Designing Free-form Optics with help from geometry and calculus of variations. Theory and computational algorithms

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FREE-FORM LENSES ANS MIRRORS

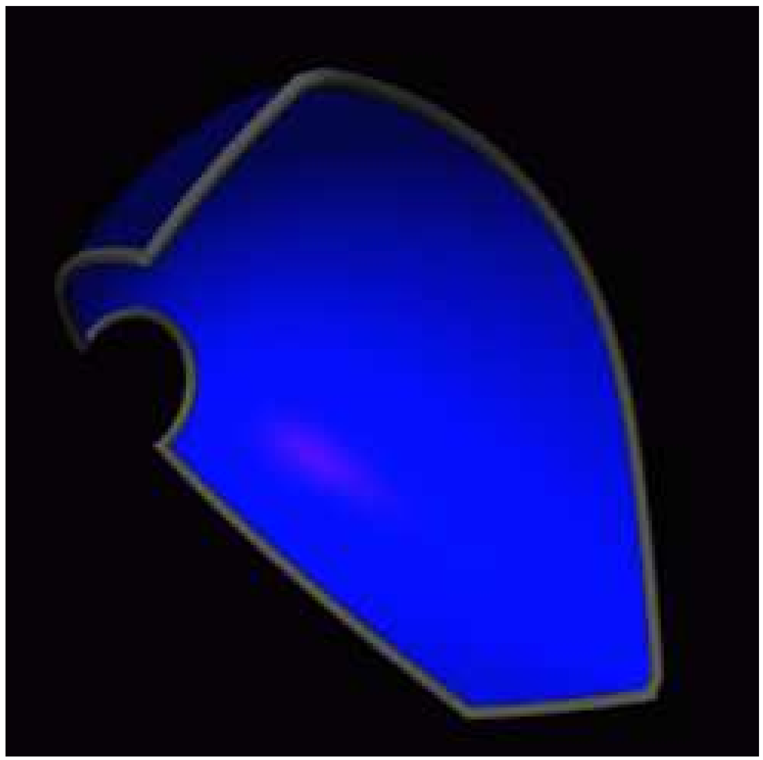
Problems of design of lenses and mirrors for redirecting and redistributing energy have been solved usually under **a priori assumption of rotational/rectangular symmetry**.

In many applications such as materials processing (welding, cutting, drilling), energy concentrators, medicine, illumination, antennas, computing lithography, laser weapons, optical data storage/imaging ... the **assumption of rotational/rectangular symmetry** is overly restrictive and leads to optical systems with low energy efficiency

Our goal:

Develop reliable and fast computational methods for design of free-form lenses/mirrors without a priori symmetry assumptions!

A free-form mirror. The mirror below transforms a sinusoidal shaped radiation pattern from a point source into a uniform planar far-field distribution; designed by V. Oliker; I/O data provided by J. C. Miñano, P. Benítez;



Design of free-form optics requires construction of maps with controlled Jacobian between input and output bundles of rays in space.

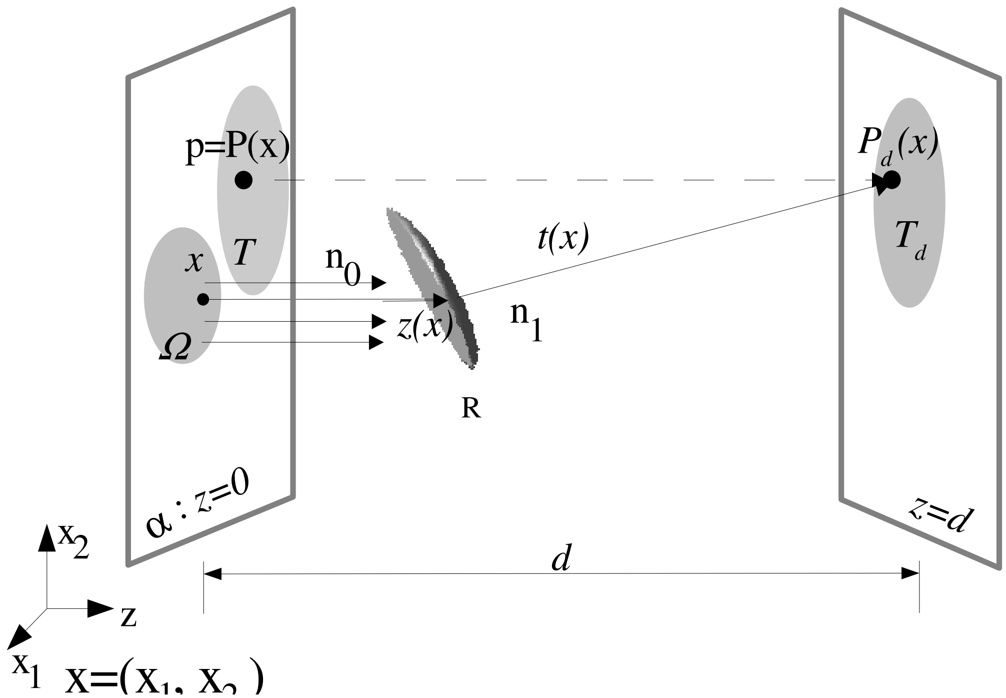
Two basic approaches: 1. **Numerical Optimization** of an ad hoc merit function(s); heuristic, designer-dependent, the solution is a local optimum

2. **Direct methods:** find a map with controlled Jacobian between given input-output 2D regions in space realizable by an optical system

- (a) The SMS method; heuristic, no control of Jacobian; J. C. Miñano, P. Benítez et al.

- (b) Geometric and variational (mass transport) methods (solving PDE's of Monge-Ampère type)), V.I. Oliker with collaborators and students: L. Caffarelli, S. Kochengin, T. Glimm, J. Rubinstein, G. Wolansky

A (sample) refractor problem



Problem: Determine R such that for given refraction indices $n_0,\ n_1$, the incoming plane wave of cross-section $\bar{\Omega}$ with intensity distribution $I(x)$ is transformed into a bundle of rays irradiating at a given $\bar{T}_d$ with prescribed intensity distribution $L(p)$.

Philosophy of direct geometric methods:

1. Recognize special surfaces (i.e. quadric(s), Cartesian ovals,...) suitable for the problem (These usually solve the problem if one of the given intensities is replaced by a measure concentrated at one point)

2. Approximate the given intensities by finite sums of Dirac masses and describe classes of free-form admissible sub-/supersolutions as lower and upper envelopes of such special surfaces (This defines admissible convex/non-convex solutions, and often very useful Fermat-like functionals!)

3. Solve the resulting problem, go to a denser sets of Dirac masses approximating the intensities and solve again. Iterate.

Provably convergent computational methods for design of free-form mirror/lenses:

(a) An iterative method based on a monotone variation of parameters defining special surfaces has been developed by V. Oliker partly in joint works with L. Caffarelli and S. Kochengin; This method is very general and intuitive and the procedure is guaranteed to converge to the true solution (in a given class a priori chosen by the user); Insufficient in problems requiring high resolution (slow when the number of required special surfaces is of order $10^6 - 10^{12}$).

This approach is currently in wide use in the optics community.

(b) A new approach is under development by V. Oliker (initially, partly with T. Glimm). The numerical scheme is guaranteed to converge to the true solution (in classes a priori chosen by the user); in principle, it allows determination of tens of thousands of data points on each mirror/lens (**required for accuracy and high image resolution.**)

MAIN CHALLENGES:

(i) **Computational Efficiency:** Problems of optimal transportation type with $10^8 - 10^{12}$ constraints must be solved efficiently

(ii) Many problems involving single refractive/reflective interfaces for incoherent output bundles of light rays lead to **highly nonlinear cost functionals**; the computational complexity is increased (as compared with (i))

We illustrate our new approach on the "Two-lens problem" arizing in computer lithography

First, we state the **Two-lens problem** as a problem in PDE's.

Second, we connect it with the optimal transport theory

Third, we describe the computational method

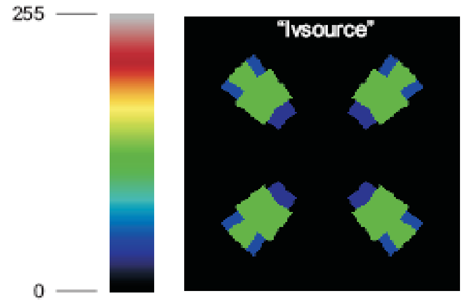
Our mathematical framework is applicable to many optics problems with single or multiple reflectors/refractors.

Test design 2. A freeform two-lens system

A computer lithography problem

Irradiance distribution for mask illumination

Is it possible to generate this pattern by reflective or refractive optics?



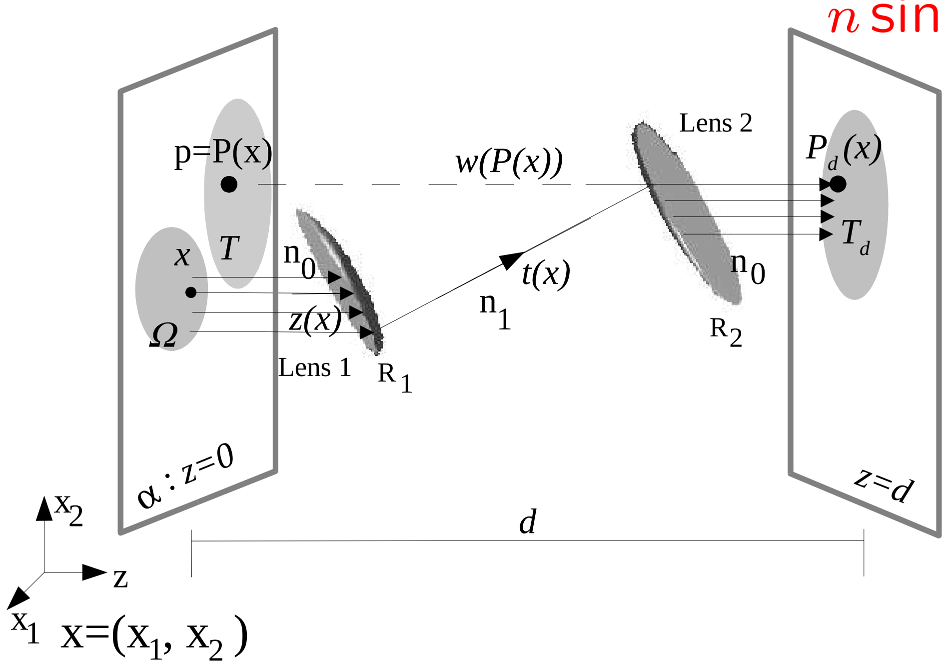
Input: uniform rectangular distribution with dimensions ~ 3mm x 6 mm

Input and out are collimated, propagate in the same direction

John Hoffnagle 16 May 2008

Problem setup and notation

$n \sin \theta_i = \sin \theta_r \left(n = \frac{n_0}{n_1} \right)$



Problem: Determine R_1 and R_2 such that for given refraction indices n_0, n_1, the incoming plane wave of cross-section $\bar{\Omega}$ with intensity distribution $I(x)$ is transformed into a plane wave irradiating at a given $\bar{T}_d$ with prescribed intensity distribution $L(p)$.

Some of the earlier (related) work on rotationally symmetric (RS) lenses: B.R. Frieden ('65), J.L. Kreuzer ('69), P.W. Rhodes & D.L. Shealy ('80), W. Jiang, D.L. Shealy & J.C. Martin ('93), J. A. Hoffnagle & C.M. Jefferson ('00–'05).

A two-lens RS system designed by Hoffnagle & Jefferson was fabricated by QED Technologies ('03?). The authors received the 2003 Kingslake Medal and Prize for this work.

Previous (most relevant) work on free-form lenses: H. Ries - J. Muschaweck, 2002 (no details); J. Rubinstein - G. Wolansky, 2007-2008 (single lens, $0 < n < 1$), - Weighted least action + use of work by Oliker-Glimm;
V. Oliker, 2005 (two- and single lens, $n > 1,\ 0 < n < 1$),- Geometric methods

FROM OPTICS TO PDE

Assuming **the geometrical optics approximation**, the following three laws are used to derive the equations for functions describing the lens surfaces:

- **Snell's (the refraction) law**
- **Conservation of Energy Along Infinitesimal Tubes of Rays**
- **Constancy of the Optical Path Length (OPL) (for coherent input and output beams)**

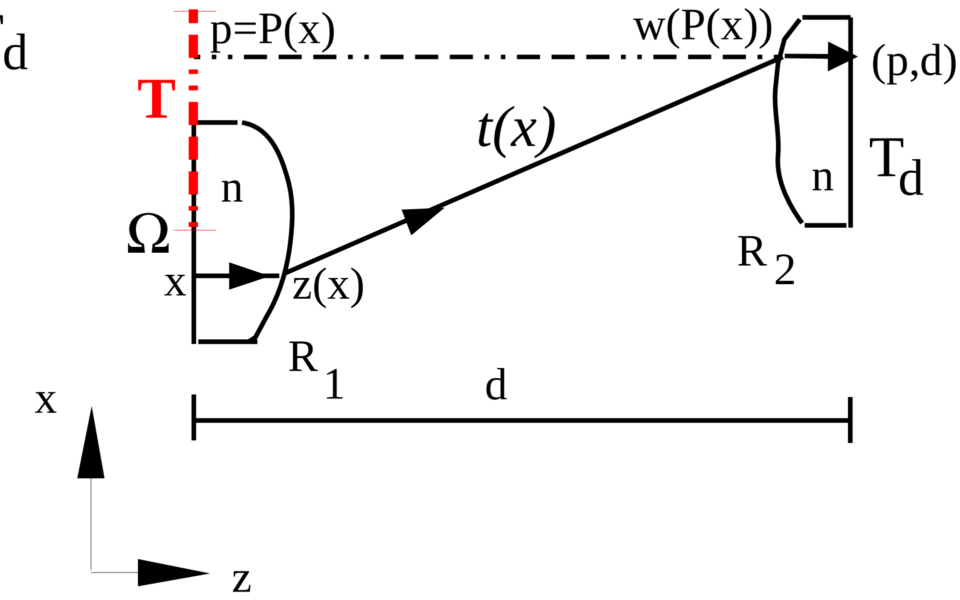
Notations:

Optical path length (OPL): $l = nz(x)+t(x)+n[d-w(P(x)]=\text{const}$

Reduced OPL: $\beta := l-nd = n[z(x)-w(p)] + \sqrt{(x-p)^2 + [w(p)-z(x)]^2}$

$T = \mathrm{proj}_{\{z=0\}} T_d$

$n = n_0/n_1$



Let $(R_1, R_2) = (z(x), w(p))$**. Put** $M(z) := \sqrt{1 + (1 - n^2)|\nabla z|^2}$.

The refraction law gives the refracted direction at the lens surface R_1**:**

$$\omega(x) = n\mathbf{k} + \frac{\mathbf{-n + M(z(x))}}{\sqrt{1 + |\nabla z(x)|^2}}\mathbf{N(x)}.$$

The refraction law and $OPL = const$ **give the refractor map:**

$$P(x) = x - \frac{\beta \nabla z(x)}{M(z(x))} : \bar{\Omega} \to \bar{T}$$

The energy conservation law:

$$L(P(x))|J(P(x)| = I(x), \quad (J \text{ is the Jacobian}).$$

Due to constancy of the OPL, **the second lens is given by**

$$w(P(x)) = -\frac{\beta}{n^2 - 1}\left[n + \frac{1}{M(z(x))}\right] + z(x).$$

The PDE problem

For bounded planar regions $\Omega, T \subset \alpha$, input intensity I, defined on $\bar{\Omega}$, and output intensity L, defined on $\bar{T}$, **find** $z \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that, the map

$$P(x) = x - \frac{\beta \nabla z(x)}{M(z(x))} : \bar{\Omega} \to \bar{T} \text{ is onto,}$$

and

$$L(P)\frac{\det\left\{M(z)\left[\mathrm{Id} + (1-n^2)\nabla z \otimes \nabla z\right] - \beta\, \mathrm{Hess}(z)\right\}}{M^4(z)} = I \quad \text{in} \quad \Omega.$$

This PDE is of Monge-Ampère type. Such equations are notoriously hard to investigate theoretically and solve numerically.

We were able to reduce this PDE problem to a variational problem and establish that:

I. For $n > 1$ and $\beta < 0$ there are two classes of (weak) solutions for the same data and these solutions can be constructed as lower and upper envelopes of suitable hyperboloids of revolution. One class of solutions always consists of two lenses whose active surfaces are, respectively, concave and convex;

II. The solutions in the second class give two lenses with active surfaces which may be neither convex nor concave.

III. **For $0 < n < 1$ and $\beta > 0$ there are two classes of solutions for the same data and these solutions can be constructed as lower and upper envelopes of suitable ellipsoids of revolution. One class of solutions always consists of a lens one side of which is concave and the other convex**

IV. **The solutions in the second class are lenses whose sides may be neither convex nor concave.**

Theorem 1. Let $\Omega, T \subset \alpha$ be bounded domains, $I \in L^1(\bar{\Omega}),\ L \in L^1(\bar{T})$, $I, L \geq 0$, and $\int_{\bar{\Omega}} I(x)dx = \int_{\bar{T}} L(p)dp \ \neq 0$. Assume $n > 1$ and $\beta < 0$. The problem

$$\mathcal{F}(z, w) := \int_{\bar{T}} w(p)L(p)dp - \int_{\bar{\Omega}} z(x)I(x)dx \longmapsto \text{min on } Adm_A$$

has a minimum $(z_{\text{min}}[concave], w_{\text{min}}[convex])$ which is of type A:

$$z_{\text{min}}(x) = \inf_{p \in \bar{T}} \left\{ \frac{\beta n - c(x,p)}{n^2 - 1} + w_{\text{min}}(p) \right\}, \ w_{\text{min}}(p) = \sup_{x \in \bar{\Omega}} \left\{ -\frac{\beta n - c(x,p)}{n^2 - 1} + z_{\text{min}}(x) \right\}.$$

The minimizer is unique on connected components of $spt I$ (up to an additive constant). In addition,

$z \in Lip(\bar{\Omega})$, $w \in Lip(\bar{T})$, $|\nabla z|, |\nabla w| < 1/\sqrt{n^2 - 1}$ and a.e. in $\bar{\Omega}$

$$P(x) = x - \frac{\beta \nabla z(x)}{M(z(x))} : \bar{\Omega} \to \bar{T} \quad \left(M(z) := \sqrt{1 + (1 - n^2)|\nabla z|^2} \right).$$

Theorem 2. A two-lens system (z, w) of type A with the refractor map P defined by (z, w) is a weak solution (of type A) of the two-lens problem **iff** it is a minimizer of the above variational problem.

Notes. **(i)** The Fermat-like functional $\mathcal{F}(z, w)$ is the mean horizontal distance between the lenses with the average weighted by intensities.

(ii) The special class of maps (refractor maps) is "built" into the constraints.

Computational algorithms

With the optimal transport approach, using the dual formulation one is required to solve a min (or max) problem with $M * N$ number of linear constraints, where M and N is, respectively, the number of vertices in domains $\bar{\Omega}$ and $\bar{T}$. For example, in the two-lens problem for the grids with $M\ 40,000$ and $N\ 40,000$ the full set of constraints is of order $1.6 * 10^9$. In the primal formulation that number is the number of variables.

In the refractor problem (see p. 4) for a suitably defined primal problem the cost, in addition to the variables $(x, p) \in \bar{\Omega} \times \bar{T}$, depends nonlinearly also on the (unknown) phase. For the dual problem this nonlinearity enters the constraints.

For the two-lens problem treated in Theorem 1 we have:

Assume that both $\bar{\Omega}$ and $\bar{T}$ are rectangular and discretized so that $\bar{\Omega}^D$ has $M = M_1 \times M_2$ nodes $\{x_{ij}\}$ and $\bar{T}^D$ has $N = N_1 \times N_2$ nodes $\{p_{kl}\}$. Let I_{ij} and L_{kl} be some suitable discretizations of intensities I and L. Then the discrete versions of the functional and constraints are, respectively,

$$\text{Minimize} \quad \sum_{k,l} w_{kl} L_{kl} - \sum_{i,j} z_{ij} I_{ij} \tag{1}$$

under constraints

$$w_{kl} - z_{ij} \geq -c(x_{ij}, p_{kl}) \ \text{ for all } i, j, k, l. \tag{2}$$

This is is a linear programming problem on $M \times N$ grid with $M + N$ unknowns w_{kl}, z_{ij} and $M \times N$ constraints in (2). In particular, if $M = N$ then the number of constraints grows quadratically with M.

To overcome this difficulty we introduce progressively refined sequences of grids combined with effective means for recomputing the solution at different levels of resolution, similar to numerical schemes in multi-resolution methods. The key observation is that for an optimal solution only a fraction of the constraints in (2) are active. This is true for a grid of any resolution. Consequently, when passing from a grid with M nodes to a grid with $M'(> M)$ nodes we keep the set of constraints from the previous run and add to it the constraints for nodes only in predefined neighborhoods of active nodes. In this scheme the number of nodes grows linearly and the number of constraints grows only with order $\approx M^{1+\varepsilon}$ with some $0 < \varepsilon \ll 1$. A similar idea was used for solving linear programming problems with a fixed finite number of variables and infinite number of constraints. Since in our case the number of variable does not remain fixed with grid refinement, the application of this idea is not immediate.

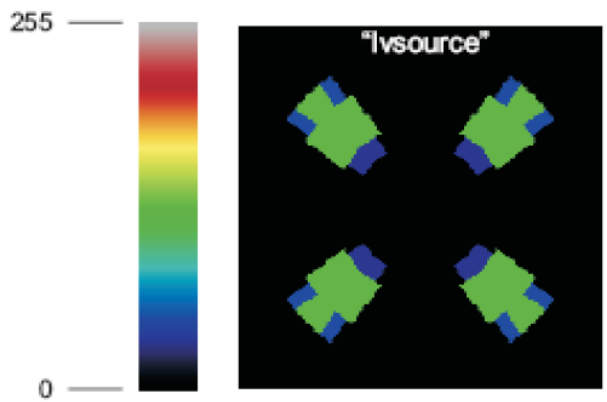
One can include in this scheme an intermediate step at which constraints are added for the same grid until a certain tolerance is achieved. After that the grid is refined and the process is repeated. An obvious advantage of this approach is that for each grid only a small fraction of admissible solution candidates is examined and the solution variables are updated at each step on the entire grid. However, more iteration steps are needed to achieve high accuracy.

Investigation of convergence and efficient implementations of this approach as well as its extension to problems of the "refracting lens" type is a part of the current work.

Test case. Determination of lenses for a two-lens optical system for reshaping the irradiance distribution of a laser beam.

Irradiance distribution for mask illumination

Is it possible to generate this pattern by reflective or refractive optics?



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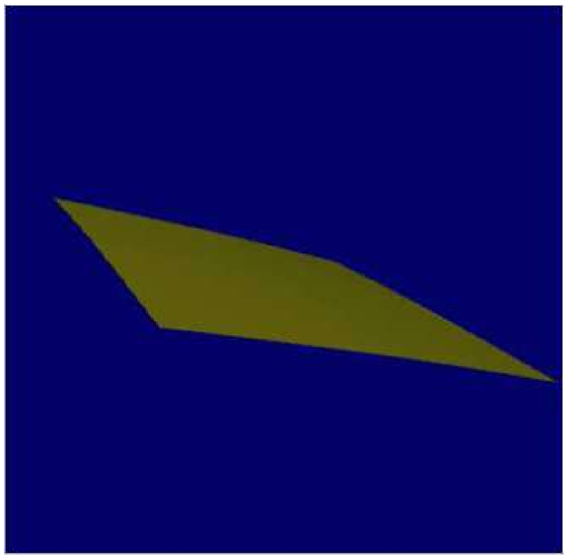
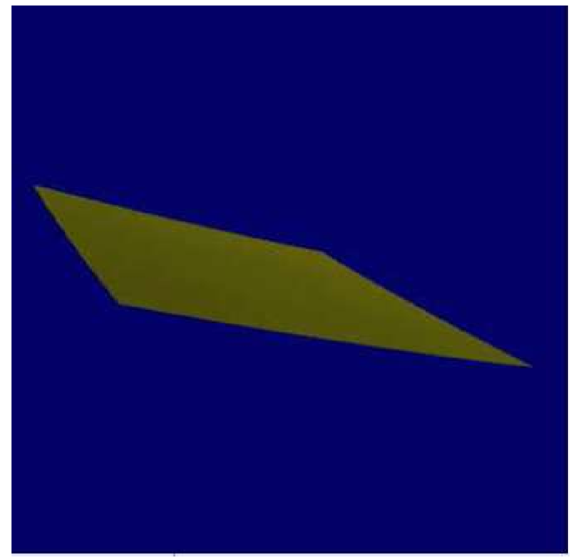
John Hoffnagle 16 May 2008



Lens 1

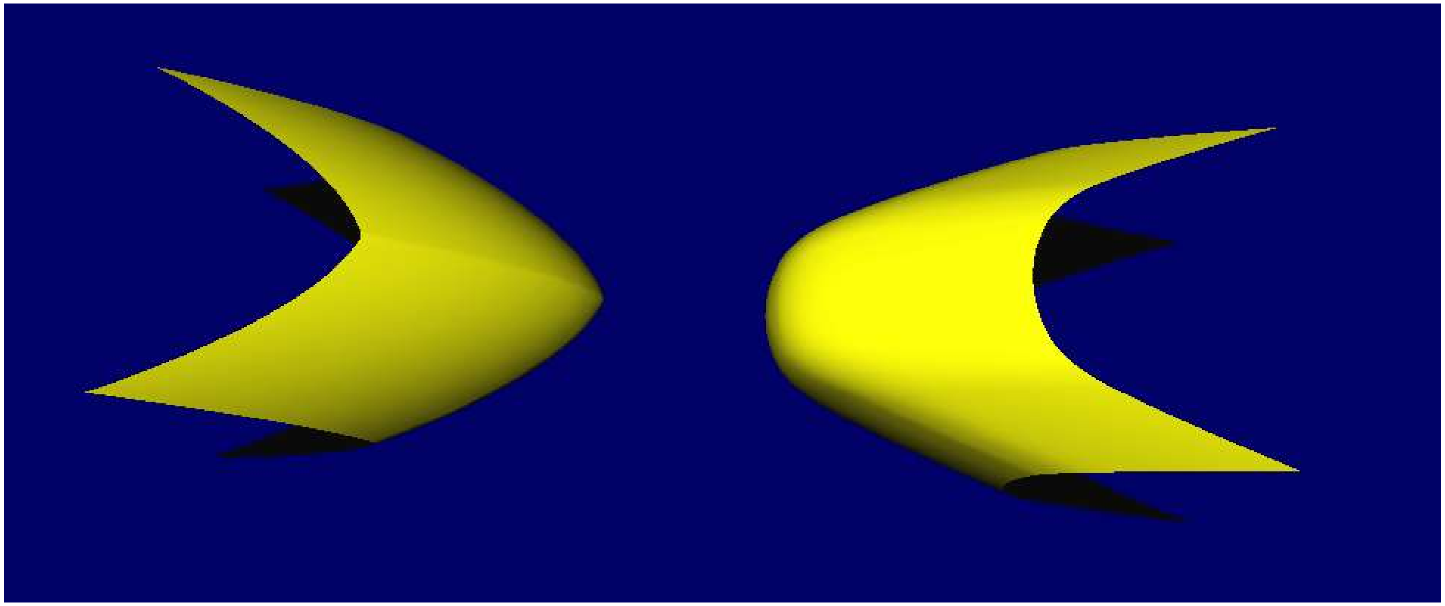


Lens 2

Computed lenses.

Grids: 198×198 square grids over cross sections of the input and output beams.

About $40,000$ grid points on each set.



Lenses 1 and 2 (times 100)

Parts of the work on the two-lens system described above appeared in: V. Oliker, Arch. Rational Mech. Anal., 201(2011), pp.1013-1045.

Parts of the work on using Optimal Transport methods in the near-field reflector problem appeared in: T. Graf and V. Oliker, Inverse Problems, 28 (2012), pp. 1-15.

Current work: In addition to the work on development of fast computational algorithms for constructing numerical solutions of free-form design problems, we are working also on development of a variational approach to design of free-form single refractor systems producing required irradiance on a given target in the near-field (as shown on p. 4).

Papers submitted for publication and in preparation for submission:

1. V. Oliker, J. Rubinstein, G. Wolansky, Ray mapping and illumination control, Submitted.

In this paper we investigate the ray mappings for systems with special symmetries and show that in such cases the same ray mapping found independently of the optical problem can be realized by several different reflective and refractive optical systems.

2. V. Oliker, J. Rubinstein, G. Wolansky, Determination of free-form refractive interfaces from near-field intensity data, In preparation.

We investigate the problem of design of a single refracting surface $u(x),\ x \in \bar{\Omega}$, that refracts an incident collimated beam of irradiance distribution $I(x)$

such that the irradiance distribution created by the refracted beam on a given plane is $L(p), p \in \bar{T}$; see the illustration on p. 4. Here, unlike the earlier example of a two-lens design, we do not prescribe the phase ($\equiv$ optical path length) of the refracted beam (otherwise the problem is overdetermined). The design goal can be achieved via a single refractor, but the phase is now an additional unknown function. Sufficient conditions for existence have been established. These conditions are physically meaningful and linked with excluding lenses with total internal reflections. This case will be considered separately. Development of efficient computational algorithms for this class of problems is a part of the current work.

The End