Compositional Reactive Planning for Complex Tasks using Topological Invariants of Strategy Spaces

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AFOSR Dynamical Systems and Control Theory Review

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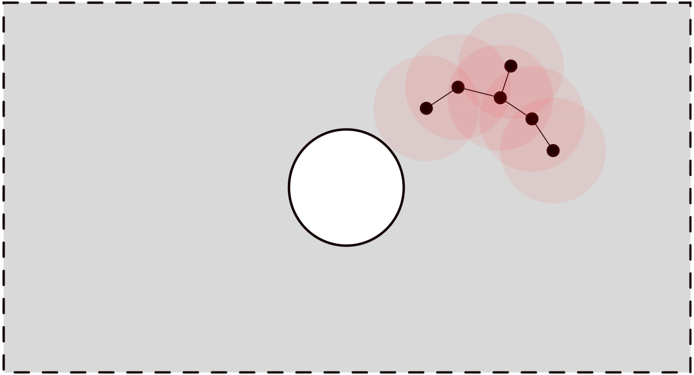
Special Thanks

Dan Koditschek	$\rightsquigarrow$ *for raising and suffering me as a father would, for 8 years;*
Warren Dixon	$\rightsquigarrow$ *for cheering and pushing me as a big brother would, for 4 years and counting;*
Federico Zegers	$\rightsquigarrow$ *for being a good friend (and teaching me network control);*
Fred Leve	$\rightsquigarrow$ *for making possible this program and intense learning experiences like this past week*
Yuliy Baryshnikov	$\rightsquigarrow$ *for introducing me to these special people and this community.*

A Motivation Slide From 3 Years Ago

Challenge: Autonomous generation of complex distributed cooperative behaviors requires reasoning over very large combinatorial structures.



For example, in networks where comms are constrained by distance,

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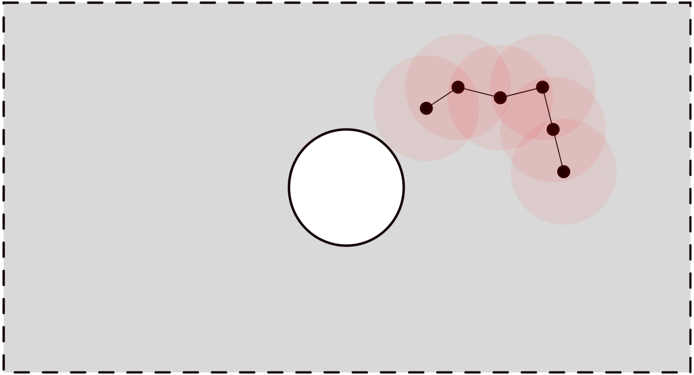

Switching between comms structures (e.g. spanning trees) is useful.

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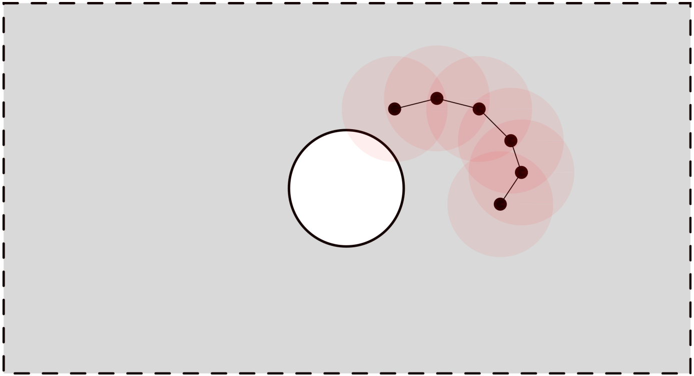

Coordinated motion under a fixed conroller. . .

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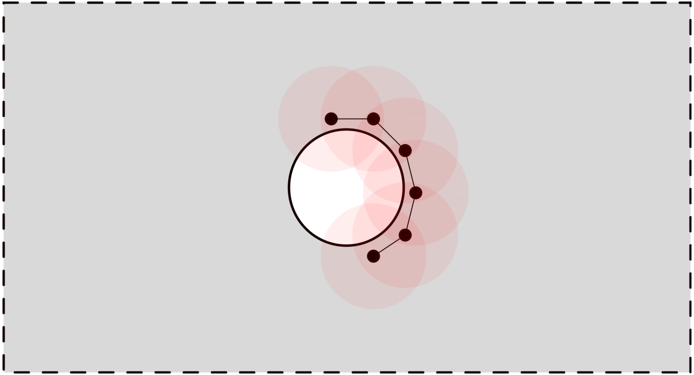

. . . may run into obstacles. . .

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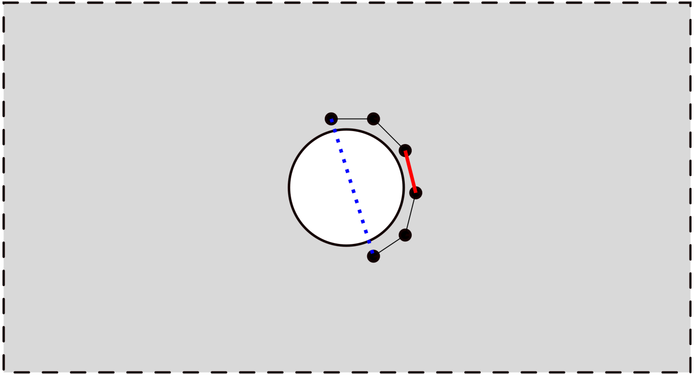

. . . suggesting a reassessment of the comms structure. . .

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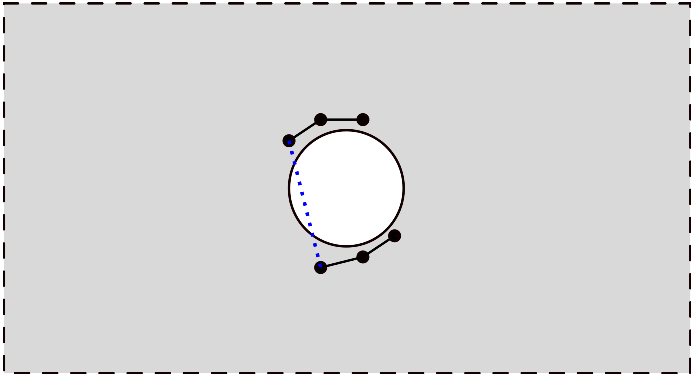

...including temporary disconnects with the aim of reconnecting soon thereafter...

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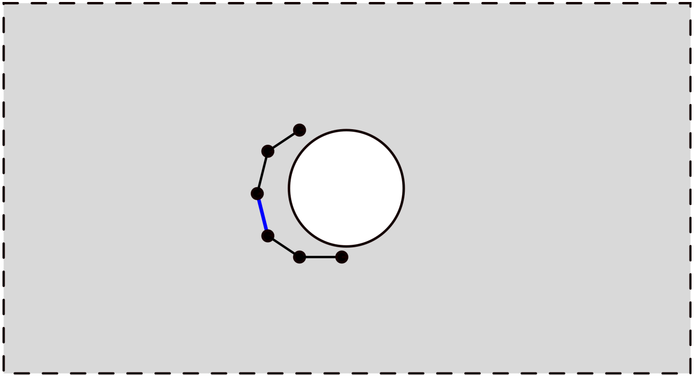

. . . using a different connectivity structure.

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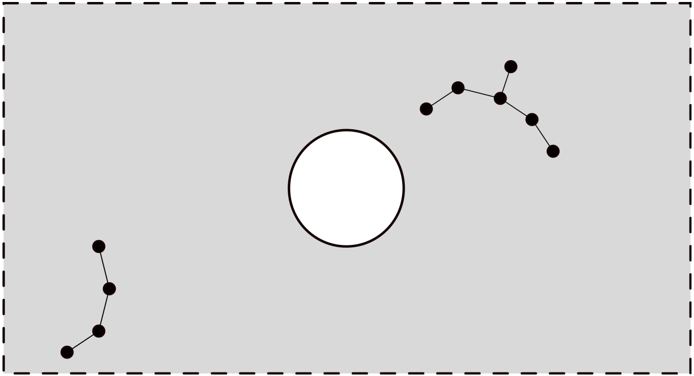

In the presence of additional resources. . .

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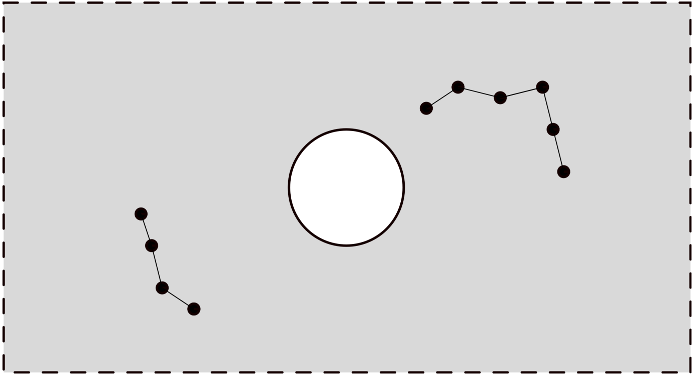

. . . a reactive control paradigm may provide alternative solutions. . .

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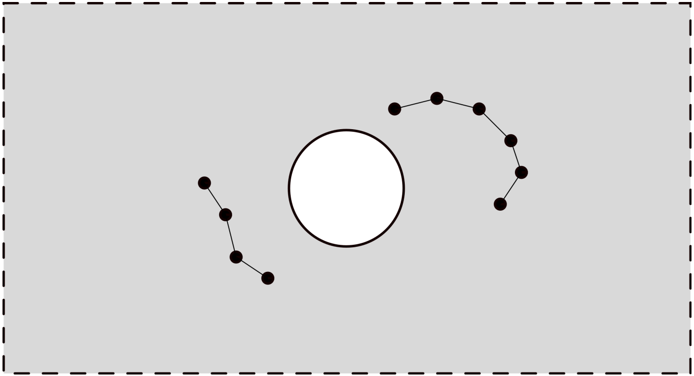

[agents move according to original plans]

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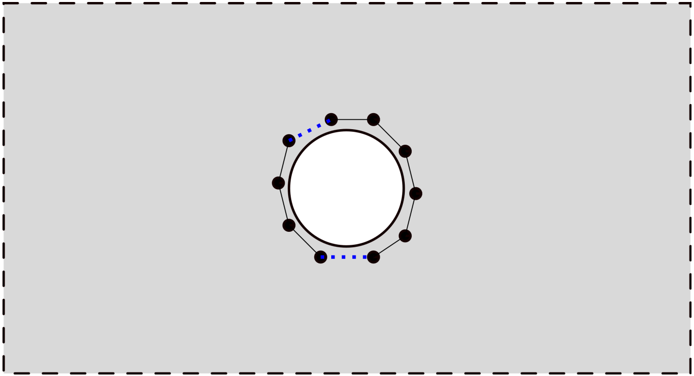

[rendevous generates new comms connections]

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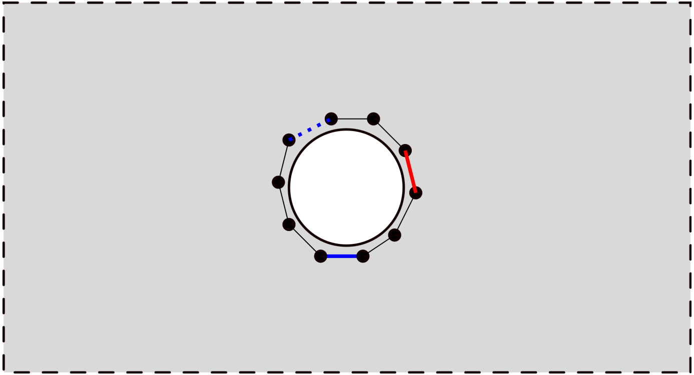

[less risky strategy becomes available]

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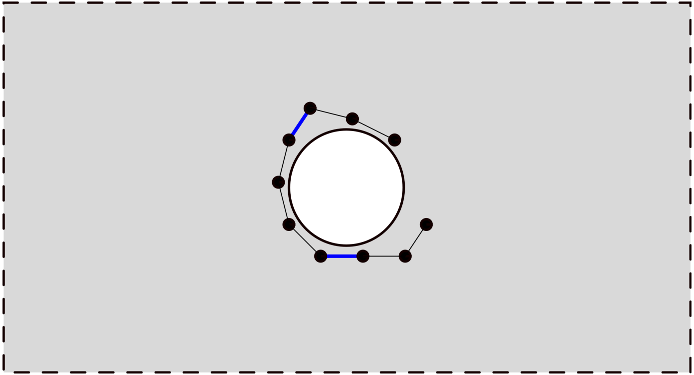

[resolution through edge-creation and edge-exchanges]

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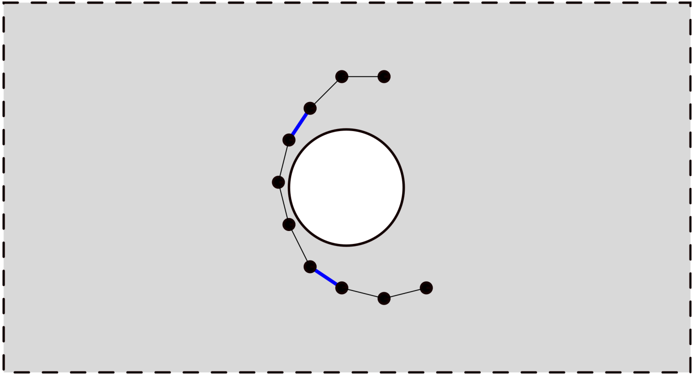

[continued motion as a group]

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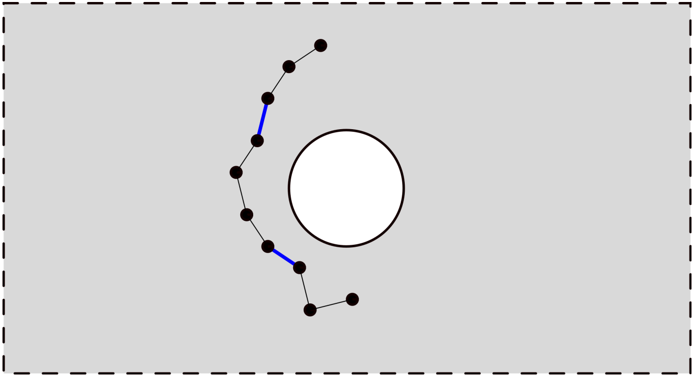

[they live happily ever after]

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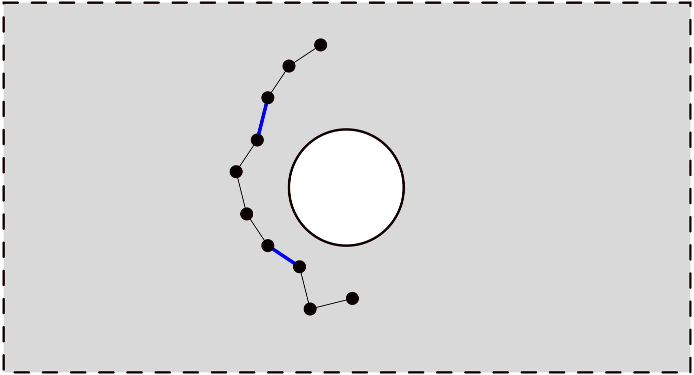

[they live happily ever after]

- "very large combinatorial structure"=the space of all spanning forests on a varying set of agents.

Why Categories of HDIs?

Emerging requirements:

- Need a rich category-theoretic “substrate” for symbolic representations of task domains
  - Do not treat tasks on a case-by-case basis
  - Instead, consider them as capabilities
  - Generate new capabilities through composition

Why Categories of HDIs?

Biginnings of An Example Structure: Provided the data

- finite set of agent labels $\mathcal{V}$,
- compact workspace with piecewise smooth boundary $\Omega \subset \mathbb{R}^d$,
- spanning forest $F \in \binom{\mathcal{V}}{2}$,

form the space $X(\mathcal{V}, F) \subset \Omega^F$ of particle configurations $(x_v)_{v \in \mathcal{V}}$ in which any two F-neighbors are R-close.

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- Expanding or shrinking Ω (e.g., adding/removing obstacles, digging tunnels) alters the safety constraints on all agents.

Why Categories of HDIs?

Example Capability $C(\mathcal{V}, F, \ell, x^*)$**:** For specified $\ell(f) \in f$, f ranging over the components of the forest F, execute a controller that brings each $x_{\ell(f)}$ to a designated target $x_f{}^* \in \Omega$ while keeping the system state in $X(\mathcal{V}, F)$, from almost all initial positions in Ω.

$\rightsquigarrow$ *components of F follow their designated leaders*

- $C(\mathcal{V}, F, \ell, x^*)$ is a compositional extension of $C(\{*\}, \varnothing, *, x^*)$: if $\mathfrak{n}_:\Omega \times \Omega \to \mathbb{R}^d$ is a continuously varying navigation field[2] with target y, then $C(\mathcal{V}, F, \ell, x^*)$ is realized by the closed-form distributed controller

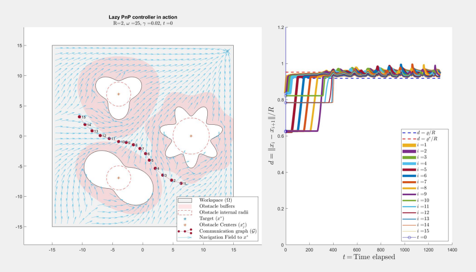
$$u_p \triangleq \sum_{q \sim_F p} \xi(x_q, x_p)\mathfrak{n}(x_q, x_p), \quad \xi(y, z) \triangleq \frac{r(\|y-z\|)\|y-z\|^2}{\langle \mathfrak{n}(y,z), y-z \rangle} \tag{1}$$

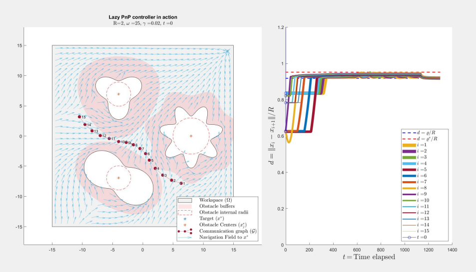
  if p is not a leader, and $u_p \triangleq \gamma\mathfrak{n}(x_f{}^*, x_p)$ if p is the leader of component f of F, provided γ is sufficiently small. $\rightsquigarrow$ *If one agent can navigate Ω, so can a tree of agents*

[2]subject to some conditions

Why Categories of HDIs?

Instances of $C(\mathcal{V}, F, \ell, x^*)$.





- This capability did not exist in the literature prior to our work in [1, 2], even for holonomic single integrator agents.
- The main point is that the composition operator requires no knowledge of Ω: the computational burden is offloaded onto $\mathfrak{n}$ and distributed between the agents.

Proposed Program

We seek a CATEGORY-THEORETIC **framework combining:**

- differential inclusions
- jump/reset relations (discontinuous/switched dynamics)
- sequential and parallel composition (concatenation/coupling)
- maps between [open] hybrid systems
- trajectories have to be maps
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HDIs on manifolds may be the right collection of objects:

- Can easily define a “graph of HDIs”, to see that this collection is “self-hybridizing”.
- Especially in the open setting, a discrete action defines a non-deterministic quiver of edges (more below).
- We aim to explore trisps (Gelfand-Manin triangulated spaces=generalized simplicial sets) as an alternative discrete skeleton.

$\rightsquigarrow$ *Is there a forgetful functor to the discrete structure?*

Proposed Program

"No Abstract Nonsense". The proposed framework must operationalize the following:

- refinement/coarsening arguments to identify behaviors/tasks

  $\rightsquigarrow$ *e.g., Template–Anchor pairs [3, 4]*

  $\rightsquigarrow$ *Other hierarchical compositions [5, 6, 7]*

- Stability and robustness arguments,

  $\rightsquigarrow$ *The hybrid differential inclusions framework [8] is an example*

- Computable invariants of task achievability

  $\rightsquigarrow$ *Connect to computational Conley theory*

  $\rightsquigarrow$ *Homological invariants à-la Erdmann*

- Analysis of temporal tameness properties (noZeno / goodZeno / badZeno & worse. . .)

  $\rightsquigarrow$ *Generalized hybrid time domains / hybrid arcs*

  $\rightsquigarrow$ *Weaker topology on the space of hybrid arcs?*

Proposed Program

Directions of Study:

- Fuse the Hybrid Differential Inclusions (HDI) framework [8] with the categorical formulations of Culbertson *et. al.* [4] and Lerman–Schmidt [9].

$\rightsquigarrow$ *toolkit for systematic reasoning about compositions of capabilities*

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- Develop a functorial theory of strategy spaces over this category.

  $\rightsquigarrow$ *Characterize tasks for which homotopy types of strategy spaces are still dichotomic*

  $\rightsquigarrow$ *For these tasks, develop a compositional complexity theory*

Strategy Spaces

Erdmann's Strategy Spaces [10].

- FINITE state space X
- FINITE set $\mathcal{A}$ of actions:
  - Pairs (x, A_x), $A_x \subset X$ (non-deterministic)
  - Pairs (x, π_x), $\pi_x \in \Delta(X)$ (Markovian)

We focus on the non-deterministic case. The Markovian case is more involved, but has analogous results.

An acyclic strategy on $(X, \mathcal{A})$ is a collection $\mathcal{B} \subset \mathcal{A}$ containing no directed cycles.

The strategy complex $\mathfrak{S}(X, \mathcal{A})$ is the simplicial complex of all acyclic strategies.

A guaranteed strategy for reaching $x \in X$ is an acyclic strategy $\sigma \in \mathfrak{S}(X, \mathcal{A})$ such that every $y \neq x$ has at least one action in σ exiting y.

For $x \in X$, denote $\mathfrak{S}_x(X, \mathcal{A}) \triangleq \mathfrak{S}(X, \mathcal{A}_x)$, where $\mathcal{A}_x$ is obtained from $\mathcal{A}$ by removing all actions based at x and adding all the deterministic actions $\{(x, y)\}$, for $y \neq x$.

Strategy Spaces

Theorem

A transition system $(X, \mathcal{A})$ has a guaranteed strategy for reaching x if and only if $\mathfrak{S}_x(X, \mathcal{A})$ is homotopy equivalent to $\mathbb{S}^{|X|-2}$. Otherwise, $\mathfrak{S}_x(X, \mathcal{A})$ is contractible.

An important observation is that an acyclic strategy is witnessed by a decreasing function of its supporting DAG. A guaranteed strategy for reaching x may be used to construct an open cover of $\mathbb{R}^{|X|} \setminus \mathrm{Sp}(\not\Vdash)$ whose nerve coincides with $\mathfrak{S}_x(X, \mathcal{A})$.

Question. What are appropriate category structures on transition systems (matched by a view of strategy spaces) such that $\mathfrak{S}, \mathfrak{S}_x$ (and other variants) are functorial?

Question. What compositions (of transition systems) yield explicit computations of the homotopy type of $\mathfrak{S}_*$?

Question. Extensions of $\mathfrak{S}_*$ to the coveted category of hybrid open systems+compositional "formulae" as above?

Such extensions would give rise to a constructive approach to the design of reactively controllable open hybrid systems.

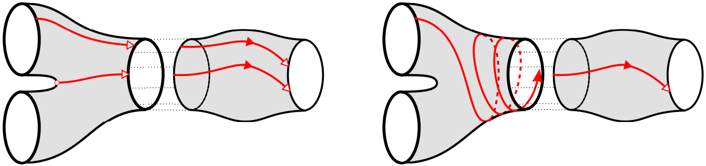
Existing Categorical Frameworks

General mantra: "Hybrid System=Graph of Dynamical Systems"

- Ames [11]:
  - general "hybridization" construction for *any* category;
  - applies to smooth dynamical systems (no composition).
- Haghverdi–Tabuada–Pappas [12]:
  - an open system version (both discrete and continuous control).
  - weakened notion of equivalence: bisimulation.
- Lerman, Lerman–Schmidt [13, 9]:
  - open systems as hybrid submersions;
  - interconnections via hybrid submersions between products.
- Culbertson–Gustafson–Koditschek–Stiller [4]:
  - hybrid semiconjugacies to construct template-anchor pairs;
  - Sequential composition using weakened notion of trajectory.

Informal Tidbits: Compositions

Sequential composition may be thought of as a concatenation operator on the trajectories of a pair of systems:

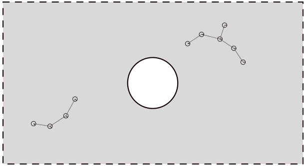


CGKS [4]: discuss difficulties with sequential composition of piecewise smooth (hybrid) trajectories, establishing the need for coarse notions of (1) hybrid trajectory and/or (2) hybrid time domains.

Informal Tidbits: Compositions

Parallel compositions.

- The simplest example is a decoupled Cartesian product of systems.
- In mobile agent networks, interconnection may be intermittent.



[While far apart, the two agent coalitions do not interact]

Ames [11] and Lerman–Schmidt [9]: enable interconnections, but need to be reconciled with HDI and sequential compositions.

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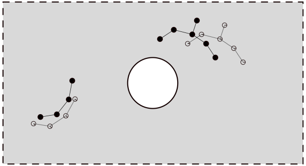

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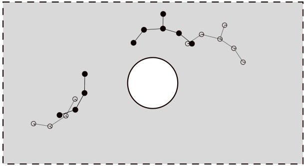

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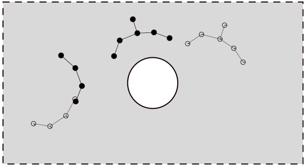

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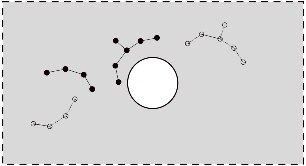

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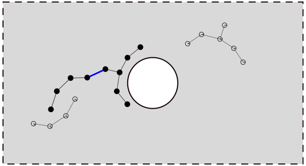

[Once communication is established, their motion may be coupled. . .]

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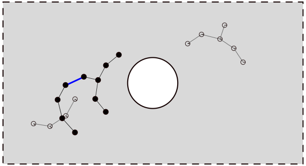

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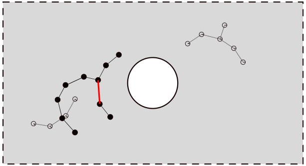

[...then decoupled, if the need arises]

Ames [11] and Lerman–Schmidt [9]: enable interconnections, but need to be reconciled with HDI and sequential compositions.

Informal Tidbits: Compositions

Parallel compositions.

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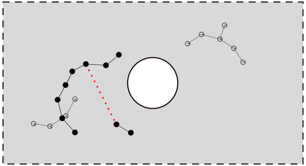

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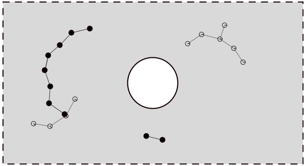

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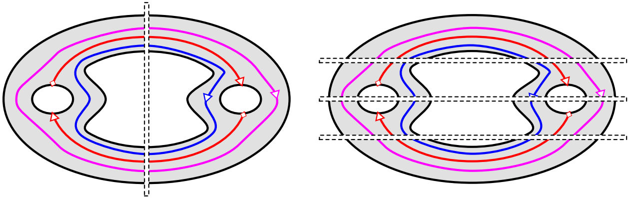
Informal Tidbits: Refinement/Coarsening

Refinement: Splitting and recombining continuous modes is useful:



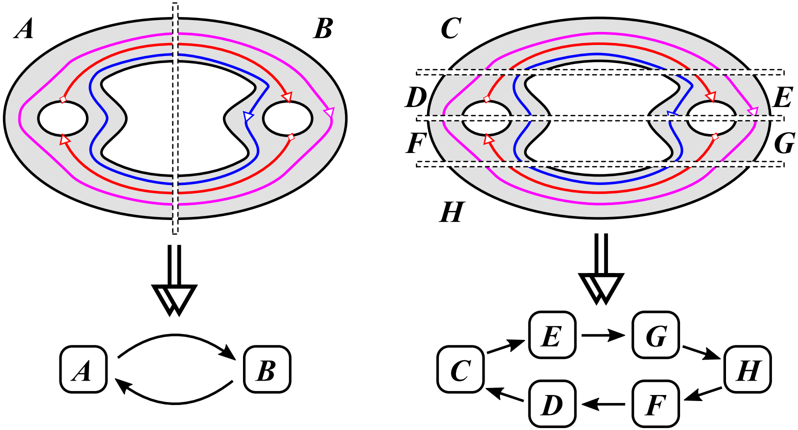
- Time as a hybrid system, trajectories as maps of time into a state space.

  $\rightsquigarrow$ *A central principle in all approaches*

- Need *generalized* trajectories to support ill-behaved time sudivisions

Informal Tidbits: Refinement/Coarsening

Coarsening: When is "projection" of a HS to the underlying discontinuous structure *more* informative?

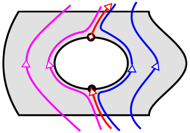


- Methods for bringing topology and hybrid structure into sync?

$\rightsquigarrow$ *This is precisely what happened to us in [14]!*

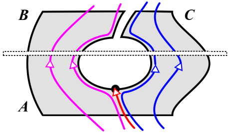
Informal Tidbits: Refinement/Coarsening

- Moving away from graphs as discrete models of hybrid structure? (a "Conley decomposition"?)



Fixed points are two-dimensional simplices?

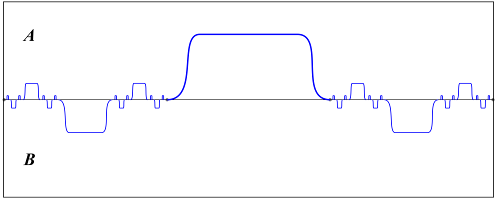
- Probabilistic aspects of hybrid structure? (Entropy??)



Probability of arrival in B given A or given C?

Informal Tidbits: It's About Time

- Generalized Hybrid Time Domains (HTD)?



A smooth "Cantor-themed" curve between two domains...

- Reformulate HTDs to facilitate trajectories of this form?

$\rightsquigarrow$ *MORE admissible solutions!*

- Then we need to replace graphs-of-modes with covers-by-modes!

$\rightsquigarrow$ *An additional vote for replacing graphs with complexes?*

Thank You!

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