconvolvers $\{\widehat{\nu}_i(\zeta)\}$ which satisfy

$$\sum_i \widehat{\mu}_i(\zeta) \cdot \widehat{\nu}_i(\zeta) = 1.$$

- Taking inverse transforms of both sides of this equation gives

$$\sum_i \mu_i * \nu_i(t) = \delta(t).$$



Multichannel Deconvolution, Cont'd

- This in turn gives

$$\begin{aligned} & s_1 * \nu_1 + \dots + s_n * \nu_n \\ = & (f * \mu_1) * \nu_1 + \dots + (f * \mu_n) * \nu_n \\ = & f * (\mu_1 * \nu_1) + \dots + f * (\mu_n * \nu_n) \\ = & f * (\mu_1 * \nu_1 + \dots + \mu_n * \nu_n) \\ = & f * \delta = f. \end{aligned}$$

- These methods are both linear (convolution with deconvolvers) and realizable (the support of the deconvolvers being contained in the bounded support of the kernels of the convolution equations). Thus, deconvolution at a point $t \in \mathbb{R}$ depends only on data near t . The theory assumes no *a priori* information about the input signals.



Examples of Strongly Coprime Systems

- Creation of strongly coprime sets requires clever manipulation of decay and zeroes in the transform domain. *The tricks for these manipulations come from number theory.*
- **Cubes in \mathbb{R}^d** (Petersen and Meisters) Let $0 < r_1 < \dots < r_m$, $m \geq d + 1$ satisfy r_i/r_j is poorly approximated by rationals whenever $i \neq j$. Then the collection $\{\chi_{[-r_i, r_i]^d}\}_{i=1}^m$ is a strongly coprime set.
- **B-Splines in \mathbb{R}^d** (C) Let $0 < r_1 < \dots < r_m$, $m \geq d + 1$ satisfy r_i/r_j is poorly approximated by rationals whenever $i \neq j$. Then the collection of B-splines created by convolutions of $\{\chi_{[-r_i, r_i]^d}\}_{i=1}^m$ is a strongly coprime set.

Examples of Strongly Coprime Systems, Cont'd

- **Balls in \mathbb{R}^d** (Berenstein and Yger) Let μ_1 and μ_2 be the characteristic functions of the disks $B(0, r_1)$ and $B(0, r_2) \subseteq \mathbb{R}^2$ respectively. Then the system $\{\mu_1, \mu_2\}$ is strongly coprime if and only if there is a constant $A > 0$ such that

$$|r_2/r_1 - \xi/\eta| \geq (1/A) |\eta|^{-A}$$

for any pair $\xi, \eta > 0$ with $J_1(\xi) = J_1(\eta) = 0$ where J_1 is the Bessel function of order 1. This is true if $r_2/r_1 \in \mathbb{Q}$.

Constructing Deconvolvers

- While Hörmander's Theorem supplies necessary and sufficient conditions for the solution to deconvolution locally, it does not supply any explicit formulas for constructing deconvolvers. Finding such explicit formula involves *interpolation techniques from the theory of entire functions with restricted growth*.
- To construct numerical solutions to the problem we modify the δ equation to solve for an approximate identity $\psi_\lambda(x)$.
- Then,

$$f = \lim_{\lambda \rightarrow 0^+} (f * \psi_\lambda) = \sum_{i=1}^m \lim_{\lambda \rightarrow 0^+} (f * \mu_i * \nu_{i,\psi_\lambda}) = \sum_{i=1}^m \lim_{\lambda \rightarrow 0^+} (s_i * \nu_{i,\psi_\lambda}).$$



Constructing Deconvolvers, Cont'd

- Let $r_1 = 1$ and $r_2 = \sqrt{p}$, and let $i, j \in \{1, 2\}, i \neq j$. Given

$$\mu_1(t) = \chi_{[-1,1]}(t), \quad \mu_2(t) = \chi_{[-\sqrt{p}, \sqrt{p}]}(t),$$

the deconvolvers $\nu_{i,\psi}$ such that

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

•

$$\begin{aligned} \nu_{i,\psi}(t) &= \sum_{z \in \mathbb{Z}_j} \frac{\widehat{\psi}(z)}{\widehat{\mu}_i(z) \frac{d}{d\zeta} \widehat{\mu}_j(z)} \left(\frac{1}{z} \left(e^{2\pi i z(t+r_j)} - 1 \right) \chi_{[-r_j, r_j]}(t) \right) \\ &= \left(\frac{1}{2r_j} \sum_{n \neq 0} \frac{\widehat{\psi}(n/(2r_j))}{\widehat{\mu}_i(n/(2r_j))} e^{\pi i (n/r_j)t} \right) \chi_{[-r_j, r_j]}. \end{aligned}$$

Constructing Deconvolvers, Cont'd

- **Sketch of Proof:** We want to solve

$$\mu_1 * \nu_{1,\psi_\lambda} + \mu_2 * \nu_{2,\psi_\lambda} = \psi_\lambda$$

or equivalently, the modified Bezout equation

$$\widehat{\mu_1}(\omega)\widehat{\nu_{1,\psi_\lambda}}(\omega) + \widehat{\mu_2}(\omega)\widehat{\nu_{2,\psi_\lambda}}(\omega) = \widehat{\psi_\lambda}(\omega).$$

- Let $\mathcal{Z}_i = \{ \text{zeros of } \widehat{\mu_i} \}$. These are simple zeros.
- When $z \in \mathcal{Z}_1$, $\widehat{\nu_{2,\psi_\lambda}} = \frac{\widehat{\psi_\lambda}(z)}{\widehat{\mu_2}(z)}$. When $z \in \mathcal{Z}_2$, $\widehat{\nu_{1,\psi_\lambda}} = \frac{\widehat{\psi_\lambda}(z)}{\widehat{\mu_1}(z)}$.

Constructing Deconvolvers, Cont'd

- Construct circular paths $\{\Gamma_n\}$ with radii ρ_n such that $\mathcal{Z}_i \cap \Gamma_n = \emptyset$ and $\lim_{n \rightarrow \infty} \rho_n = \infty$.
- We now apply the Jacobi Interpolation Formula.

$$\begin{aligned}\widehat{\psi}(\zeta) &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\widehat{\psi}(z)}{z - \zeta} dz \\ &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\widehat{\psi}(z)[\widehat{\mu}_1(z)\widehat{\mu}_2(z) - \widehat{\mu}_1(\zeta)\widehat{\mu}_2(\zeta)]}{(z - \zeta)[\widehat{\mu}_1(z)\widehat{\mu}_2(z)]} dz \quad (1)\end{aligned}$$

$$+ \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\widehat{\psi}(z)[\widehat{\mu}_1(\zeta)\widehat{\mu}_2(\zeta)]}{(z - \zeta)[\widehat{\mu}_1(z)\widehat{\mu}_2(z)]} dz \quad (2).$$

Constructing Deconvolvers, Cont'd

- Now, for $z \in \mathcal{Z}_1$ or $z \in \mathcal{Z}_2$, $(\widehat{\mu}_1(z)\widehat{\mu}_2(z)) = 0$, but $\frac{d}{d\zeta}(\widehat{\mu}_1(z)\widehat{\mu}_2(z)) \neq 0$. Thus, the function

$$\left(\frac{1}{\widehat{\mu}_1(z)\widehat{\mu}_2(z)} \right)$$

has simple poles at $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$.

Constructing Deconvolvers, Cont'd

- Therefore, by the Cauchy Residue Theorem,

$$\begin{aligned}
 (1) &= \oint_{\Gamma_n} \frac{\widehat{\psi}(z)[\widehat{\mu}_1(z)\widehat{\mu}_2(z) - \widehat{\mu}_1(\zeta)\widehat{\mu}_2(\zeta)]}{(z - \zeta)[\widehat{\mu}_1(z)\widehat{\mu}_2(z)]} dz \\
 &= \widehat{\mu}_1(\zeta) \sum_{\substack{z \in \mathcal{Z}_2 \\ |z| < \rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu}_1(z) \frac{d}{d\zeta} \widehat{\mu}_2(z)} \left(\frac{\widehat{\mu}_2(\zeta)}{\zeta - z} \right) \\
 &\quad + \widehat{\mu}_2(\zeta) \sum_{\substack{z \in \mathcal{Z}_1 \\ |z| < \rho_n}} \frac{\widehat{\psi}(z)}{\widehat{\mu}_2(z) \frac{d}{d\zeta} \widehat{\mu}_1(z)} \left(\frac{\widehat{\mu}_1(\zeta)}{\zeta - z} \right).
 \end{aligned}$$

Strongly coprime $\oplus \text{supp } \psi \implies |(2)| \longrightarrow 0$ as $n \longrightarrow \infty$.

Sampling via Cauchy and Jacobi

Theorem (C-W-W-K-S-R-K ... Sampling Theorem)

Let $f \in \mathbb{PW}(\Omega)$, $\delta_{n\sigma}(t) = \delta(t - n\sigma)$ and $\text{sinc}_\sigma(t) = \frac{\sin(\frac{2\pi}{\sigma}t)}{\pi t}$.

a.) If $\sigma \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) \frac{\sin(\frac{2\pi}{\sigma}(t - n\sigma))}{\pi(t - n\sigma)} = \sigma \left(\left[\sum_{n=-\infty}^{\infty} \delta_{n\sigma} \right] \cdot f \right) * \text{sinc}_\sigma.$$

b.) If $\sigma \leq 1/2\Omega$ and $f(n\sigma) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

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b.) If $\sigma \leq 1/2\Omega$ and $f(n\sigma) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

Now, $f(t)$ is real analytic and has an analytic continuation $f(z)$ to \mathbb{C} .

Moreover, $f(z)$ satisfies Paley-Wiener, i.e., $\exists K = K(n)$ such that $\forall n \in \mathbb{N}$

$$|f(z)| \leq K(n)(1 + |z|)^{-n} e^{2\pi\Omega|\Im z|}.$$

Sampling via Cauchy and Jacobi, Cont'd

- Since dilation is an isometry on L^2 , we may assume

$$\Omega = \frac{\sigma}{2} \leq \frac{1}{2}.$$

- Let $T = 1$. Then $T \leq \frac{1}{2\Omega}$. Also let

$$G(\zeta) = \sin(\pi\zeta).$$

This has zeros $\mathcal{Z} = k$, $k \in \mathbb{Z}$. To avoid these zeros, let Γ_m be a circular contour with radius $(m + \frac{1}{2})$ for $m \in \mathbb{N}$.



Sampling via Cauchy and Jacobi, Cont'd

- We now apply the Jacobi Interpolation Formula.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{G(\zeta) - G(z)}{G(\zeta)} d\zeta \quad (\mathbf{S}) \\ &\quad + \frac{1}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{G(z)}{G(\zeta)} d\zeta \quad (\mathbf{R}) \end{aligned}$$

Sampling via Cauchy and Jacobi, Cont'd

- For $0 < \sigma < 1$,

$$|(\mathbf{R})| = O\left(e^{m(\sigma-1)}\right)$$

which $\rightarrow 0$ exponentially as $m \rightarrow \infty$.

- For $\sigma = 1$,

$$|(\mathbf{R})| = O\left(\frac{1}{[1+m]^n}\right)$$

which $\rightarrow 0$ with polynomial decay as $m \rightarrow \infty$.

Sampling via Cauchy and Jacobi, Cont'd

•

$$\sin \pi \zeta = \lim_{N \rightarrow \infty} s_N(\zeta) = \lim_{N \rightarrow \infty} \pi \zeta \prod_{j=1}^N \left(1 - \frac{\zeta^2}{j^2}\right)$$

is the Weierstrass product representation of a sine function.

•

$$\frac{1}{(\zeta - z)s_N(\zeta)} = \frac{1}{(\zeta - z)s_N(z)} + \sum_{|n| \leq N} \frac{1}{(\zeta - n)(n - z)s'_N(n)}$$

is the Mittag-Leffler partial fractions decomposition,

Sampling via Cauchy and Jacobi, Cont'd

- Let $R_m = (\mathbf{R})$ on Γ_m . Then, if $m < N$,

$$f(z) = \frac{G(z)}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{\zeta - z} \cdot \left[\frac{1}{G(z)} - \frac{1}{G(\zeta)} \right] d\zeta + R_m \quad (1.)$$

$$= \lim_{N \rightarrow \infty} \frac{s_N(z)}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{\zeta - z} \cdot \left[\frac{1}{s_N(z)} - \frac{1}{s_N(\zeta)} \right] d\zeta + R_m \quad (2.)$$

•

$$= \lim_{N \rightarrow \infty} \frac{s_N(z)}{2\pi i} \oint_{\Gamma_m} \cdot \sum_{|n| \leq N} \frac{f(\zeta)}{(\zeta - n)(n - z)s'_N(n)} d\zeta + R_m \quad (3.)$$

$$= \lim_{N \rightarrow \infty} \frac{s_N(z)}{2\pi i} \sum_{|n| \leq N} \cdot \oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - n)(n - z)s'_N(n)} d\zeta + R_m \quad (4.)$$

Sampling via Cauchy and Jacobi, Cont'd

$$= \lim_{N \rightarrow \infty} s_N(z) \sum_{|n| \leq m} \frac{f(n)}{(n-z)s'_N(n)} + R_m \quad (5.)$$

$$= \sin \pi z \sum_{|n| \leq m} f(n) \cdot \frac{(-1)^n}{\pi(z-n)} + R_m \quad (6.)$$

$$= \sum_{|n| \leq m} f(n) \operatorname{sinc}(z-n) + R_m \quad (7.)$$

(1.) Jacobi interpolation, **(2.)** Weierstrass product, **(3.)** Mittag-Leffler, **(4.)** linearity, **(5.)** Cauchy Residue, **(6.)** Weierstrass and **(7.)** sinc function. Since $R_m \rightarrow 0$ as $m \rightarrow \infty$, we get sampling.



Sampling on Non-Commensurate Lattices

- Now let

$$\Lambda_1 = \left\{ \frac{\pm k}{2} \right\}, \quad \Lambda_2 = \left\{ \frac{\pm k}{2\alpha} \right\},$$

for $k \in \mathbb{N}$, and let

$$\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2.$$

- **Theorem (Walnut)** : Let α be an irrational, and let f be a $(1 + \alpha)$ -band-limited function. Then f is uniquely determined by

$$\{f(\lambda_k)\} \cup \{f(0), f'(0)\}.$$

Sampling on Non-Commensurate Lattices, Cont'd

- **Theorem (C)** : Let α be an irrational that is poorly approximated by rationals. Let

$$\Lambda_1 = \left\{ \frac{\pm k}{2} \right\}, \Lambda_2 = \left\{ \frac{\pm k}{2\alpha} \right\},$$

for $k \in \mathbb{N}$, and let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2$. Let f be a $(1 + \alpha)$ -band-limited function. Then f can be conditionally reconstructed from

$$\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$$

by the formula

$$f(t) \approx \sum_{\lambda_k \in \Lambda} f(\lambda_k) \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)} + [f(0)K_1(t) + f'(0)K_2(t)].$$



Sampling on Non-Commensurate Lattices, Cont'd

- Here

$$G(t) = \sin(2\pi t) \cdot \sin(2\alpha\pi t) \text{ and}$$

$$K_1(t) = \frac{G(t)}{(G''(0)/2!)(t^2)} = \frac{G(t)}{(4\pi^2\alpha)(t^2)},$$

$$K_2(t) = \frac{G(t)}{(G''(0)/2!)(t)} = \frac{G(t)}{(4\pi^2\alpha)(t)}.$$

- Let δ be given, $0 < \delta < \frac{1}{4\alpha}$. Let

$$\Lambda_\delta = \{\lambda \in \Lambda : \text{dist}(\lambda, \Lambda \setminus \{\lambda\}) < \delta\}.$$

Elements in Λ_δ occur in pairs. Then if $\Lambda_1 = \{\frac{\pm k}{2}\}$, $\Lambda_2 = \{\frac{\pm k}{2\alpha}\}$, for $k \in \mathbb{N}$, and let

$$\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 = \Lambda_\delta \cup \Lambda_\sigma.$$



Sampling on Non-Commensurate Lattices, Cont'd

- Let δ be given, $0 < \delta < \frac{1}{4\alpha}$. Let

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$$\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 = \Lambda_\delta \cup \Lambda_\sigma.$$

- For

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi\alpha t), \text{ let } H_{\lambda_k} = \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)}.$$

- Theorem :**

$$\text{Span} \left\{ \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)} \right\}$$

is dense in $L^2_{(1+\alpha)}$.



Sampling on Non-Commensurate Lattices, Cont'd

- **Lemma :**

$$\left\{ \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)} \right\}_{\lambda_k \in \Lambda_{\delta'}}$$

is a Bessel sequence.

- **Lemma :**

$$\frac{G(t)}{4\pi\alpha t}, \frac{G(t)}{4\pi\alpha t^2} \in \text{Closure}(\text{Span}\{\cdot\}).$$

Sampling on Non-Commensurate Lattices, Cont'd

- **Lemma :** For $\{\lambda_1, \lambda_2\} \subset \Lambda_\delta$,

$$\begin{aligned} & [f(\lambda_1)H_{\lambda_1}(t) + f(\lambda_2)H_{\lambda_2}(t)] \\ = & f(\lambda_1)H_{\lambda_1}(t) + f'(\xi_{\lambda_1}) \left[\frac{G(t)}{\left(\frac{G'(\lambda_1) - G'(\lambda_2)}{2(\lambda_1 - \lambda_2)} \right) (t - \lambda_1)(t - \lambda_2)} \right] \\ & + \mathcal{R}(\lambda_1 - \lambda_2)^2, \end{aligned}$$

where $\xi_{\lambda_1} \rightarrow \lambda_1$ and $\mathcal{R}(\lambda_1 - \lambda_2)^2 \rightarrow 0$ as $\delta \rightarrow 0$.

- **Lemma :**

$$\left\{ \frac{G(t)}{(G''(\lambda_k)/2!)(t - \lambda_k)^2} \right\}_{\lambda_k \in \Lambda_\delta}$$

is a Bessel sequence.

Sampling on Non-Commensurate Lattices, Cont'd

- **Theorem (B. Ya. Levin (1961) – Kharkov Inst.)** : Let α be an irrational. Let

$$\Lambda_1 = \left\{ \frac{\pm k}{2} \right\}, \Lambda_2 = \left\{ \frac{\pm k}{2\alpha} \right\},$$

for $k \in \mathbb{N}$, and let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2$. Let f be a $(1 + \alpha)$ -band-limited function. Then f can be conditionally reconstructed from

$$\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$$

- Boris Yakovlevich Levin “On Bases of Exponential Functions in L_2 ,” *Zap. Mekh.-Mat. Fak. i Khar'kov. Mat, Obshch.*, **27**, 39–48 (1961).
- **Moral of the Story** : Always Read the Masters.



Generalized Sampling



$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \frac{G(\zeta) - G(z)}{G(\zeta)} d\zeta + R_m \\ &= \frac{G(z)}{2\pi i} \oint_{\Gamma_m} \frac{f(\zeta)}{(\zeta - z)} \cdot \left[\frac{1}{G(z)} - \frac{1}{G(\zeta)} \right] d\zeta + R_m. \end{aligned}$$

$$G(z) = \sin(\pi z) \text{ and } \lambda \in \mathbb{Z}$$

- Let f be π -band-limited. Then

$$f(z) = \left(\frac{\sin(\pi z/2)}{\pi} \right)^2 \sum_{n=1}^{\infty} \left[\frac{f(2n)}{(z - 2n)^2} + \frac{f'(2n)}{(z - 2n)^2} \right]$$

$$G(z) = \sin^2\left(\frac{\pi z}{2}\right) \text{ and } \lambda \in 2\mathbb{Z}$$

Generalized Sampling

- For general $K \in \mathbb{N}$,

$$f(t) = \sum_{n \in \mathbb{Z}} \left[f(KnT) + (t - KnT)f'(KnT) + \frac{(t - KnT)^2}{2!}f''(KnT) + \dots + \frac{(t - KnT)^{(K-1)}}{(K-1)!}f^{(K-1)}(KnT) \right] \left[\operatorname{sinc} \left(\frac{(t - 2nT)}{KT} \right) \right]^K.$$

$$G(z) = \sin^K \left(\frac{\pi z}{K} \right) \text{ and } \lambda \in K\mathbb{Z}$$

Multi-Rate Sampling

- Let α be an irrational that is poorly approximated by rationals. Let

$$\Lambda_1 = \left\{ \frac{\pm k}{2} \right\}, \Lambda_2 = \left\{ \frac{\pm k}{2\alpha} \right\},$$

for $k \in \mathbb{N}$, and let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2$. Let f be a $(1 + \alpha)$ -band-limited function. Then f can be conditionally reconstructed from $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$ by the formula

$$f(t) \approx \sum_{\lambda_k \in \Lambda} f(\lambda_k) \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)} + [f(0)K_1(t) + f'(0)K_2(t)].$$

$$G(z) = \sin(2\pi z) \cdot \sin(2\pi\alpha z)$$

- Generalization – $\{r_i\}_{i=1}^{\ell}$ such that (r_i/r_j) is irrational for $i \neq j$

$$G(z) = \prod_{k=1}^{\ell} \sin(2\pi r_k z)$$

Radial and Radial Multi-Rate Sampling

$$G(z) = \frac{J_\nu(\pi z)}{\pi(z)^{1-\nu}}$$

- Generalization – $\{r_i\}_{i=1}^\ell$ such that (r_i/r_j) is **rational** for $i \neq j$

$$G(z) = \prod_{k=1}^{\ell} \frac{J_\nu(\pi r_k z)}{\pi(r_k z)^{1-\nu}}$$

Summary

- Multichannel Deconvolution : Reformulate ill-posed problems into well-posed problems. Leads to multirate sampling.
- CAVEAT : “In theory, there is no difference between theory and practice. In practice, there is.” *Anon.*
- Some References
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 - S.D. Casey, “Shannon Sampling via Poisson, Cauchy, Jacobi and Levin,” Birkhauser Research Monographs, 43 pp. (to appear) (2022).

