UNIVERSITY OF Southampton

# THE VIRTUAL FIELDS METHOD

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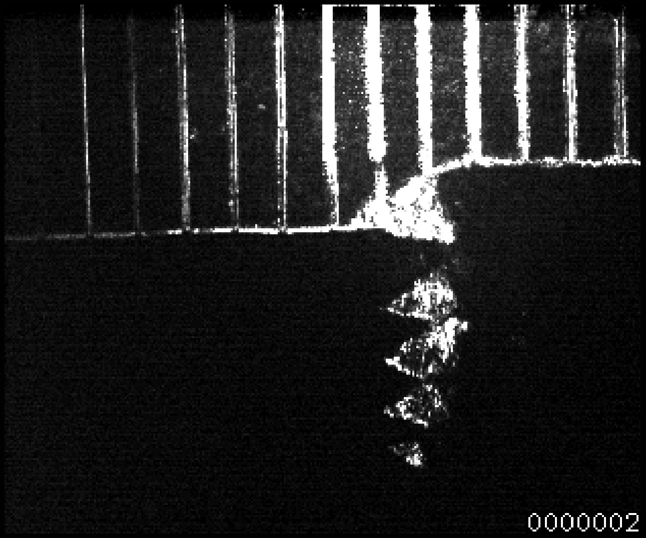
# Course overview

- Introduction and basic idea
- Principle of virtual work
- The Virtual Fields Method (VFM): principle in elasticity
- Complements on the VFM
- Conclusion

# Introduction and basic idea

# Introduction

- Huge progress in computational mechanics
  - Simulation of machining
    - Large strains elasto-plasticity
    - Large strain rates
    - Localization
    - Friction/thermal behaviour




- Problem
  - Many material parameters required
  - How to obtain them?

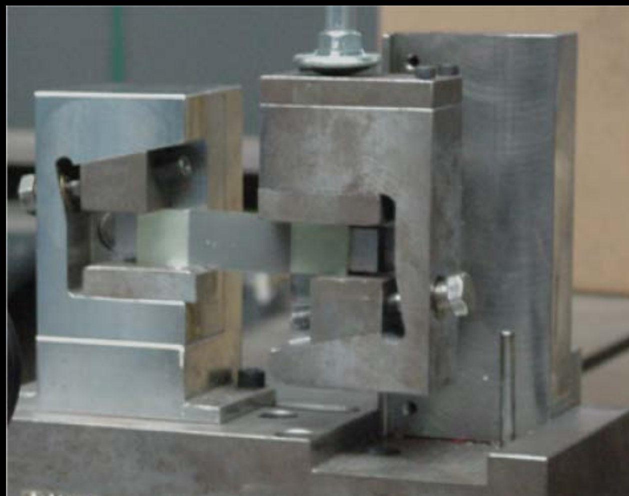
# Introduction

- Standard tests: Tensile test on rectangular specimen
  - Uniform stress state
  - Uniaxial stress strain curve


  - Very poor information
  - Very restrictive assumptions (constraints)

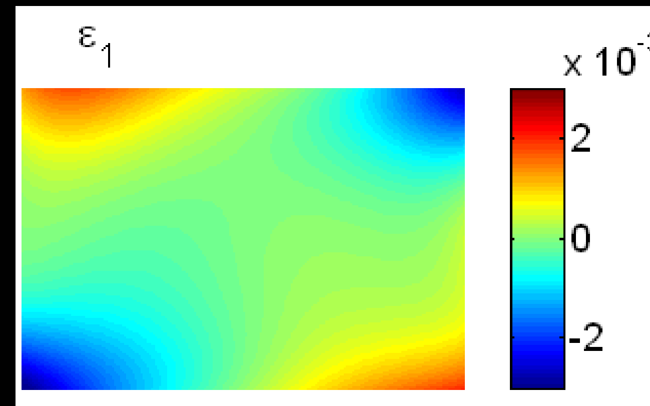
Develop the experimental identification procedures of the future !

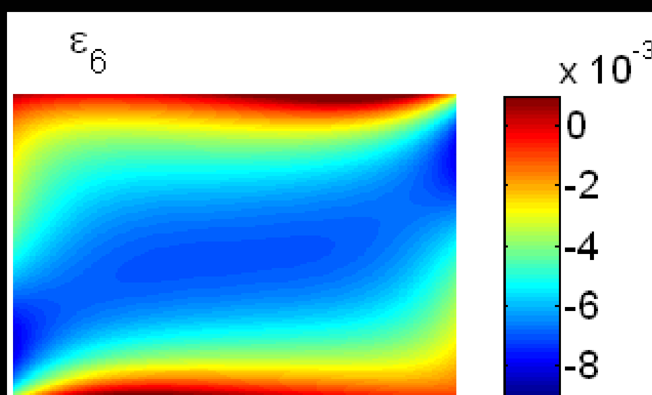
# Alternative

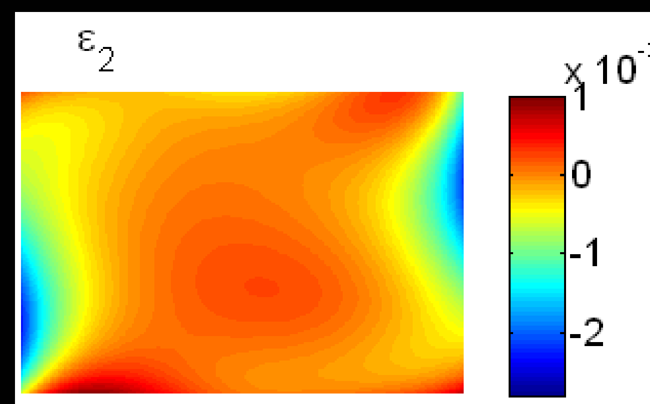


Composite specimen

Grid method



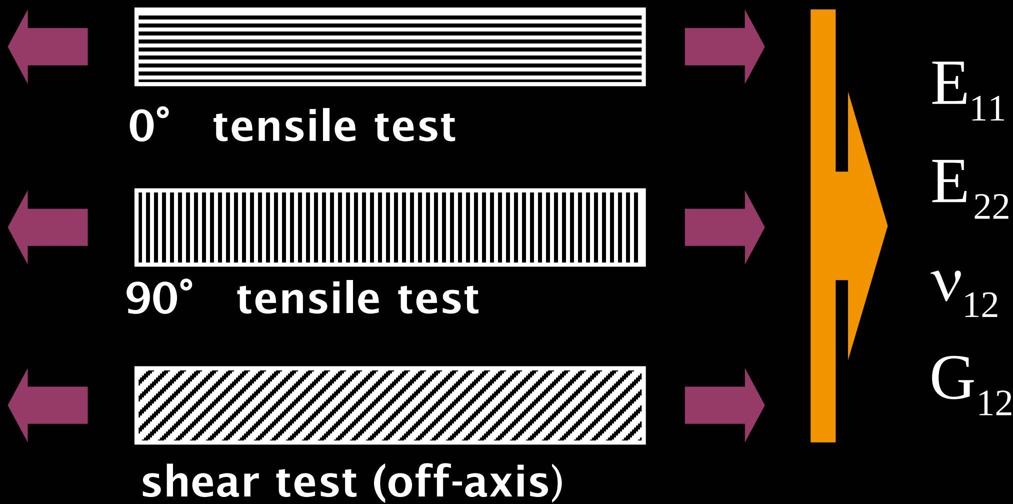




Extract material properties ($Q_{11}$, $Q_{22}$, $Q_{12}$, $Q_{66}$) from strain fields and load

# Motivation

- Extract more information from 1 test

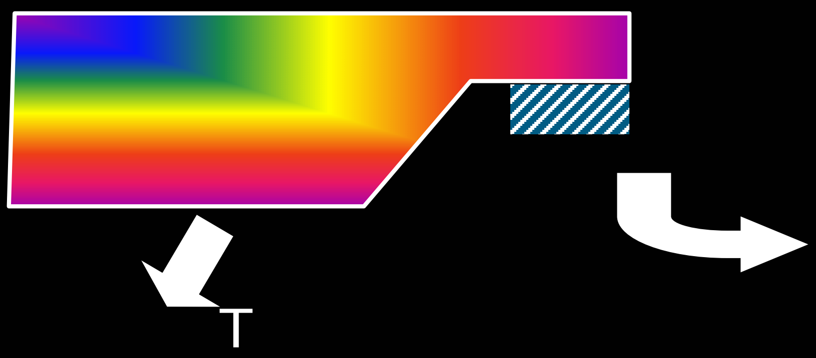



3 tests
4 parameters

Local strain measurements
Uniform stress fields (closed-form solution)
Spatial stress distribution

# Motivation

- Novel strategy for orthotropic elastic moduli measurements




$E_{11}$

$E_{22}$

$\nu_{12}$

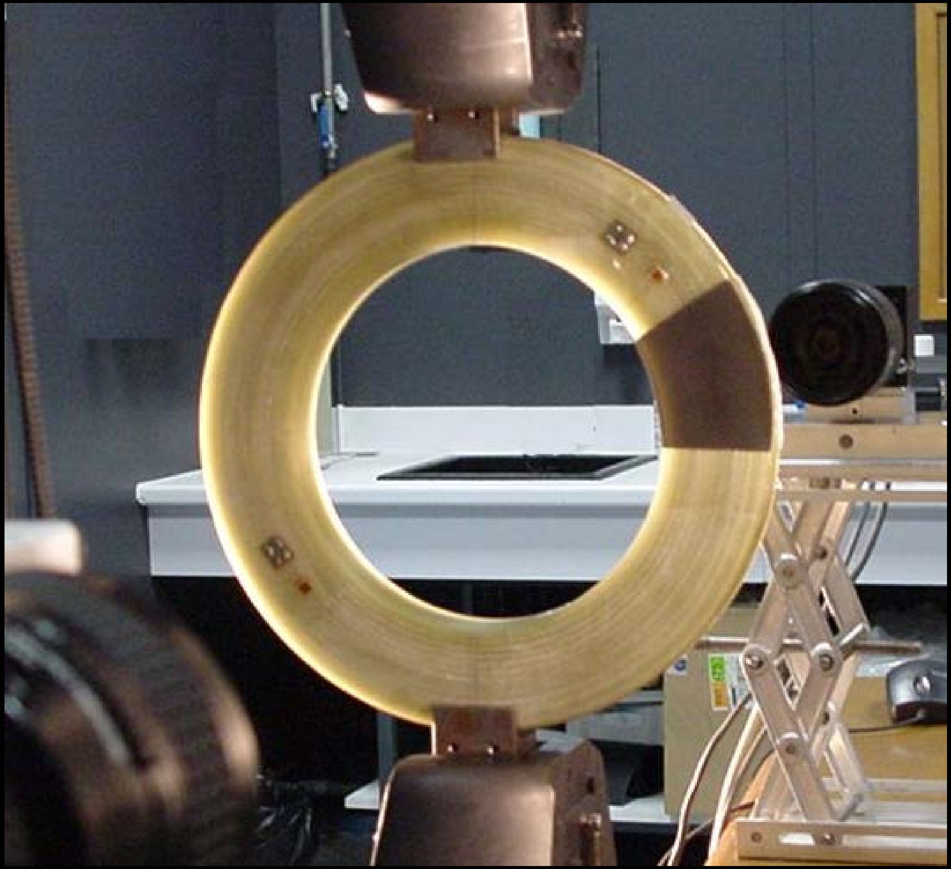
$G_{12}$

**1 test**

Heterogeneous stress fields
(no closed-form solution)

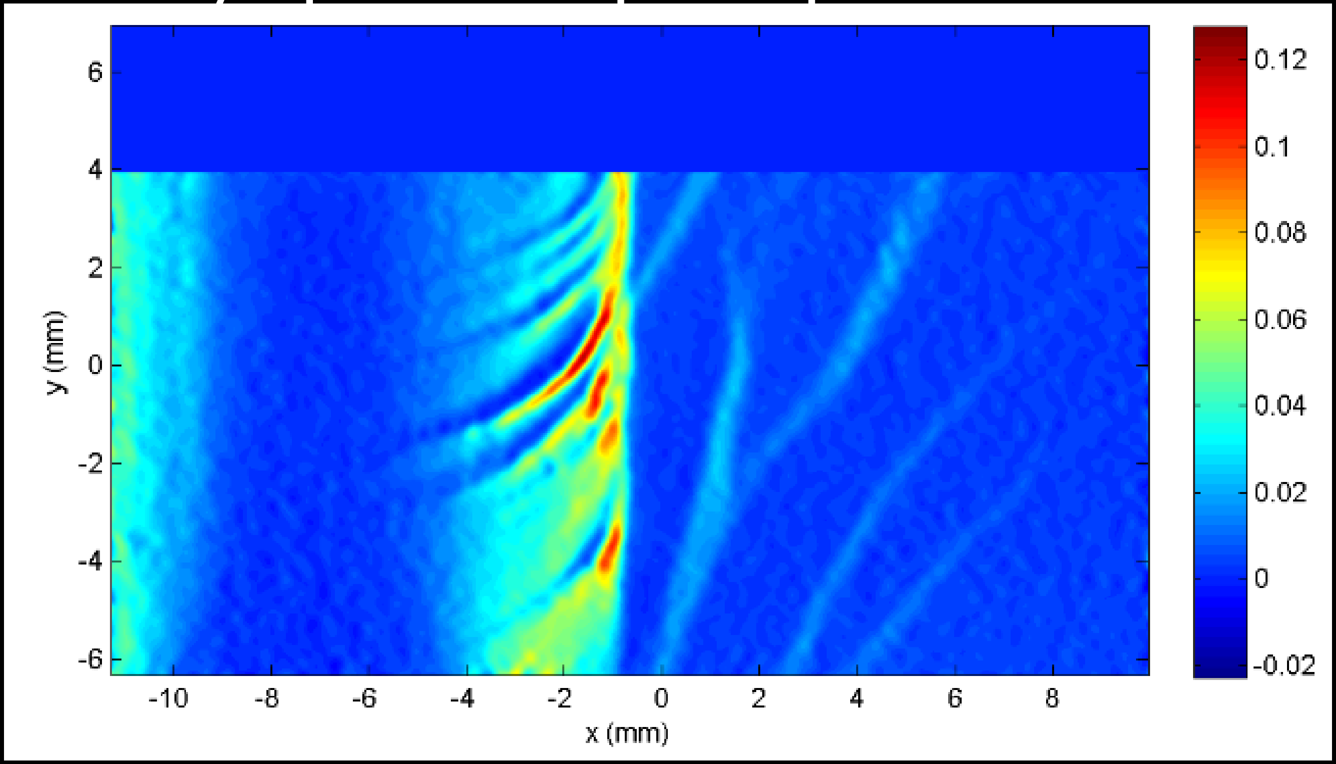
# Motivation

- Complex test geometry



# Motivation

- Heterogeneous materials
  - Identify spatial maps of parameters




Longitudinal strain in a magnesium FSW weld (speckle interferometry)

# Key issues: full-field measurements

- Why?
  - Complex strain state
  - Need for more experimental information
- Interferometry (moiré, speckle, etc…)
  - High sensitivity, high spatial resolution
  - Small displacements (rigid body), low strains and high strain gradients
- White light techniques
  - Image correlation : large displacements, 3D
  - Grid method : intermediate

# Key issues: full-field measurements

- Extraordinary potential
  - From a few measurements points to a few 100.000's!!
- Qualitative / quantitative ?
  - Metrological properties not fully assessed yet
  - Often reduced to visual information
  - Very few quantitative use of the data!
- International effort
  - DIC Challenge (Society for Experimental Mechanics) https://www.sem.org/dic-challenge/

# Key issues: inverse resolution

- Extensive literature on inverse problems (mathematical)
- Very few tools targeted at full-field processing
- Very few groups with expertise in measurements and inverse problems at the same time
- Inversion closely related to measurement performance!
- Huge investigation area, critically underexplored

# Inverse resolution

- Basic equations

I Equilibrium equations (static)

$$\sigma_{ij,j} + f_i = 0$$ + boundary conditions strong (local)

or

$$-\int_V \sigma_{ij}\varepsilon^*_{ij} dV + \int_{\partial V_f} T_i u^*_i dS + \int_V f_i u^*_i dV = 0$$ weak (global)

II Constitutive equations (elasticity) $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

III Kinematic equations (small strains/displacements)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

# Inverse resolution

| | Known | Unknown |
|---|---|---|
| Direct problem | $C_{ijkl}$<br>Geometry<br>Boundary conditions | $\sigma_{ij}, \varepsilon_{ij}, u_i$ |

- Tools for solving this problem
  - Direct integration (closed-form solution)
  - Approximate solutions
  - Galerkin, Ritz
  - Finite elements, boundary elements…
  - etc…

# Inverse resolution

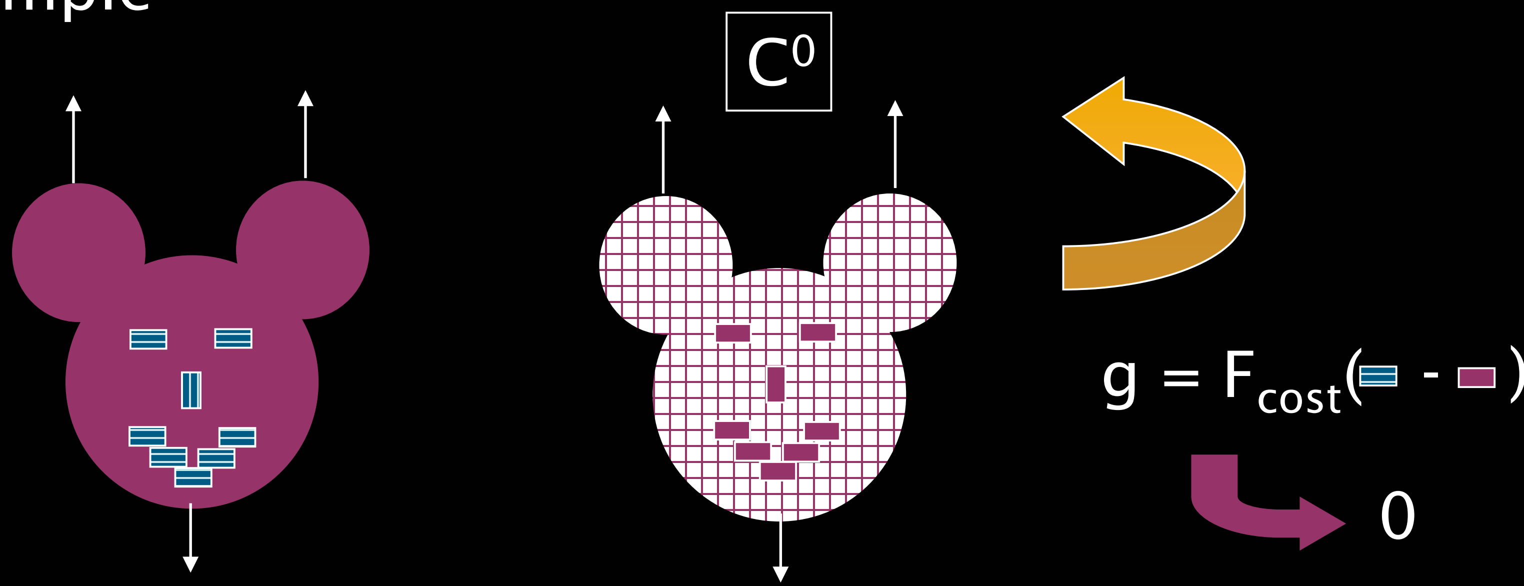
| | Known | Unknown |
|---|---|---|
| Inverse problem | $\varepsilon_{ij}, u_i$ (measured)<br>Geometry<br>Some information on the boundary conditions (load cell) | $C_{ijkl}$<br>$\sigma_{ij}$ |

- Tools for solving this problem
  - Statically determined tests:
    Closed form solution of Eq. I (uncoupled system)
    Force BC, simple geometry
    Ex.: tensile test, bending tests (on rect. beams) etc…

# Inverse resolution

- Tools for solving this problem
  - Model updating
    Idea: iterative use of tool for direct problem (analytical or approximate)

Example



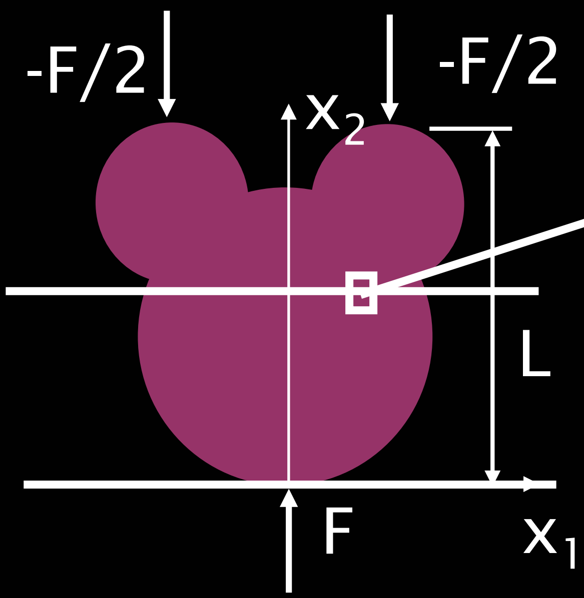
# Inverse resolution

- Model updating
  - Advantages
    - General method (full-field measurements not compulsory)
    - Tools already developed
  - Shortcomings
    - Sensitive to boundary conditions (generally badly known)
    - CPU intensive (for numerical approximations and non-linear equations…)
    - Not fully dedicated to full-field measurements

➡ Alternative tool: the Virtual Fields Method

# Basic idea

- Use global equations (and not local)




$$\varepsilon_{22}(x_1^0, x_2^0) \quad \sigma_{22}(x_1^0, x_2^0)?$$

$$\int_S \sigma_{22} dx_1 dx_3 = -F$$

Integrate over $x_2$ $$\int_V \sigma_{22} dx_1 dx_2 dx_3 = -FL$$

# Basic idea

- Constitutive behaviour

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{11} & 0 \\ 0 & 0 & \dfrac{Q_{11}-Q_{12}}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

In-plane linear elastic isotropy

$$\int_V \sigma_{22} dV = -FL \quad \Rightarrow \quad \int_V (Q_{12}\varepsilon_{11} + Q_{11}\varepsilon_{22}) dV = -FL$$

Material is homogeneous

$$Q_{11}\int_V \varepsilon_{22} dV + Q_{12}\int_V \varepsilon_{11} dV = -FL$$

# Basic idea

- Surface measurements only

Constant strains through the thickness

$$Q_{11}\int_S \underset{?}{\varepsilon_{22}dx_1dx_2} + Q_{12}\int_S \varepsilon_{11}dx_1dx_2 = -\frac{FL}{e}$$

$$\int_S \varepsilon_{22}dx_1dx_2 \approx \sum_{i=1}^{n}\varepsilon_{22}^{i}s^{i}$$

$\mathbf{s^i}$ is the surface of each pixel

n is the number of strain data points

If all pixels have the same size s (usually the case for CCD/CMOS based measurements)

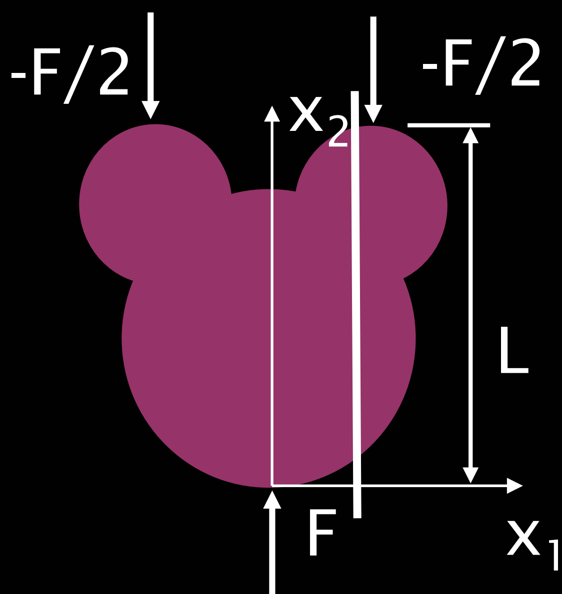
$$\sum_{i=1}^{n}\varepsilon_{22}^{i}s^{i} = s\sum_{i=1}^{n}\varepsilon_{22}^{i} = \frac{S_d}{n}\sum_{i=1}^{n}\varepsilon_{22}^{i} = S_d\overline{\varepsilon_{22}} \qquad \overline{\varepsilon_{22}} = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_{22}^{i}$$

$\mathbf{S_d}$ is the surface of the disc

# Basic idea

- Finally

$$Q_{22}\overline{\varepsilon}_{22} + Q_{12}\overline{\varepsilon}_{11} = \frac{-FL}{eS_d} \qquad Q_{11}\overline{\varepsilon}_{11} + Q_{12}\overline{\varepsilon}_{22} = 0$$




Integrate over $x_1$

$$\int_S \sigma_{11} dx_2 dx_3 = 0 \qquad \int_V \sigma_{11} dV = 0$$

$$\begin{bmatrix} \overline{\varepsilon}_{11} & \overline{\varepsilon}_{22} \\ \overline{\varepsilon}_{22} & \overline{\varepsilon}_{11} \end{bmatrix} \begin{pmatrix} Q_{11} \\ Q_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{-FL}{eS_d} \end{pmatrix}$$

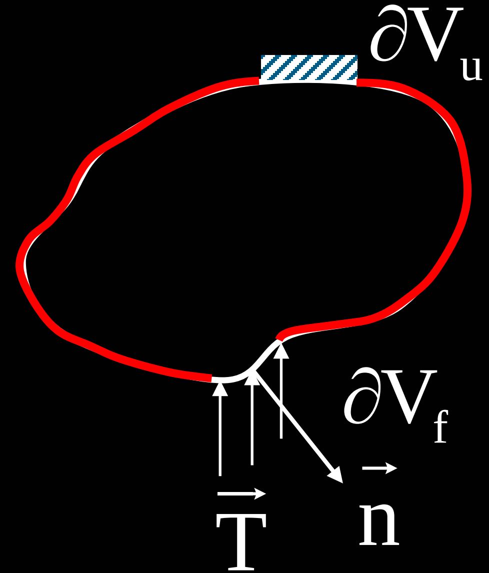
$$Q_{11} = \frac{-FL\overline{\varepsilon}_{22}}{eS_d(\overline{\varepsilon}_{22}^2 - \overline{\varepsilon}_{11}^2)} \qquad Q_{12} = \frac{FL\overline{\varepsilon}_{11}}{eS_d(\overline{\varepsilon}_{22}^2 - \overline{\varepsilon}_{11}^2)}$$

# Basic idea

- What if the disc material is anisotropic?
  - Need for more equations
- Tool to generate integral equilibrium equations: <u>principle of virtual work</u>

# The principle of virtual work

# The Principle of Virtual Work




Equilibrium equation

$$\overrightarrow{\mathrm{div}\,\overline{\overline{\sigma}}} + \vec{f} = \vec{0}$$

$$\sigma_{ij,j} + f_i = 0$$

Boundary conditions

$$T_i = \sigma_{ij} n_j \text{ on } \partial V_f$$

$$u = \overline{u} \text{ on } \partial V_u$$

Let $u_i^*$ be a vectorial function

$$\sigma_{ij,j} u_i^* + f_i u_i^* = 0$$

Integration over the whole domain

$$\int_V \sigma_{ij,j} u_i^* dV + \int_V f_i u_i^* dV = 0$$

# The Principle of Virtual Work

- Integration by part and use of divergence theorem

$$\boxed{-\int_V \sigma_{ij}\varepsilon^*_{ij}dV + \int_{\partial V_f} T_i u^*_i dS + \int_V f_i u^*_i dV = 0}$$

$u^*_i$ Virtual displacement field (cont. and diff)

$\varepsilon^*_{ij} = \frac{1}{2}(u^*_{i,j} + u^*_{j,i})$ virtual strain tensor

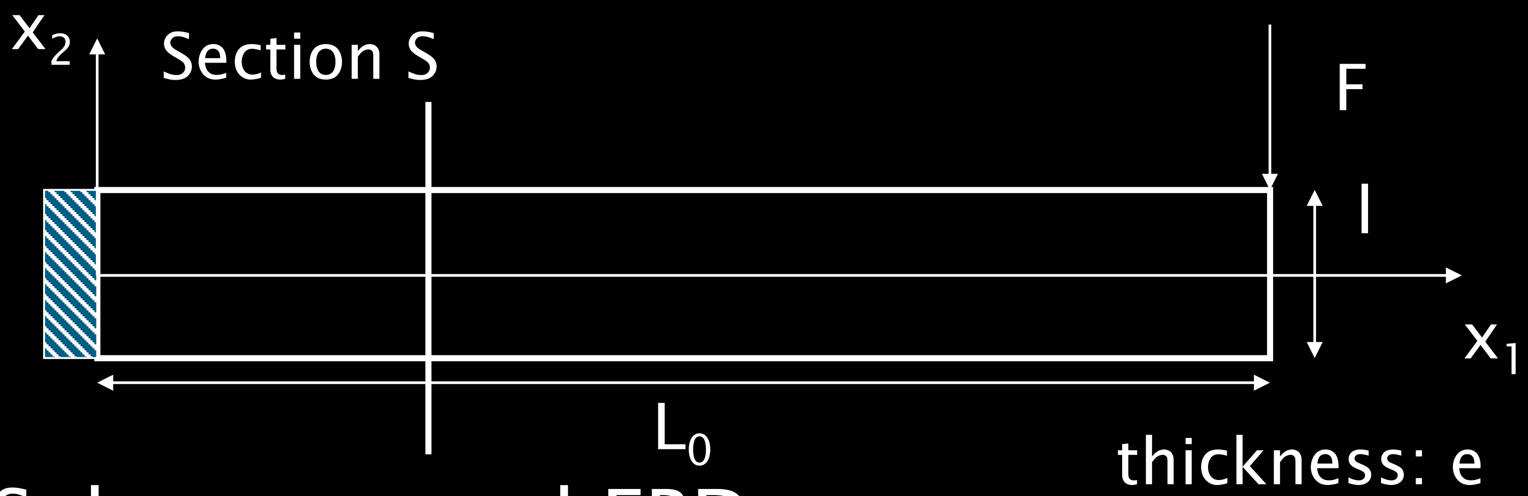
$T_i$ Applied stress vector on solid boundary $\partial V_f$

$f_i$ External volume forces

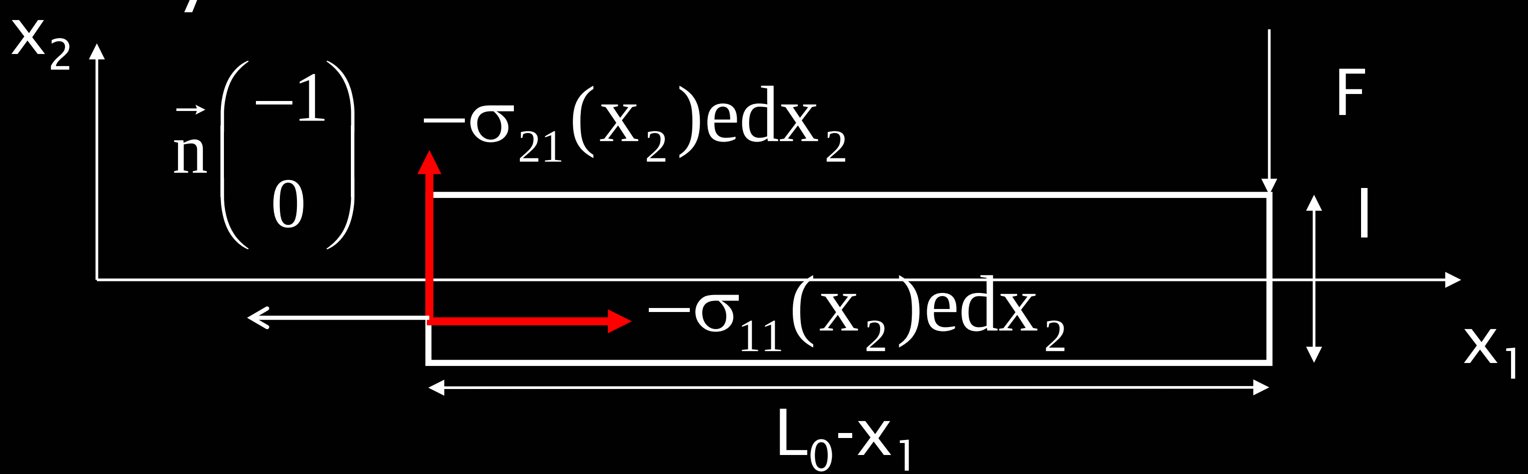
$\sigma_{ij}$ Stress tensor

# Illustration of the PVW

- Cantilever beam

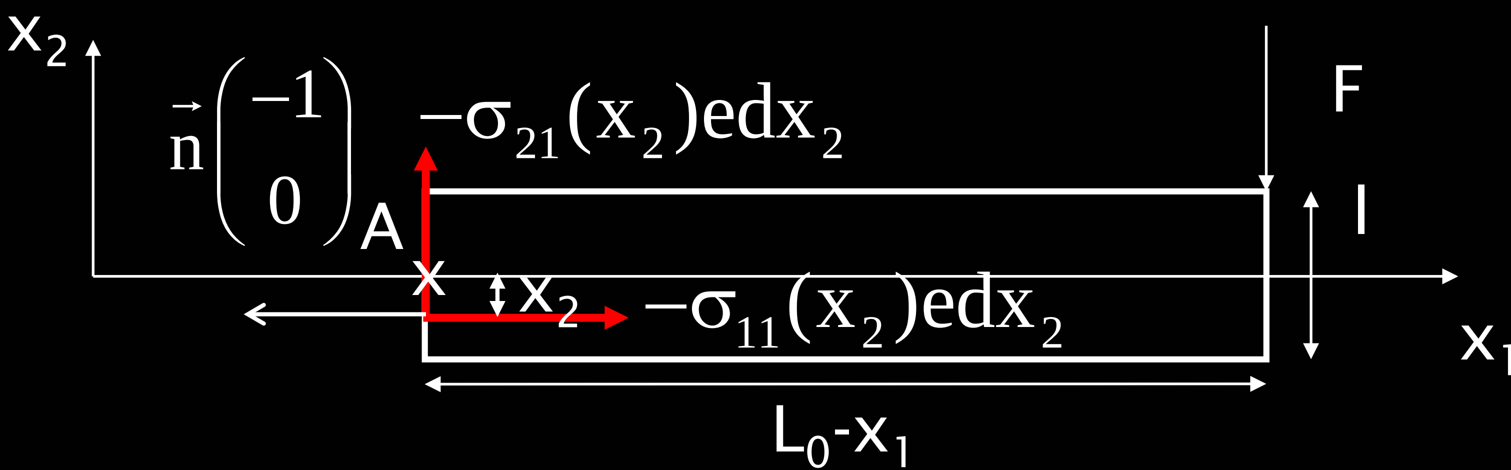



- Subsystem and FBD



# Illustration of the PVW

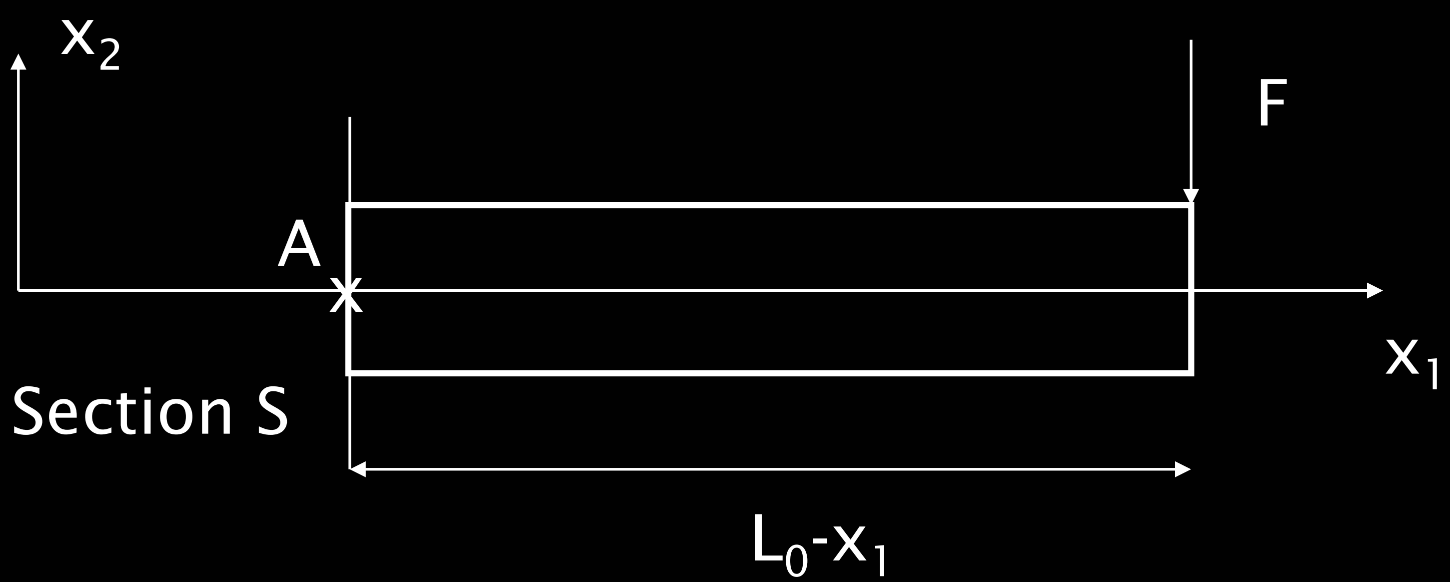
- Resultant of internal forces




$$\vec{F}_i = \begin{pmatrix} -e\int_{-l/2}^{l/2}\sigma_{11}dx_2 \\ -e\int_{-l/2}^{l/2}\sigma_{21}dx_2 \end{pmatrix} \qquad M_{i/A}^{x_3} = e\int_{-l/2}^{l/2}\sigma_{11}x_2dx_2$$

# Illustration of the PVW

- Resultant of external force




$$\vec{F}_e = \begin{pmatrix} 0 \\ -F \end{pmatrix} \; ; \; M_{e/A}^{x_3} = -F(L_0 - x_1)$$

# Illustration of the PVW

- Equilibrium

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \begin{pmatrix} -e\int_{-l/2}^{l/2} \sigma_{11} dx_2 \\ -e\int_{-l/2}^{l/2} \sigma_{21} dx_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum M^{x3} = 0 \quad \Rightarrow \quad F(L_0 - x_1) - e\int_{-l/2}^{l/2} \sigma_{11} x_2 dx_2 = 0$$

$$\int_{-l/2}^{l/2} \sigma_{11} dx_2 = 0$$

$$e\int_{-l/2}^{l/2} \sigma_{21} dx_2 = -F$$

$$e\int_{-l/2}^{l/2} \sigma_{11} x_2 dx_2 = F(L_0 - x_1)$$

# Illustration of the PVW

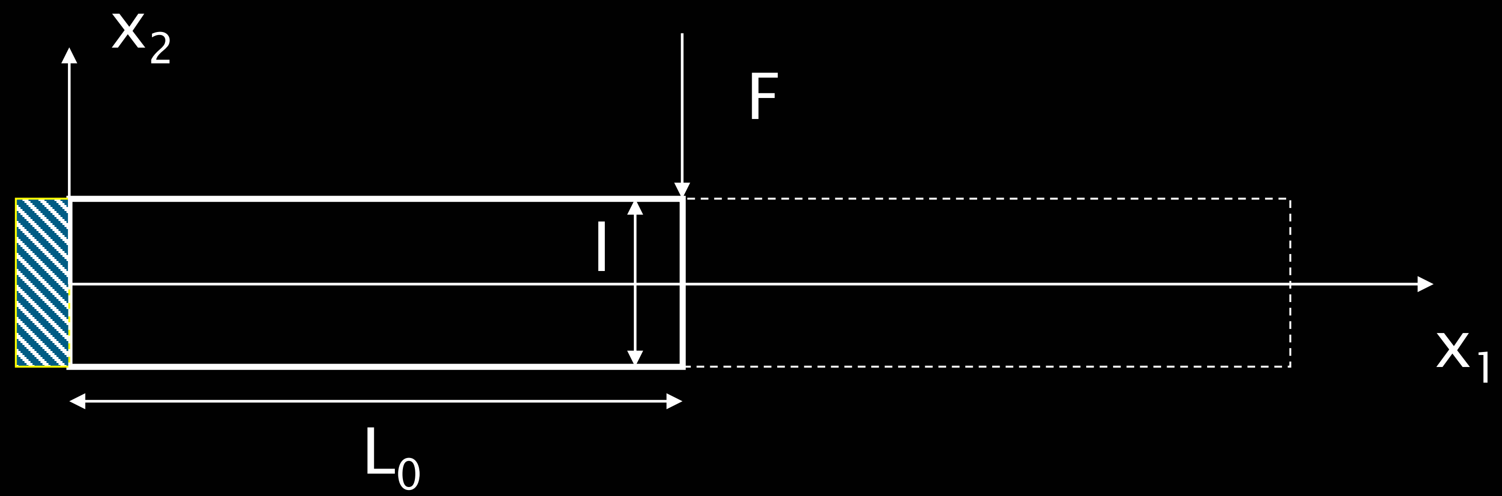
- Valid over any section S of the beam:
  - integration over $x_1$

$$\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}dx_1dx_2 = 0 \qquad \text{Eq. 1}$$

$$e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{21}dx_1dx_2 = -FL_0 \qquad \text{Eq. 2}$$

$$e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}x_2dx_1dx_2 = \frac{FL_0^2}{2} \qquad \text{Eq. 3}$$

# Illustration of the PVW




Principle of virtual work (static, no volume forces)

$$-\int_V \sigma_{ij}\varepsilon^*_{ij}dV + \int_{\partial V_f} T_i u^*_i dS = 0$$

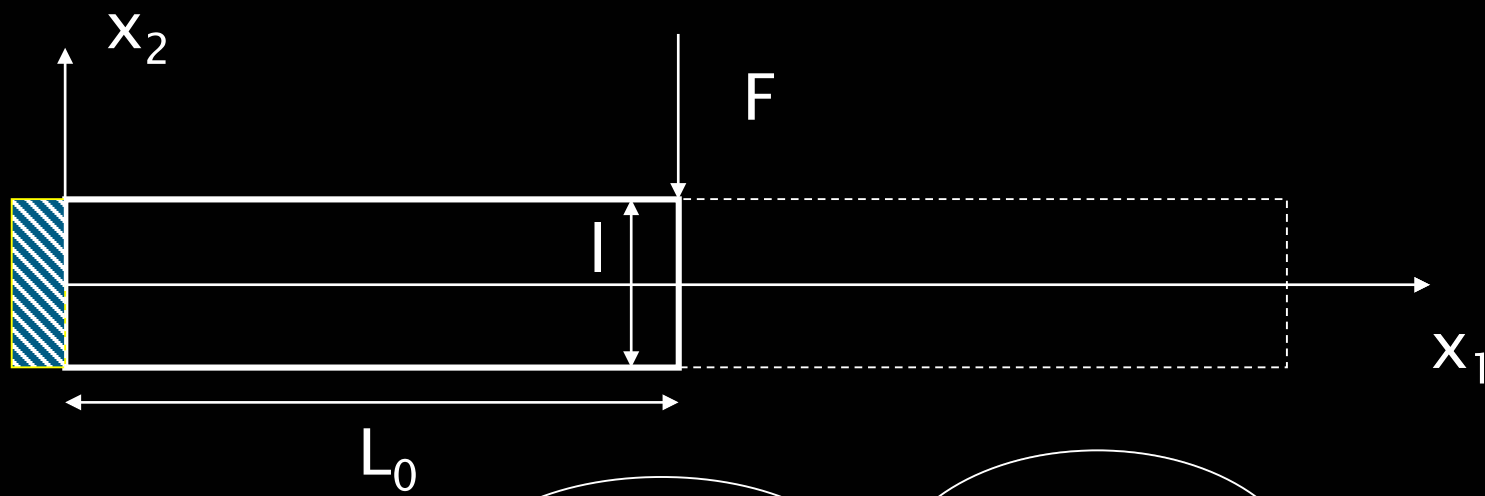
Let us write a virtual field:

$$u^*_1 = x_1 \qquad \varepsilon^*_{11} = 1$$

$$u^*_2 = 0 \quad \Rightarrow \quad \varepsilon^*_{22} = 0$$

$$\varepsilon^*_{12} = 0$$

# Illustration of the PVW



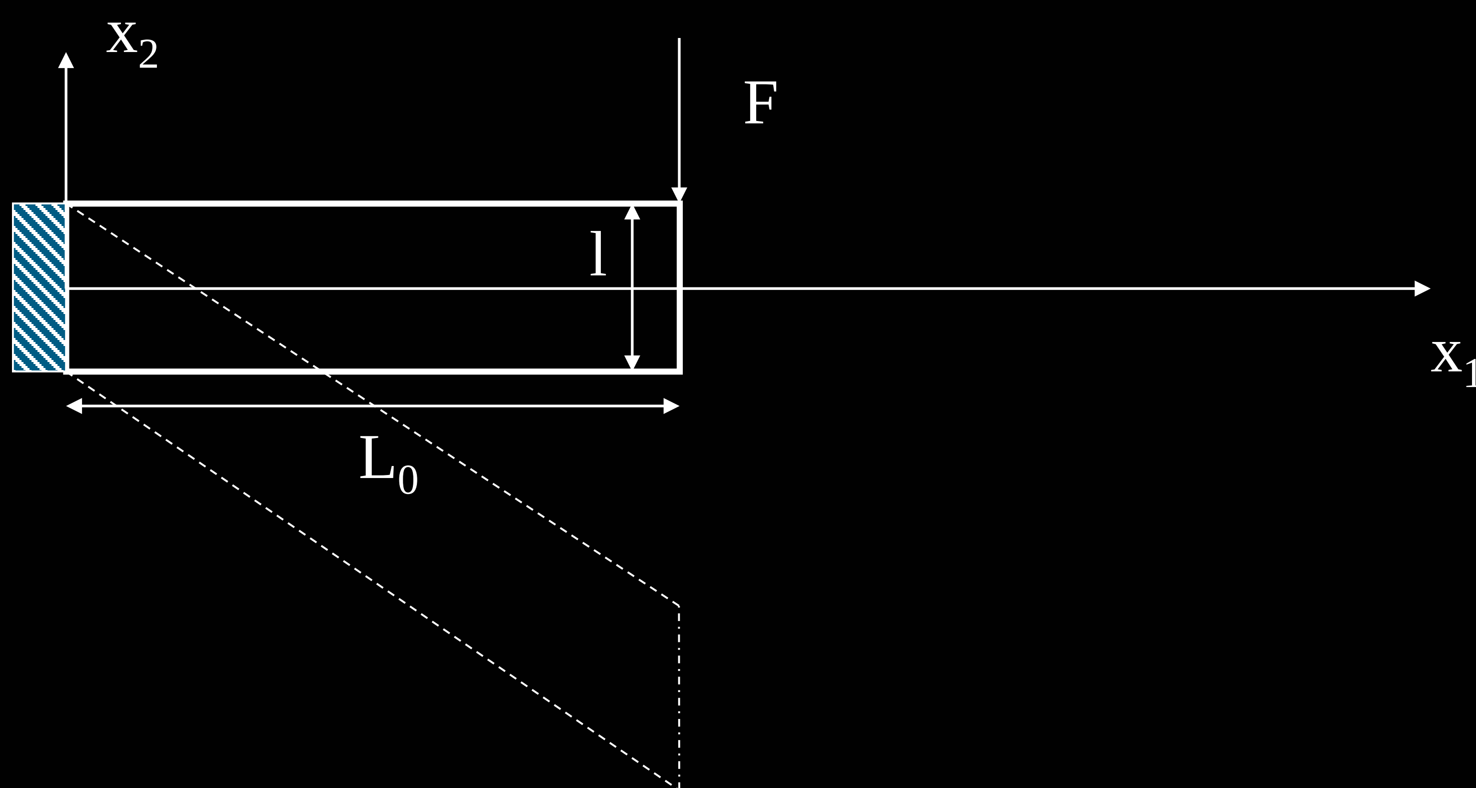

$$-\int_V \sigma_{ij}\varepsilon^*_{ij}\,dV + \int_{\partial V_f} T_i u^*_i\,dS = 0$$

$$-\int_V \sigma_{11}\varepsilon^*_{11}\,dV = -e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}\,dx_1\,dx_2$$

$$=0$$

$$\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}\,dx_1\,dx_2 = 0 \qquad \text{Eq. 1}$$

# Illustration of the PVW

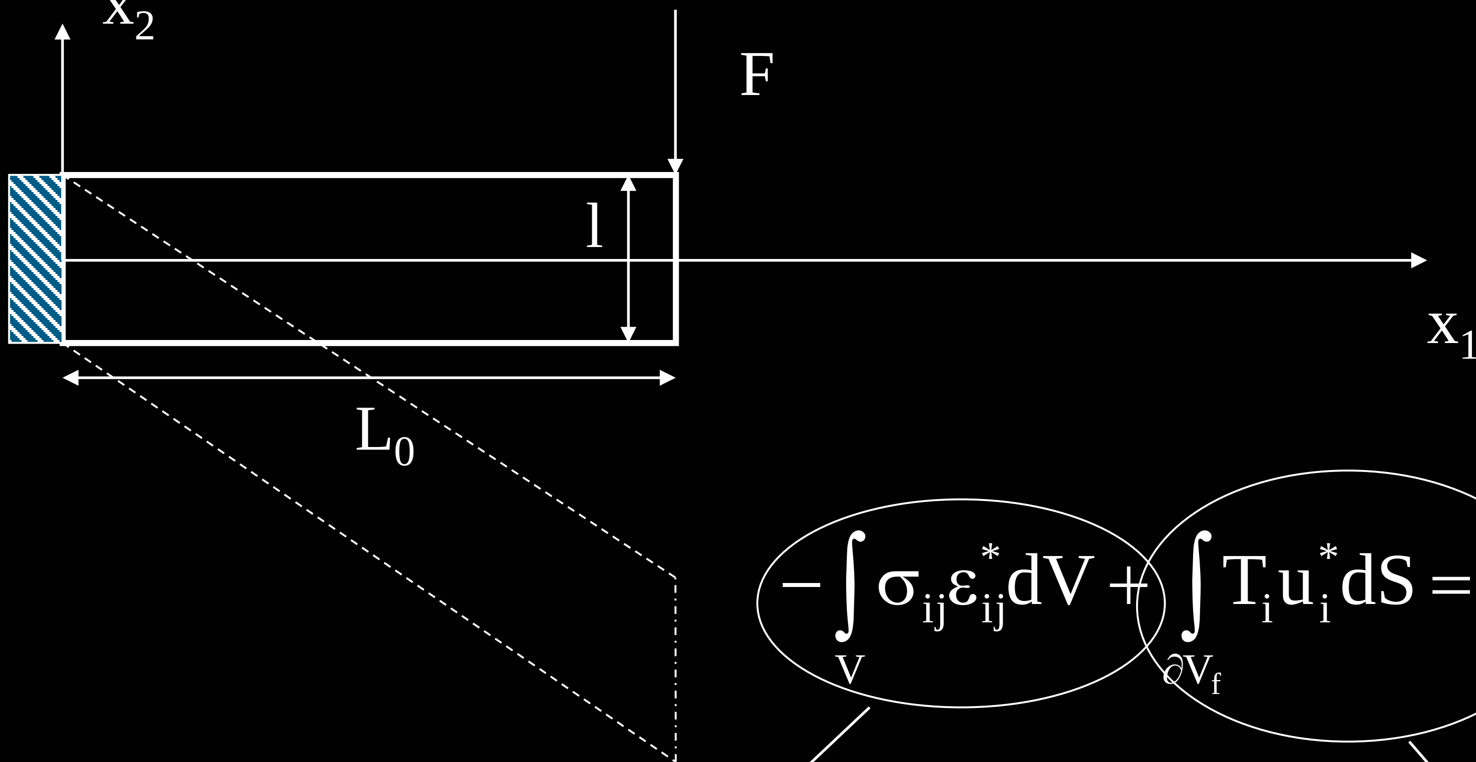



Let us write another virtual field:

$$u_1^* = 0$$
$$u_2^* = -x_1$$

$$\varepsilon_{11}^* = 0$$
$$\varepsilon_{22}^* = 0$$
$$\varepsilon_{12}^* = -1/2$$

# Illustration of the PVW



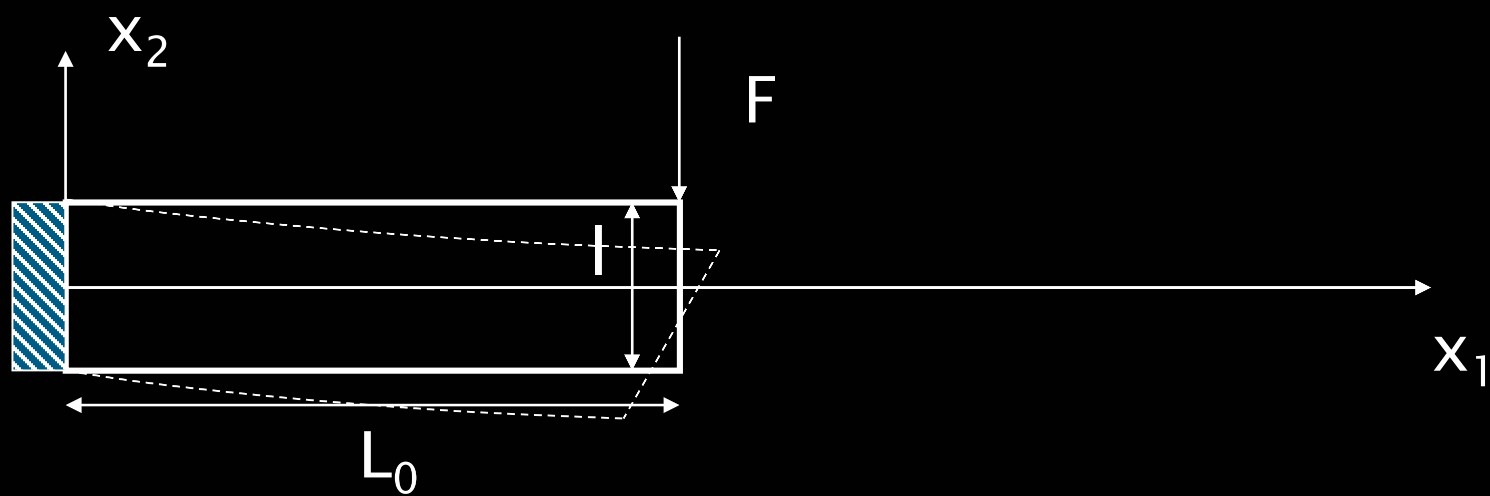

$$-\int_V \sigma_{ij}\varepsilon_{ij}^{*}dV + \int_{\partial V_f} T_i u_i^{*} dS = 0$$

$$-\int_V 2\sigma_{12}\varepsilon_{12}^{*}dV = e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{12}dx_1dx_2$$

$$= FL_0$$

$$e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{12}dx_1dx_2 = -FL_0 \qquad \text{Eq. 2}$$

# Illustration of the PVW



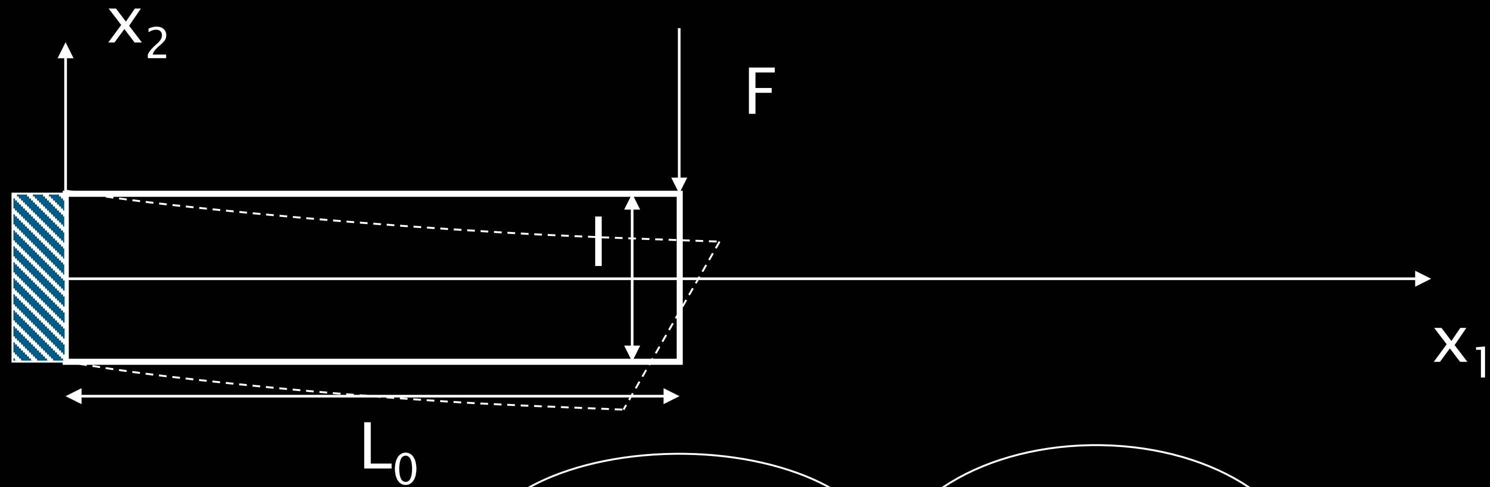

Let us write a 3rd field: virtual bending

$$u_1^* = x_1 x_2 \qquad u_2^* = -\frac{x_1^2}{2}$$

$$\Rightarrow$$

$$\varepsilon_{11}^* = x_2 \qquad \varepsilon_{22}^* = 0 \qquad \varepsilon_{12}^* = 0$$

# Illustration of the PVW




$$-\int_V \sigma_{ij}\varepsilon^*_{ij}dV + \int_{\partial V_f} T_i u^*_i dS = 0$$

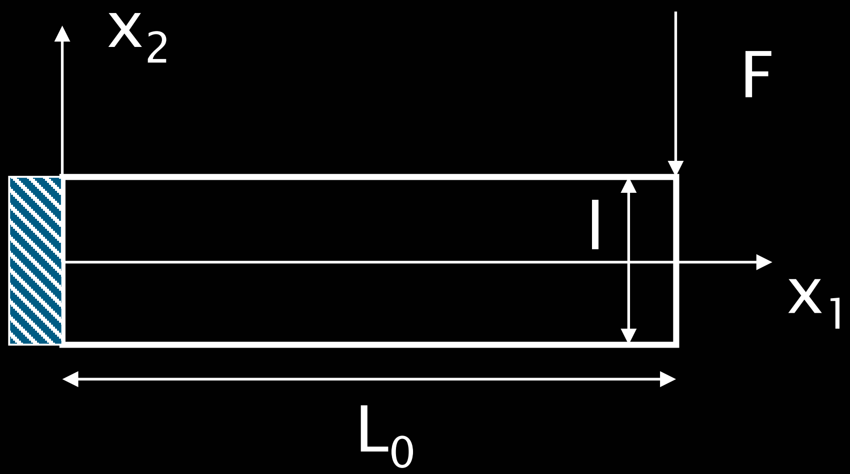
$$-\int_V \sigma_{11}\varepsilon^*_{11}dV = -e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}x_2dx_1dx_2$$

$$= \frac{F.L_0^2}{2}$$

$$e\int_0^{L_0}\int_{-l/2}^{l/2}\sigma_{11}x_2dx_1dx_2 = \frac{FL_0^2}{2}$$

Eq. 3

# Illustration of the PVW




Any virtual field can be selected

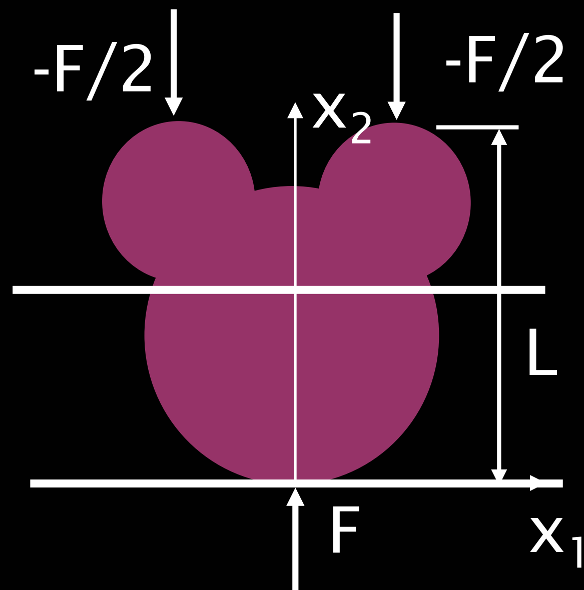
$$\begin{cases} u_1^* = 1 - e^{x_1} \\ u_2^* = 0 \end{cases} \Rightarrow \begin{array}{l} \varepsilon_{11}^* = -e^{x_1} \\ \varepsilon_{22}^* = 0 \\ \varepsilon_{12}^* = 0 \end{array}$$

$$e\int_0^{L_0}\int_{-l/2}^{l/2} \sigma_{11} e^{x_1} dx_1 dx_2 = 0$$

# The Virtual Fields Method

# The Virtual Fields Method

- Back to the funny shaped disc




- Principle of virtual work

$$-\int_V \sigma_{ij}\varepsilon_{ij}^* dV + \int_{\partial V} T_i u_i^* dS = 0$$

No volume forces, static

# The Virtual Fields Method

- Let's write two virtual fields
  - $1^{st}$ virtual field: virtual compression field

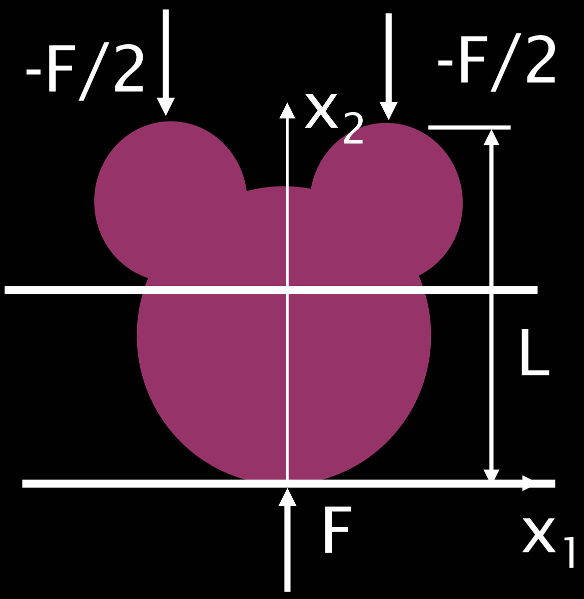
$$u_1^* = 0 \; ; \; u_2^* = -x_2$$

$$\varepsilon_{11}^* = 0 \; ; \; \varepsilon_{22}^* = -1 \; ; \; \varepsilon_{12}^* = 0$$

$$-\int_V \sigma_{ij}\varepsilon_{ij}^* dV = \int_V \sigma_{22} dV$$

$$\int_{\partial V_f} T_i u_i^* dS = FL$$

$$\int_V \sigma_{22} dV = -FL$$



# The Virtual Fields Method

- 2nd virtual field: transverse shrinking

$$u_1^* = -x_1 \,;\, u_2^* = 0 \qquad \varepsilon_{11}^* = -1 \,;\, \varepsilon_{22}^* = 0 \,;\, \varepsilon_{12}^* = 0$$

$$-\int_V \sigma : \varepsilon^* dV = \int_V \sigma_{11} dV \qquad \int_{\partial V_f} \vec{T}.\vec{u}^* dS = 0 \Rightarrow \int_V \sigma_{11} dV = 0$$

- Other virtual fields (an infinity!)

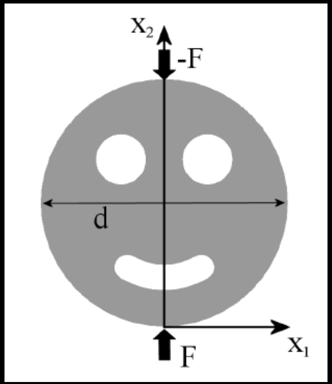
$$u_1^* = 0 \,;\, u_2^* = -\sin\left(\frac{\pi}{2L} x_2\right)$$

$$Q_{11}\overline{[\varepsilon_{22} \cos(\frac{\pi}{2L} x_2)]} + Q_{12}\overline{[\varepsilon_{11} \cos(\frac{\pi}{2L} x_2)]} = \frac{-2FL}{\pi e S_d}$$

# The Virtual Fields Method

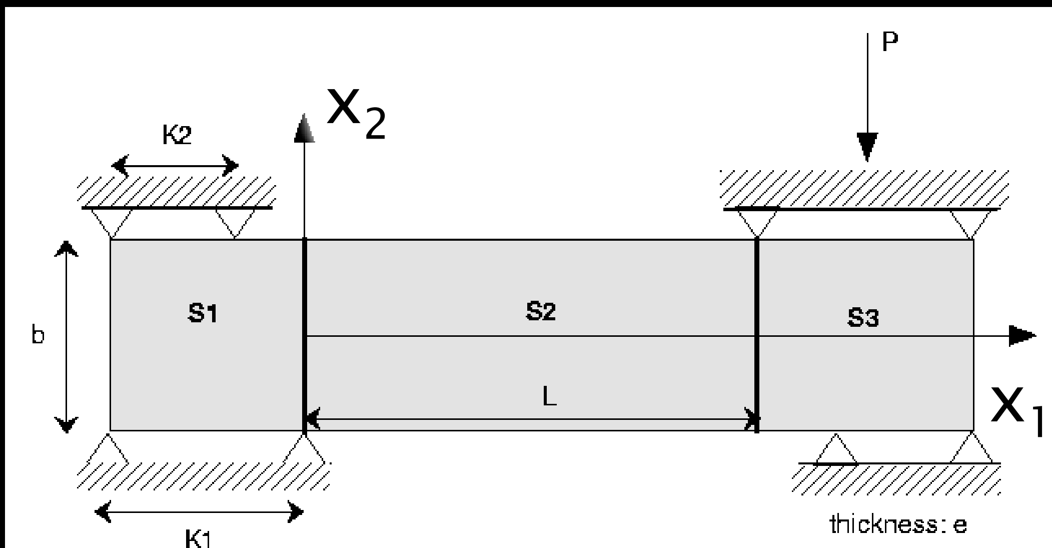
- Principal advantages
  - Independent from stress distribution; etc. geometry
  - Direct identification in elasticity (no updating)

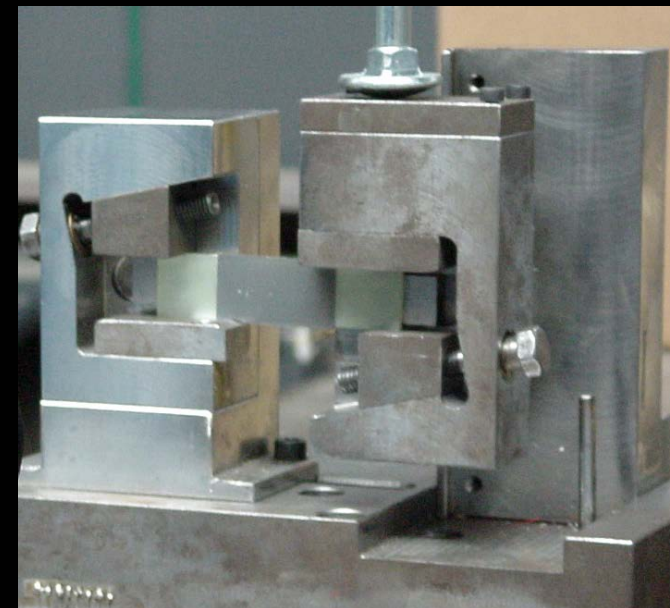
- Limitations
  - Kinematic assumption through the thickness (plane stress, plane strain, bending...)



# VFM: more complex example

- Orthotropic elasticity

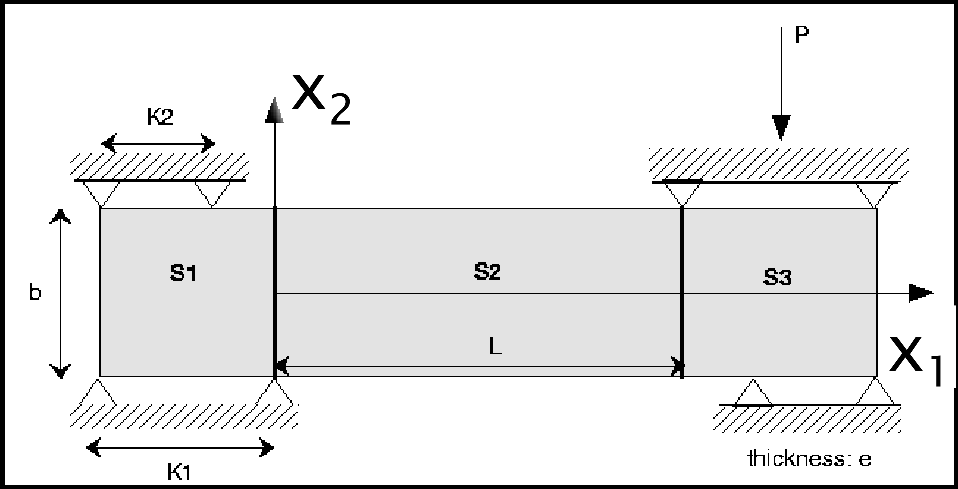


Orthotropic material

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

# VFM: more complex example




Choice of the virtual fields

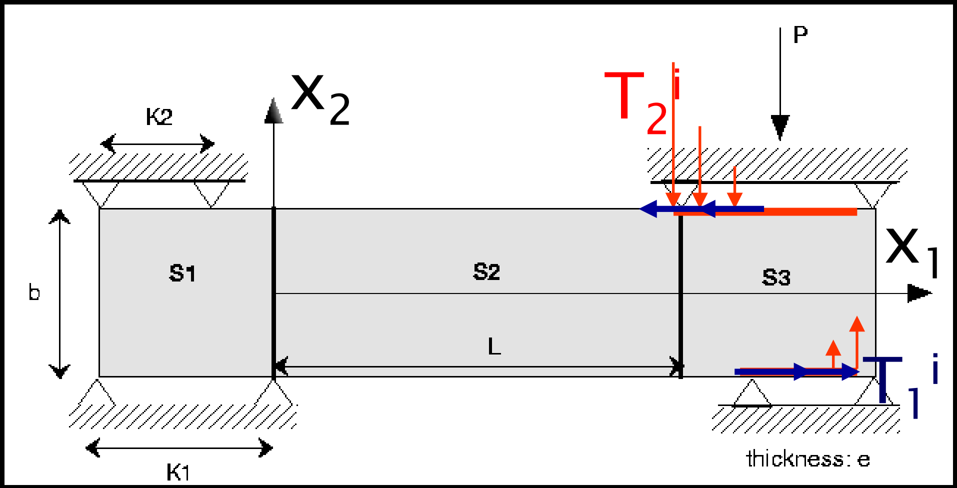
1. Measurement on $S_2$ only (optical system)

   Over $S_1$ and $S_3$: $\varepsilon^*_{11} = 0$ ; $\varepsilon^*_{22} = 0$ ; $\varepsilon^*_{12} = 0$ (rigid body)

2. A priori choice:

   over $S_1$: $u^*_1 = 0 ; u^*_2 = 0$ $\Rightarrow$ $\int_{S_1} T_i u^*_i dS = 0$

# VFM: more complex example




Unknown force distribution over $S_1$ and $S_3$.
Resultant P measured

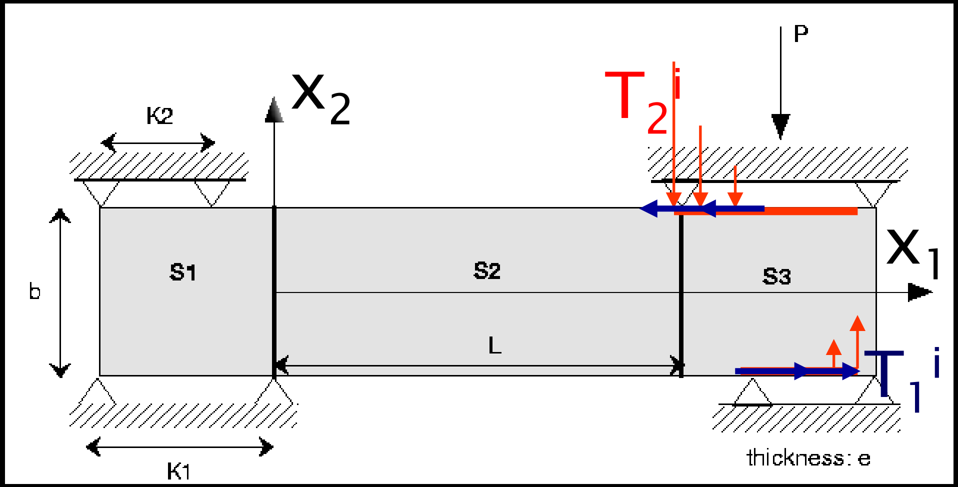
3. Over $S_3$ (rigid body) : 2 possibilities

3.1 $u_1^* = 0 ; u_2^* = 0$ $\Rightarrow$ $\int_{S_3} T_i u_i^* dS = 0$

3.2

$$\int_{S_3} T_i u_i^* dx_1 dx_3 = e\left[\int_{\text{blue}} T_1 u_1^* dx_1 + \int_{\text{red}} T_2 u_2^* dx_1\right]$$

# VFM: more complex example




No information on $t_1$ ⇨ $u_1^* = 0$

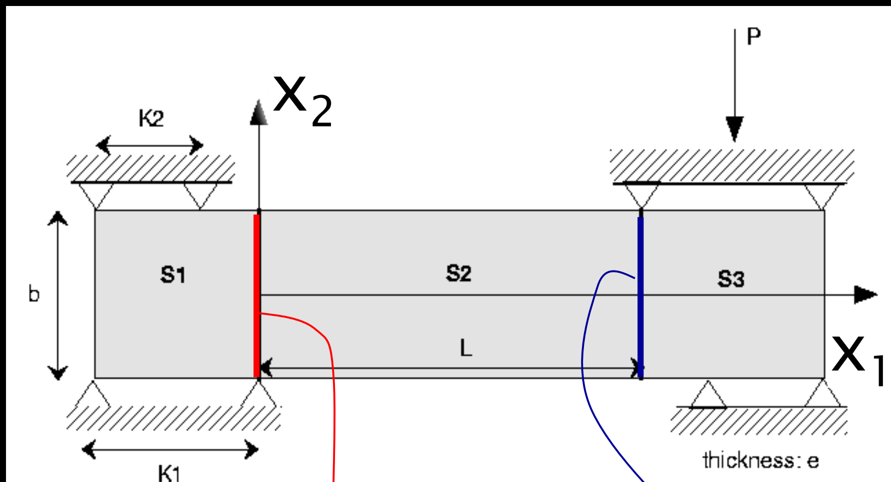
$$\int_{S_3} T_i u_i^* dx_1 dx_2 = e\left[ \qquad\qquad + \int_{red} T_2 u_2^* dx_1 \right]$$

Distribution $T_2$ unknown ⇨ $u_2^* = k$

Filtering capacity of the VF

$$\int_{red} T_2 u_2^* dx_1 = \int_{red} T_2 k dx_1 = k \int_{red} T_2 dx_1 = -kP$$

# VFM: more complex example



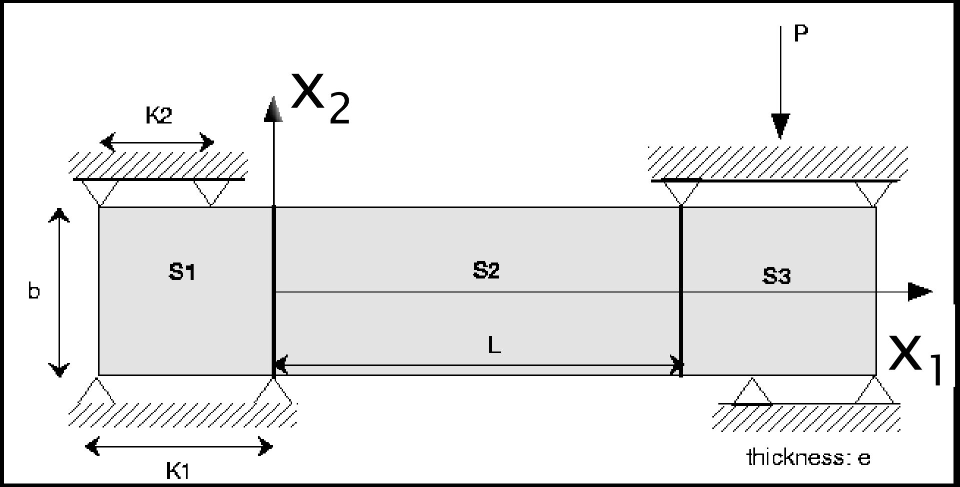

4. Continuity of the virtual displacement field
Conditions over $S_2$

$$u_1^* = u_2^* = 0 \qquad u_1^* = 0\,;\, u_2^* = k$$

Virtual strain field discontinuous

Choice of 4 virtual fields at least: example

# VFM: more complex example

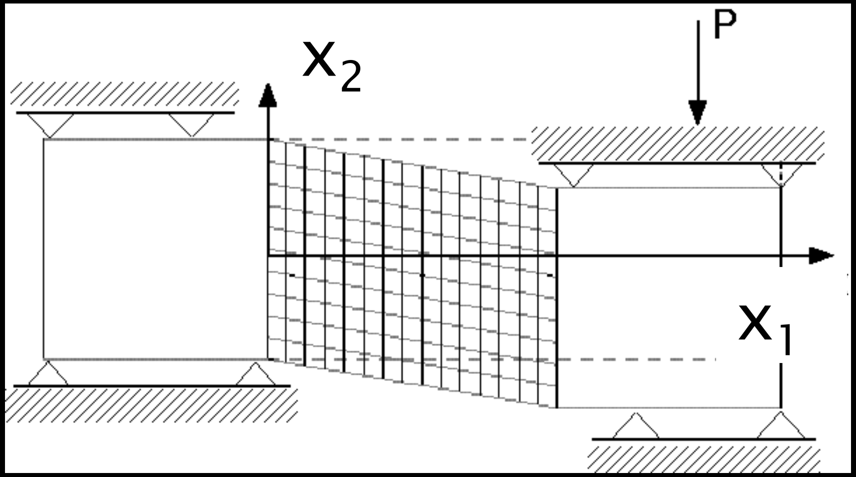



Over $S_2$

$$u_1^* = 0\,;\, u_2^* = -x_1$$

$$\varepsilon_{11}^* = 0\ ;\ \varepsilon_{22}^* = 0\ ;\ \varepsilon_{12}^* = -\frac{1}{2}$$

Over $S_3$ k = -L




Uniform virtual shear

# VFM: more complex example

$$-\int_V \sigma_{ij}\varepsilon_{ij}^* dV + \int_{\partial V_f} T_i u_i^* dS = 0$$

Plane stress

$$\int_V \sigma_{12} dV \Rightarrow e\int_S \sigma_{12} dS$$

Homogeneous material

$$2e\int_S Q_{66}\varepsilon_{12} dS \Rightarrow 2eQ_{66}\int_S \varepsilon_{12} dS$$

Plane stress orthotropic elasticity

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{pmatrix}$$

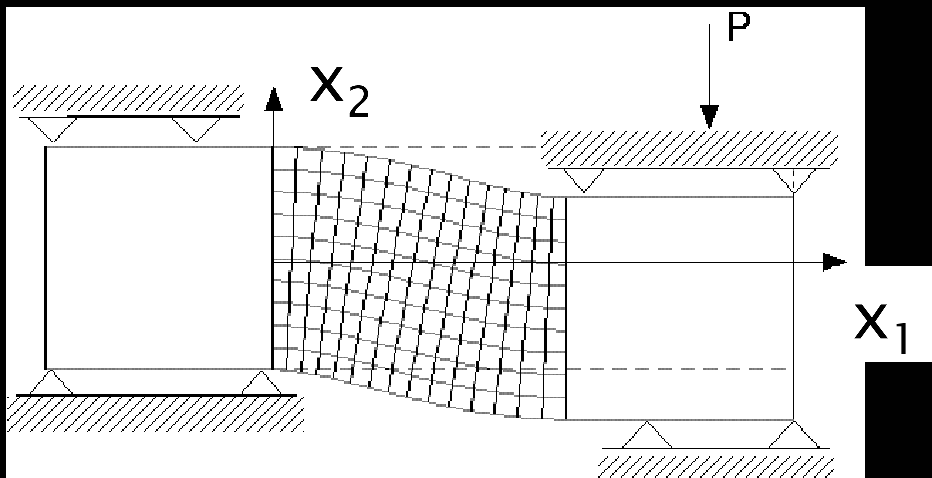
$$\int_{\partial V_f} T_i u_i^* dS \Downarrow PL$$

$$Q_{66}\int_{S_2} \varepsilon_{12} dx_1 dx_2 = \frac{-PL}{2e}$$

$$\boxed{Q_{66}\overline{\varepsilon_{12}} = \frac{-P}{2eb}}$$

# VFM: more complex example

Field n° 2: Bernoulli bending




Over $S_2$

$$\begin{cases} u_1^* = 6x_1x_2(L - x_1) \\ u_2^* = x_1^2(2x_1 - 3L) \end{cases}$$

$$\begin{cases} \varepsilon_{11}^* = 6x_2(L - 2x_1) \\ \varepsilon_{22}^* = 0 \; ; \; \varepsilon_{12}^* = 0 \end{cases}$$

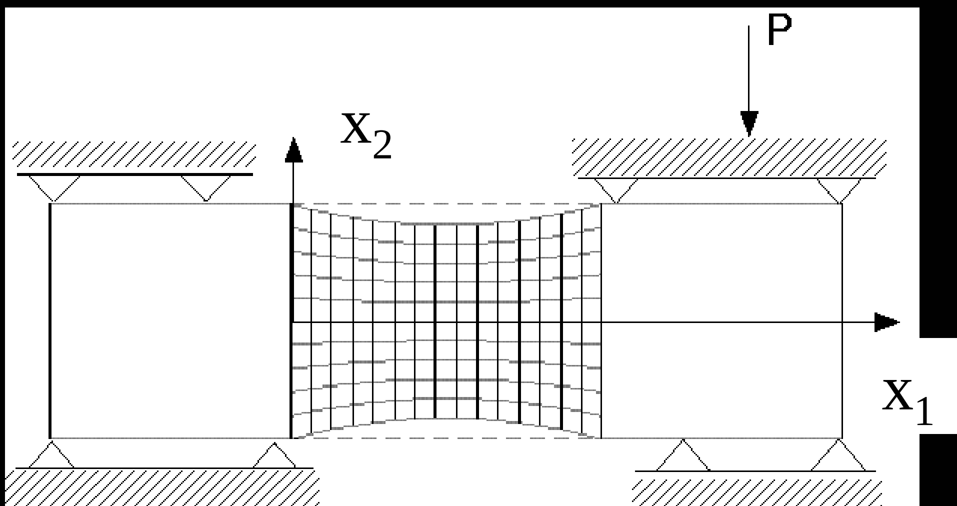
Over $S_3$

$k = -L^3$

$$\boxed{Q_{11}\overline{6x_2(L-2x_1)\varepsilon_{11}} + Q_{12}\overline{6x_2(L-2x_1)\varepsilon_{22}} = \frac{PL^2}{eb}}$$

# VFM: more complex example

Field n° 3: Global compression




Over $S_2$

$$\begin{cases} u_1^* = 0 \\ u_2^* = x_1 x_2 (x_1 - L) \end{cases}$$

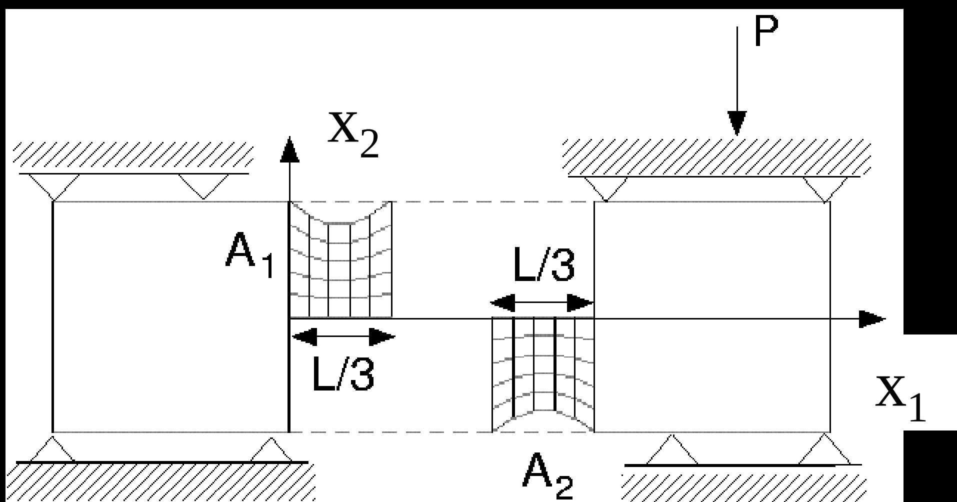
$$\begin{cases} \varepsilon_{11}^* = 0 \\ \varepsilon_{22}^* = x_1 (x_1 - L) \\ \varepsilon_{12}^* = \frac{1}{2} x_2 (2x_1 - L) \end{cases}$$

$$\boxed{\begin{aligned} &Q_{22}\overline{x_1(x_1-L)\varepsilon_{22}} + Q_{12}\overline{x_1(x_1-L)\varepsilon_{11}} \\ &+Q_{66}\overline{2x_2(2x_1-L)\varepsilon_{12}} = 0 \end{aligned}}$$

Over $S_3$
k = 0

# VFM: more complex example

Field n° 4: Local compression




Over $S_3$ k = 0

Over $A_1$

$$\begin{cases} u_1^* = 0 \\ u_2^* = x_1 x_2 (x_1 - L/3) \end{cases}$$

$$\begin{cases} \varepsilon_{11}^* = 0 \\ \varepsilon_{22}^* = x_1 (x_1 - L/3) \\ \varepsilon_{12}^* = \frac{1}{2} x_2 (2x_1 - L/3) \end{cases}$$

Over $A_2$

$$\begin{cases} u_1^* = 0 \\ u_2^* = x_2 (x_1 - L)(x_1 - 2L/3) \end{cases}$$

$$\begin{cases} \varepsilon_{11}^* = 0 \\ \varepsilon_{22}^* = (x_1 - L)(x_1 - L/3) \\ \varepsilon_{12}^* = \frac{1}{2} x_2 (2x_1 - 5L/3) \end{cases}$$

# VFM: more complex example

Field n° 4: Local compression

$$\boxed{\begin{aligned}&Q_{22}\left[\overline{x_1(x_1-L)\varepsilon_{22}}^{A_1}+\overline{(x_1-L)(x_1-L/3)\varepsilon_{22}}^{A_2}\right]\\&+Q_{12}\left[\overline{x_1(x_1-L)\varepsilon_{11}}^{A_1}+\overline{(x_1-L)(x_1-L/3)\varepsilon_{11}}^{A_2}\right]\\&+Q_{66}\left[\overline{2x_2(2x_1-L/3)\varepsilon_{12}}^{A_1}+\overline{2x_2(2x_1-5L/3)\varepsilon_{12}}^{A_2}\right]=0\end{aligned}}$$

# VFM: more complex example

Final system

$$
\begin{bmatrix}
0 & 0 & 0 & \overline{\varepsilon_{12}} \\
\overline{6x_2(L-2x_1)\varepsilon_{11}} & 0 & \overline{6x_2(L-2x_1)\varepsilon_{22}} & 0 \\
0 & \overline{x_1(x_1-L)\varepsilon_{22}} & \overline{x_1(x_1-L)\varepsilon_{11}} & \overline{x_2(2x_1-L)\varepsilon_{12}} \\
0 & \left\{ \begin{array}{l} \overline{x_1(x_1-L)\varepsilon_{22}}^{A_1} + \\ \overline{(x_1-L)(x_1-L/3)\varepsilon_{22}}^{A_2} \end{array} \right. & \left\{ \begin{array}{l} \overline{x_1(x_1-L)\varepsilon_{11}}^{A_1} + \\ \overline{(x_1-L)(x_1-L/3)\varepsilon_{11}}^{A_2} \end{array} \right. & \left\{ \begin{array}{l} \overline{x_2(2x_1-L/3)\varepsilon_{12}}^{A_1} + \\ \overline{x_2(2x_1-5L/3)\varepsilon_{12}}^{A_2} \end{array} \right.
\end{bmatrix}
\begin{pmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{pmatrix}
=
\begin{pmatrix} -\dfrac{P}{2eb} \\ \dfrac{PL^2}{eb} \\ 0 \\ 0 \end{pmatrix}
$$

$AQ = B \Rightarrow Q = A^{-1}B$ If VF independent !!

Pierron F. et Grédiac M., *Composites Part A*, vol. 31, pp. 309-318, 2000.

# Complements on the VFM

# Summary

- Virtual fields selection in elasticity
- Heterogeneous materials
- Non-linear constitutive models
- Dynamics
- Force identification

# VF selection

# VF selection

- Exact data
  - All sets of VF produce the SAME results
- Noisy data
  - Different sets provide different results
  - Optimal selection?

# VF selection

- Strategy for virtual fields selection in elasticity
  - Projection of VF on a certain basis of functions
  - Polynomial

$$\begin{cases} u_1^* = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\left(\frac{x_1}{L}\right)^i\left(\frac{x_2}{b}\right)^j \\ u_2^* = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\left(\frac{x_1}{L}\right)^i\left(\frac{x_2}{b}\right)^j \end{cases}$$

  - Choice of VF: choice of a set of m, n, $a_{ij}$,$b_{ij}$

# VF selection

- Strategy to find $a_{ij}$,$b_{ij}$ for given m and n
  - Special virtual fields
  - For instance, special virtual field to identify $Q_{11}$

$$Q_{11}\underbrace{\overline{\varepsilon_{11}\varepsilon_{11}^{*}}}_{=1}+Q_{22}\underbrace{\overline{\varepsilon_{22}\varepsilon_{22}^{*}}}_{=0}+Q_{12}\underbrace{\left(\overline{\varepsilon_{11}\varepsilon_{22}^{*}}+\overline{\varepsilon_{22}\varepsilon_{11}^{*}}\right)}_{=0}+4Q_{66}\underbrace{\overline{\varepsilon_{12}\varepsilon_{12}^{*}}}_{=0}=-\frac{Pu_{2}^{*}(x_1=L)}{Lbe}$$

$$Q_{11}=-\frac{Pu_{2}^{*(1)}(x_1=L)}{Lbe}$$

  - 4 linear equations linking the $a_{ij}$ and $b_{ij}$ coefficients
  - Virtual BCs: a few more linear equations
  - Still a very large number of VF in general!

# VF selection

- Strategy to find $a_{ij}$,$b_{ij}$ for given m and n
  - Minimization of the sensitivity to noise
  - Constrained minimization problem where $a_{ij}$ and $b_{ij}$ coefficients are the unknown
  - Provides the maximum likelihood solution, with:

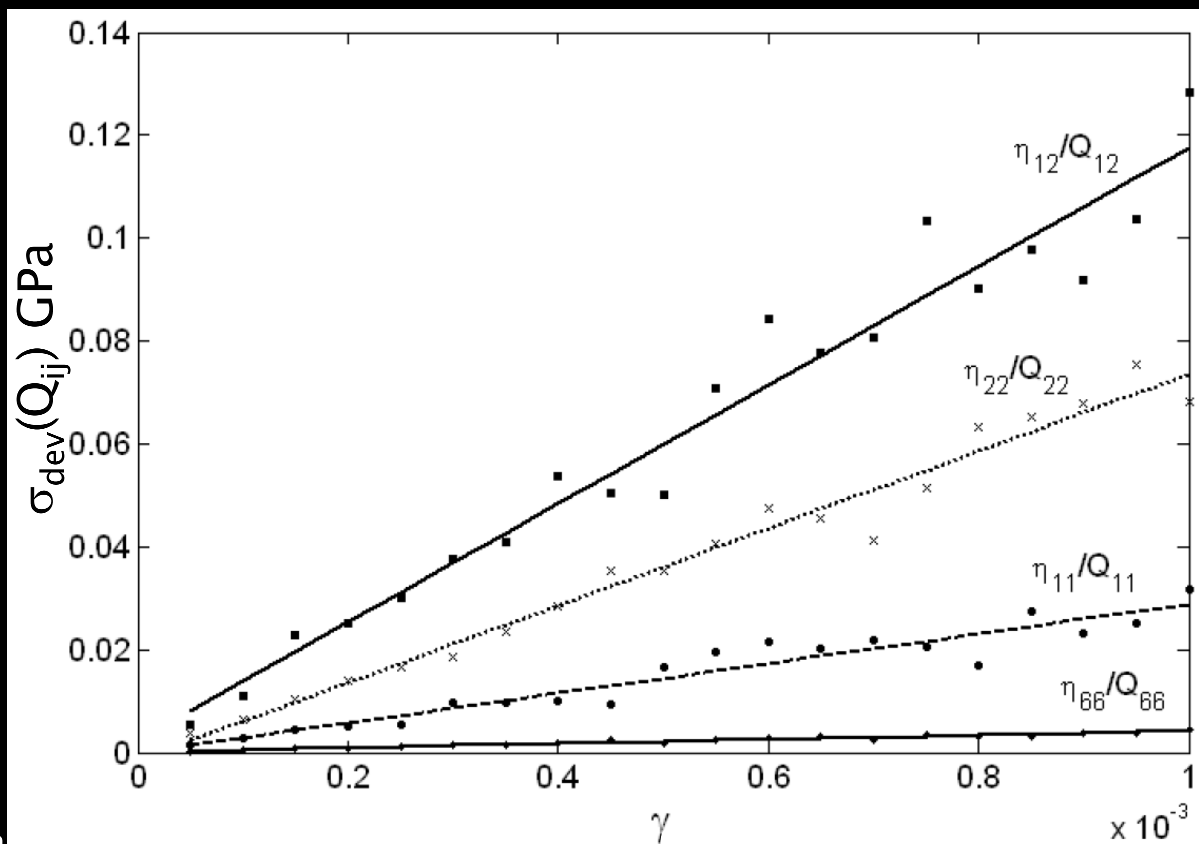
$$\begin{cases} \sigma_{dev}(Q_{11}) = \eta_{11}\gamma \\ \sigma_{dev}(Q_{22}) = \eta_{22}\gamma \\ \sigma_{dev}(Q_{12}) = \eta_{12}\gamma \\ \sigma_{dev}(Q_{66}) = \eta_{66}\gamma \end{cases}$$

$\gamma$ is the standard deviation of strain noise

$\eta_{ij}$ are the sensitivity to noise coefficients

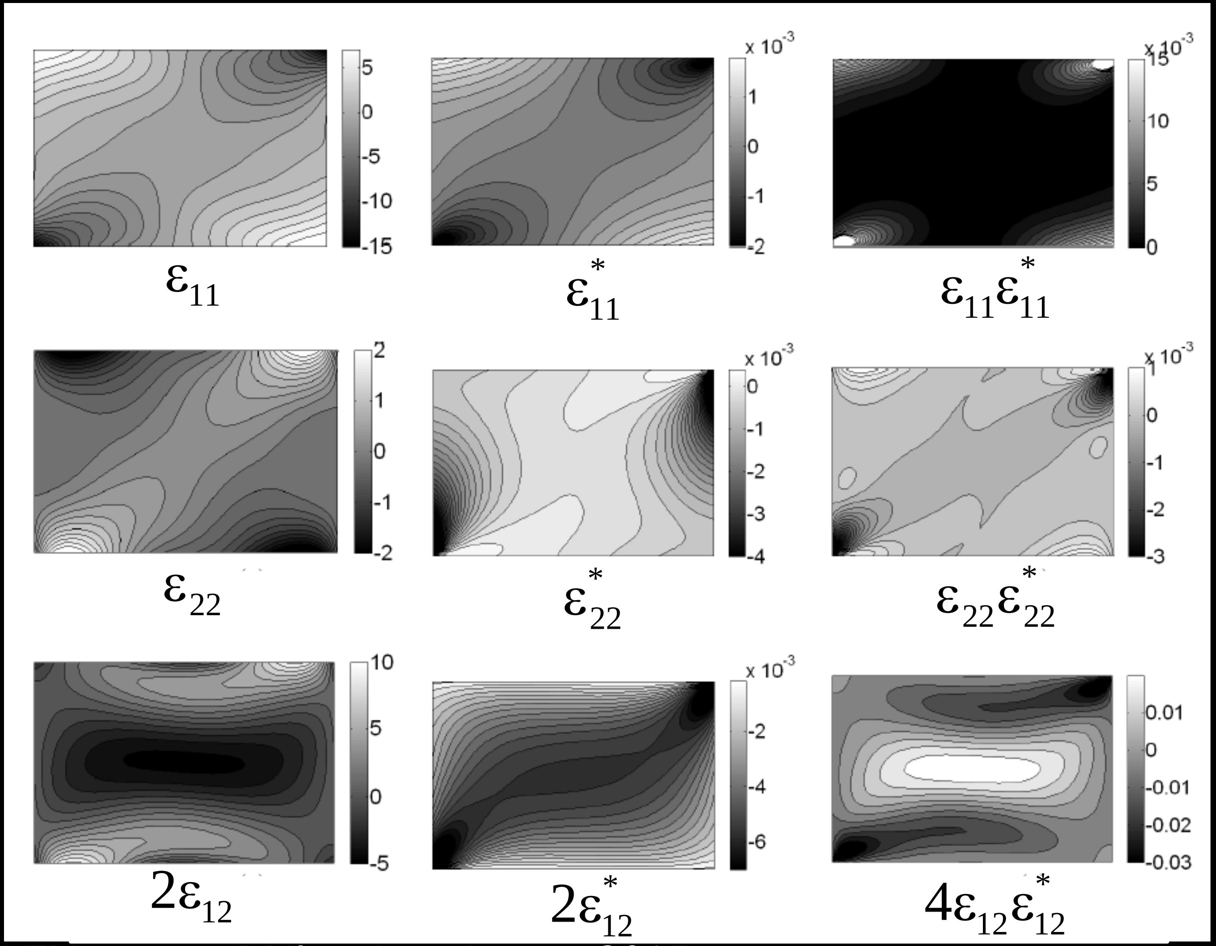
# VF selection

- Validation
  - 30 copies of noise: distribution of stiffness
  - Increasing levels of noise



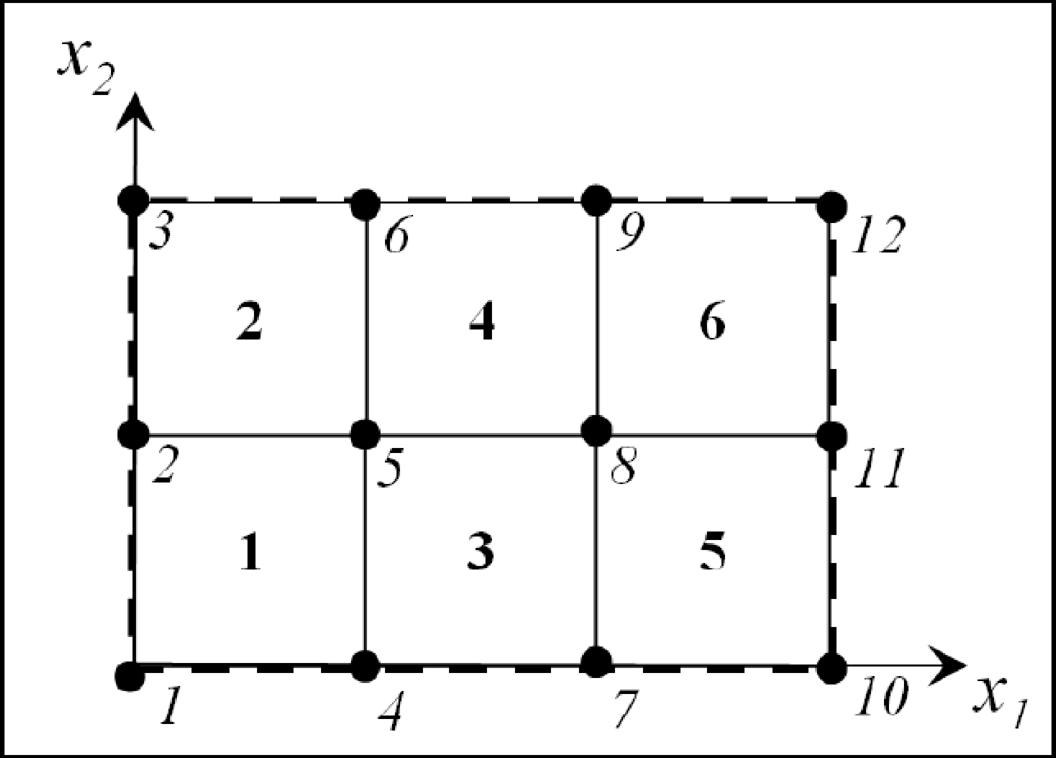
# VF selection

- Actual and virtual optimized virtual fields for $Q_{11}$ (m=4,n=5)



# VF selection

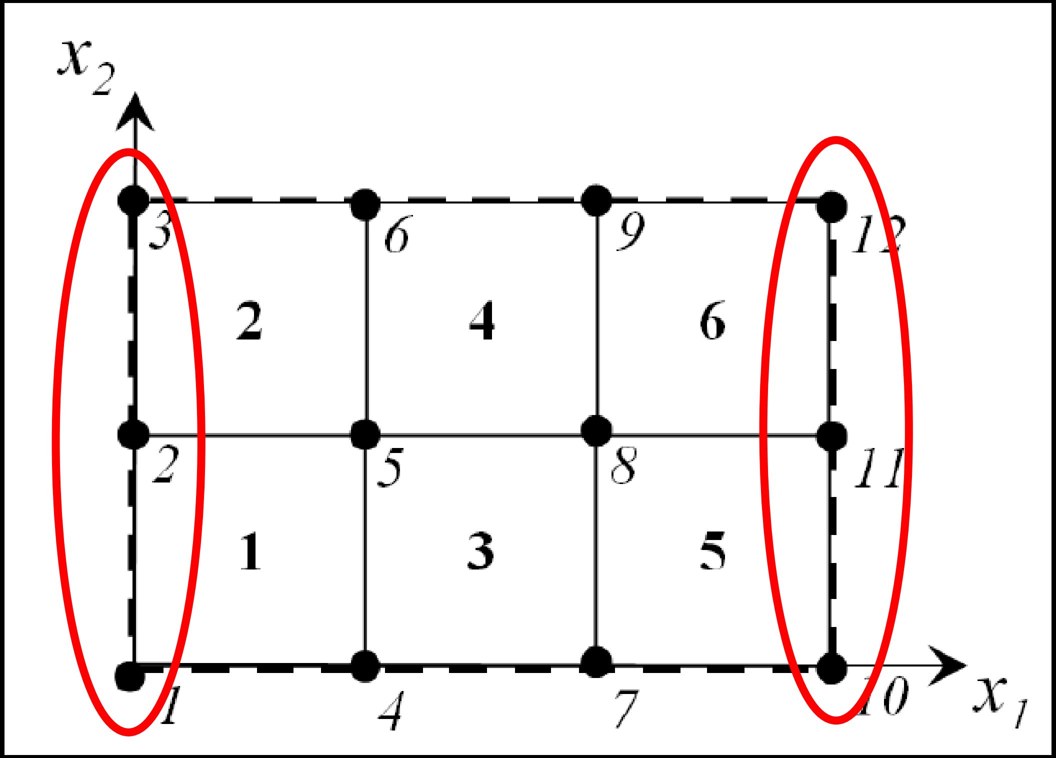
- Piecewise virtual fields
  - Finite element mesh
  - Virtual Dofs unknown of minimization problem



# VF selection

- Piecewise virtual fields
  - Advantage: more flexible
  - Easy for virtual BCs

All dofs
To zero

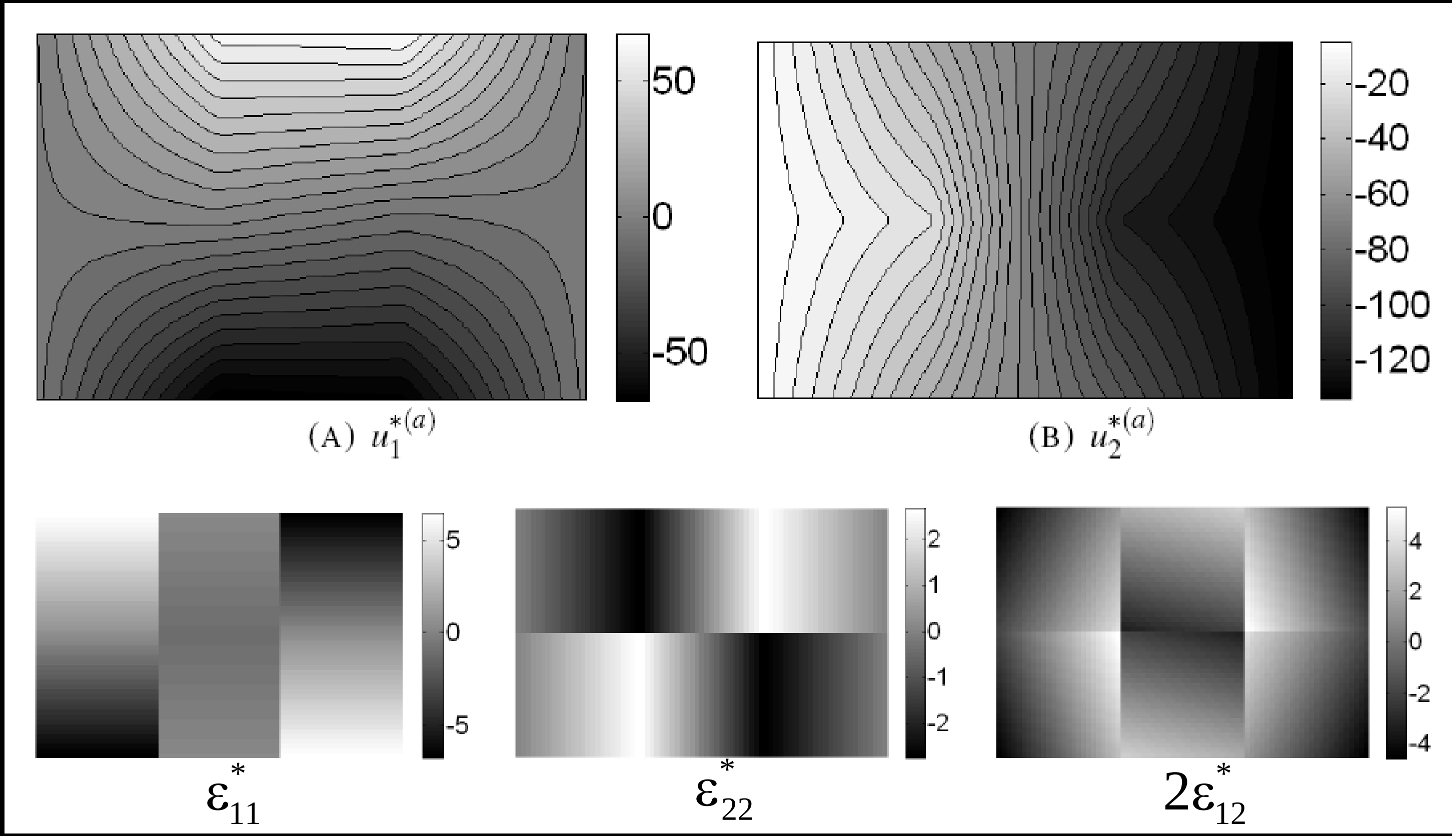



All $u_1^*=0$

All $u_2^*=cst$

# VF selection

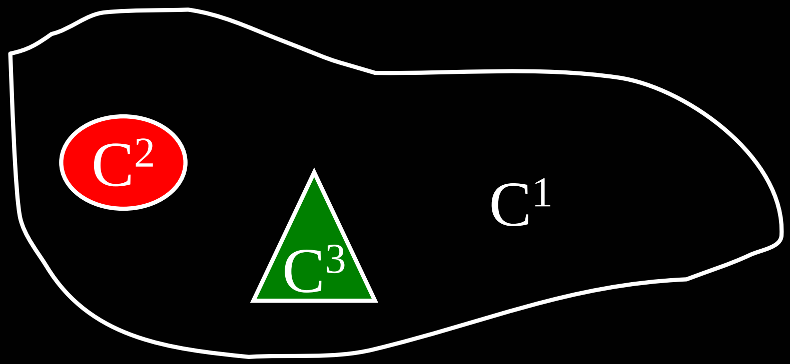
- Optimized piecewise virtual fields for $Q_{11}$
  - Virtual strains discontinuous



# Heterogeneous materials

# Heterogeneous materials

- Discrete parameterization
  - Multimaterials
  - Sharp transitions




$$-\int_{V} \sigma_{ij}\varepsilon^*_{ij}dV = -\int_{C1} \sigma_{ij}\varepsilon^*_{ij}dV - \int_{C2} \sigma_{ij}\varepsilon^*_{ij}dV - \int_{C3} \sigma_{ij}\varepsilon^*_{ij}dV$$

# Heterogeneous materials

- Discrete parameterization
  - 6 unknowns in isotropic linear elasticity
  - Contrast of stiffness: strain derivative!!
  - Subject to noise

$$
\begin{aligned}
&= -Q_{11}^{1}\int_{C1}(\varepsilon_1\varepsilon_1^{*}+\varepsilon_2\varepsilon_2^{*}+\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV - Q_{12}^{1}\int_{C1}(\varepsilon_1\varepsilon_2^{*}+\varepsilon_2\varepsilon_1^{*}-\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV \\
&\quad -Q_{xx}^{2}\int_{C2}(\varepsilon_1\varepsilon_1^{*}+\varepsilon_2\varepsilon_2^{*}+\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV - Q_{12}^{1}\int_{C2}(\varepsilon_1\varepsilon_2^{*}+\varepsilon_2\varepsilon_1^{*}-\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV \\
&\quad -Q_{xx}^{3}\int_{C3}(\varepsilon_1\varepsilon_1^{*}+\varepsilon_2\varepsilon_2^{*}+\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV - Q_{12}^{1}\int_{C3}(\varepsilon_1\varepsilon_2^{*}+\varepsilon_2\varepsilon_1^{*}-\frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV
\end{aligned}
$$

# Heterogeneous materials

- Continuous parameterization
  - Functionally graded material
  - Smooth transitions


  - Polymer/composite with moisture ingress from one face
  - Ceramic with varying pore density
  - Etc…

# Heterogeneous materials

- Continuous parameterization


$$\begin{cases} Q_{11} = Q_{11}^{0} + Q_{11}^{1}\dfrac{x_2}{w} + Q_{11}^{2}\left(\dfrac{x_2}{w}\right)^2 \\ Q_{12} = Q_{12}^{0} + Q_{12}^{1}\dfrac{x_2}{w} + Q_{12}^{2}\left(\dfrac{x_2}{w}\right)^2 \end{cases}$$

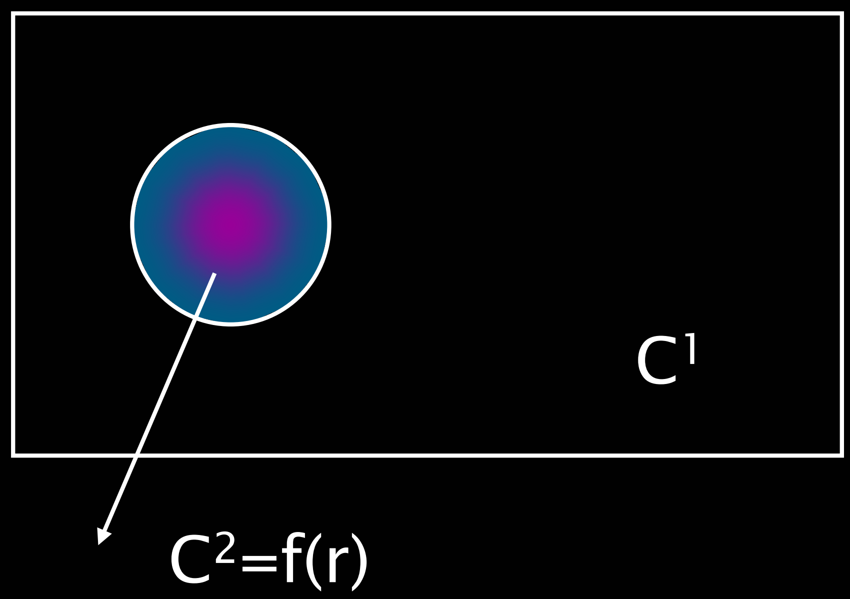
$$-Q_{11}^{0}\int_{C1}(\varepsilon_1\varepsilon_1^{*} + \varepsilon_2\varepsilon_2^{*} + \frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV - Q_{12}^{0}\int_{C1}(\varepsilon_1\varepsilon_2^{*} + \varepsilon_1\varepsilon_2^{*} - \frac{1}{2}\varepsilon_6\varepsilon_6^{*})dV$$

$$-Q_{11}^{1}\int_{C2}(\varepsilon_1\varepsilon_1^{*} + \varepsilon_2\varepsilon_3^{*} + \frac{1}{2}\varepsilon_6\varepsilon_6^{*})\frac{x_2}{w}dV - Q_{12}^{1}\int_{C2}(\varepsilon_1\varepsilon_2^{*} + \varepsilon_2\varepsilon_1^{*} - \frac{1}{2}\varepsilon_6\varepsilon_6^{*})\frac{x_2}{w}dV$$

$$-Q_{11}^{2}\int_{C3}(\varepsilon_1\varepsilon_1^{*} + \varepsilon_2\varepsilon_2^{*} + \frac{1}{2}\varepsilon_6\varepsilon_6^{*})\left(\frac{x_2}{w}\right)^2 dV - Q_{12}^{2}\int_{C3}(\varepsilon_1\varepsilon_2^{*} + \varepsilon_2\varepsilon_1^{*} - \frac{1}{2}\varepsilon_6\varepsilon_6^{*})\left(\frac{x_2}{w}\right)^2 dV$$

# Heterogeneous materials

- Mixed parameterization
  - Plate with local damage or manufacturing defect

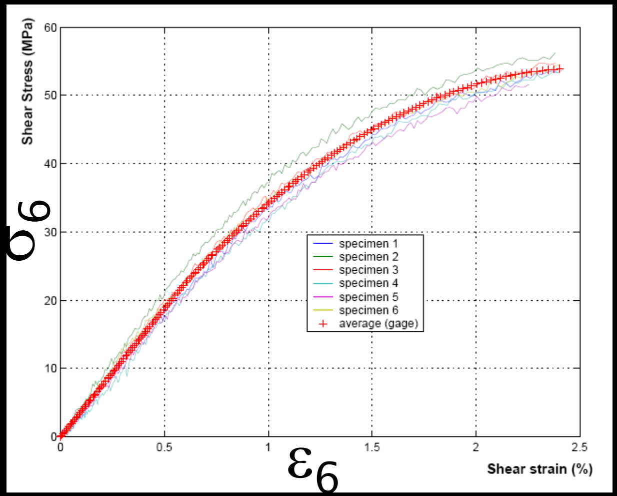


# Non-linear constitutive model

# Non-linear constitutive model

- Non-linear model with linear VFM formulation
  - Composite with shear strain softening

$$\sigma_6 = Q_{66}^0 \varepsilon_6 - K\varepsilon_6^3$$




$$Q_{11}\overline{\varepsilon_1\varepsilon_1^*} + Q_{22}\overline{\varepsilon_2\varepsilon_2^*} + Q_{12}\left(\overline{\varepsilon_1\varepsilon_2^*} + \overline{\varepsilon_2\varepsilon_1^*}\right)$$

$$+Q_{66}^0\overline{\varepsilon_6\varepsilon_6^*} - K\overline{\varepsilon_{66}^3\varepsilon_6^*} = -\frac{Pu_2^*(x_1 = L)}{Lbe}$$

# … in elasto-plasticity

- Virtual fields method for non-linear constitutive equations

$$\sigma(t) = f\left(\varepsilon, \boxed{Y_1^0, \cdots, Y_m^0}\right) \qquad F(t)$$

Measured:

$\varepsilon(t), F(t)$

$$\min\left(-\int_V \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}, \mathbf{Y}_1 \ldots \mathbf{Y}_n) : \underline{\underline{\varepsilon}}^* \, dV + \int_{\partial V} \vec{T} \cdot \vec{u}^* \, dS\right)^2 = 0$$

$$u_1^*, \cdots, u_n^* \qquad Y_1, \cdots, Y_m$$

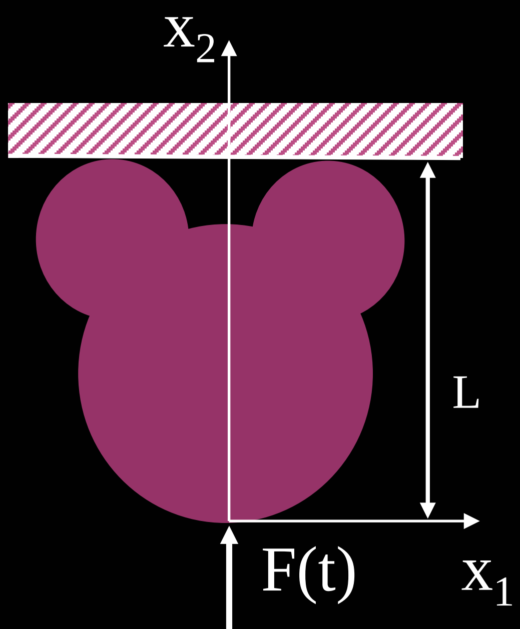
Iterative but no direct problem solved

Fast (minutes)

# Dynamics

# VFM in dynamics

- Principle of virtual work in dynamics




$$-\int_V \sigma_{ij}\varepsilon^*_{ij}\,dV + \cancel{\int_{\partial V_f} T_i u^*_i\,dS} = \int_V \rho a_i u^*_i\,dV$$

If the load is unknown

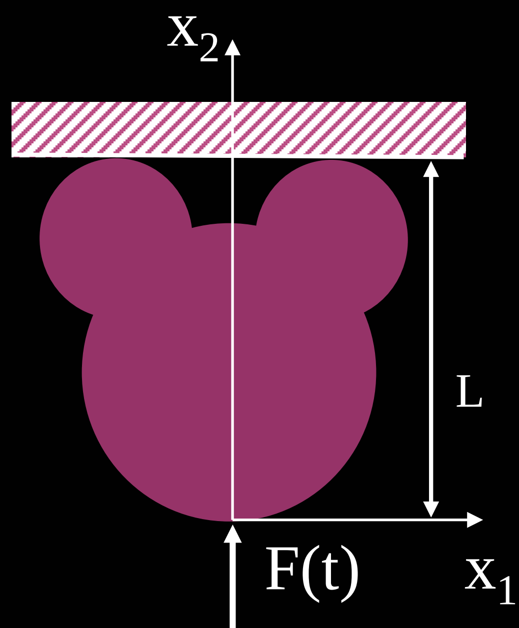
$$u^*_1 = 0\,;\, u^*_2 = x_2(x_2 - L)$$

In statics: with other virtual fields

$$[A]\{Q\} = \{0\}$$

Only stiffness ratios can be identified (Poisson’s ratio)

# VFM in dynamics




$$-\int_V \sigma_{ij}\varepsilon^*_{ij}\,dV = \int_V \rho a_i u^*_i\,dV$$

$$u^*_1 = 0\,;\; u^*_2 = x_2(x_2 - L)$$

$$\varepsilon^*_1 = 0\,;\; \varepsilon^*_2 = 2x_2 - L\,;\; \varepsilon^*_6 = 0$$

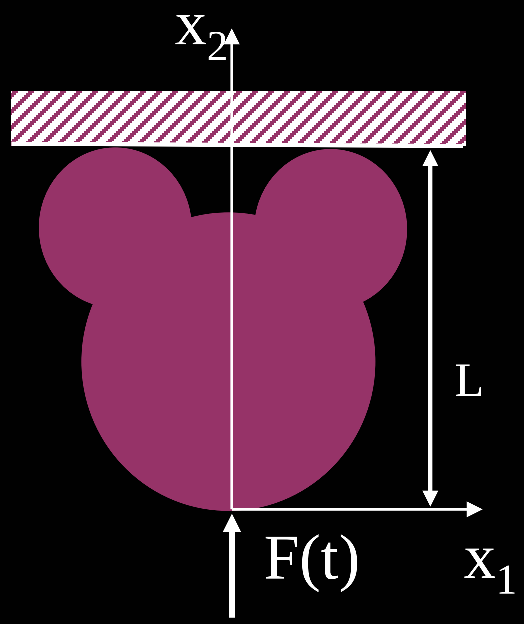
$$Q_{11}\overline{(2x_2 - L)\varepsilon_2} + Q_{12}\overline{(2x_2 - L)\varepsilon_1} = -\rho\overline{x_2(x_2 - L)a_2}$$

Measurements $\begin{cases} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \end{cases}$

Single spatial differentiation

Double temporal differentiation

# VFM in dynamics




$$-\int_V \sigma_{ij}\varepsilon^*_{ij}\,dV = \int_V \rho a_i u^*_i\,dV$$

$$u^*_1 = x_1 \,;\, u^*_2 = 0$$

$$\varepsilon^*_1 = 1; \varepsilon^*_2 = 0; \varepsilon^*_6 = 0$$

$$Q_{11}\overline{\varepsilon_1} + Q_{12}\overline{\varepsilon_2} = -\rho\overline{x_1 a_1}$$

$$[A]\{Q\} = \{B(\rho,\vec{a})\}$$

Volume distributed load cell
Embedded in the full-field data!!

## More on Tuesday!

# Force identification

# Force identification

- Standard VFM

$$-\int_V \sigma_{ij}\varepsilon^*_{ij}dV + \int_{\partial V_f} T_i u^*_i dS = \int_V \rho a_i u^*_i dV$$

Measured

$$\sigma_{ij} = f\left(\varepsilon_{ij}, \boxed{E, \nu \ldots}\right)$$

Measured

Measured

Unknown

- Force VFM

$$-\int_V \sigma_{ij}\varepsilon^*_{ij}dV + \int_{\partial V_f} \boxed{T_i} u^*_i dS = \int_V \rho a_i u^*_i dV$$

$$\sigma_{ij} = f\left(\varepsilon_{ij}, E, \nu \ldots\right)$$

Berry, A., Robin, O., & Pierron, F. (2014). *Journal of Sound and Vibration, 333(26), 7151-7164.*

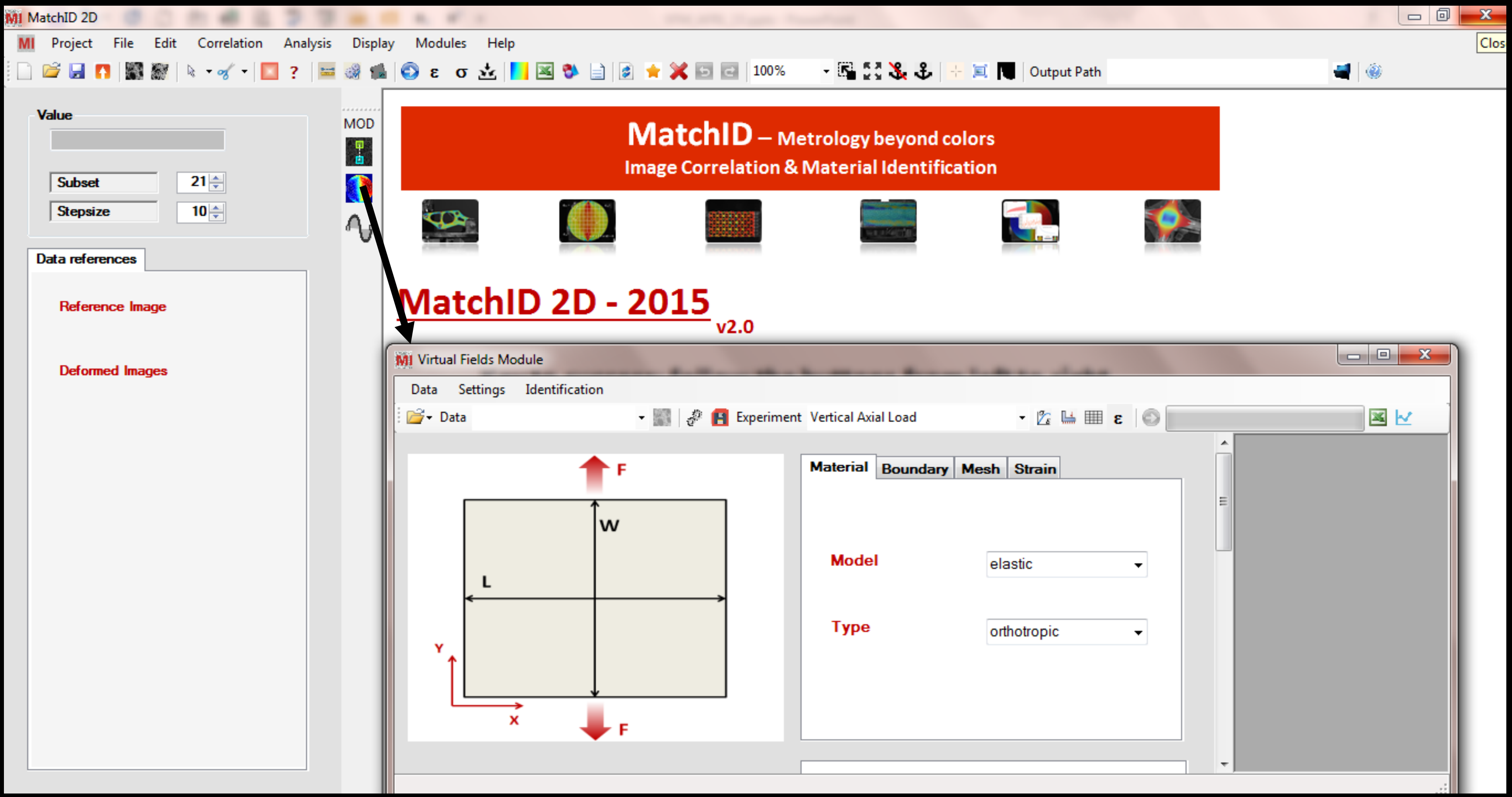
# Conclusion

# Conclusion

- VFM: operational tool, computationally simple and efficient
- Future challenges
  - Design and optimize test configurations, predict error bars (new operational standards)
  - Apply to 3D (X-ray CT + DVC)
  - High rate dynamics (see talk on Tuesday)
  - Optimization of VFs for non-linear VFM
  - Integrate with measurements: MatchID platform

# MatchID

- www.matchIDmbc.com



# More on the VFM

- Theory
- Applications
- Self-training