

Abstract

**Reduced Basis and Stochastic Modeling  
of Liquid Propellant Rocket Engine as a Complex System**

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This new research program will apply interrogation techniques using liquid rocket combustion instability models to explore the instability source by addressing the multi-injector rocket combustion chamber as a complex system with many semi-autonomous components that affect the nonlinear oscillatory macro-behavior. Here, “source” refers to both the triggering mechanism of the instability and the driving mechanism for the ultimate nonlinear oscillation. Both mechanisms can involve similar physical and chemical processes; however, the organization and relative importance of the processes can differ within the two mechanisms. The research is designed to identify (1) the physico-chemical sources that serve as “triggers” of nonlinear instabilities in liquid-propellant rocket engines as well as (2) the physico-chemical sources that serve as drivers of the limit-cycle oscillation and the transient to that cycle. The key relations amongst the initiation process, growth of the nonlinear resonant oscillation, and transient to the limit-cycle will be established. Foundations for control strategies that can inhibit the instability will be identified. New methodologies will be combined with other emerging methodologies to develop an overall approach that not only will aid in addressing the cause and effects of rocket instability but can have other applications.

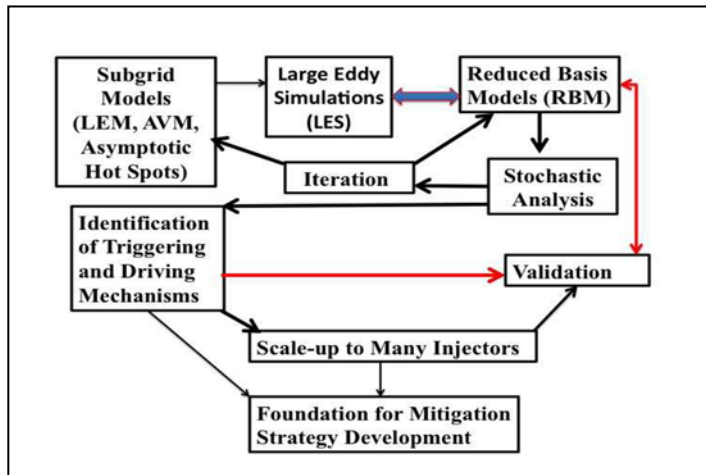
The treatment of combustion and flow processes in a liquid-propellant rocket engine as a complex system [1] using a confluence of advanced mathematical methods is aimed to understand and characterize nonlinear triggering, transient oscillations, and limit-cycle oscillations. Complex systems involve stochastic behaviors of semi-autonomous components networked in a way that allows emergent behavior to develop. Our complex system will include combustion chamber, convergent nozzle, propellant injectors, and all contained flow and thermal structures at supercritical pressure. Uncertainties that justify stochastic approach relate to magnitude, duration, and location of triggering disturbances; property values in supercritical domain; chemical kinetics data; and turbulent flow. An important aspect of this project is the development of advanced mathematical methods to reduce computational burden in detailed CFD calculation of turbulent combustion dynamics. The and stochastic modeling of combustion dynamics [2,3] and Reduced Basis Method (RBM) [4,5] have the potential to perform parametric studies of combustion physics in an affordable time frame, as well as complement a deterministic prediction with stochastic estimates when faced with challenges in modeling.

**Formulation as a Complex System:** Complex systems [1] are usually defined through their characterizing properties. For example, complex systems are characterized by a large number of entities/components that interact locally (as in a network) in a way that allows emergent behavior/patterns to develop. The components of complex systems can span across many different spatial scales and the interaction amongst them is highly nonlinear and occurs over many different time scales. In a rocket engine, complex system components include hardware elements and/or flow/thermal structures.

Uncertainty is ubiquitous in complex systems but emergent behaviors are adaptive and resilient. Stochastic processes may apply to fluctuations of propellant flow rates, fluctuations in fluid properties, and flow turbulence [2]. Stochastic terms may enter analysis as initial conditions, boundary conditions, or directly into differential equations as forcing functions or coefficients. A statistical analysis of complex systems typically reveals the existence of power laws that indicate more than one characteristic scale. Emergent structures of special interest are large-amplitude acoustic oscillations which result from combustion instability. Combustion in liquid propellant rocket engines (LPRE) is a turbulent phenomenon characterized by eddies across a large range of scales interacting nonlinearly, and indeed exhibiting power-law distributions in variables across these scales. The combustion process can generate spontaneously large-amplitude acoustic oscillations (triggered instabilities).

Thus, our system consisting of the combustion chamber, convergent nozzle, propellant injectors, and all contained flow and thermal structures can be viewed as a complex system. The challenge of the project is to develop reliable prediction of such (undesirable) emergent system behavior under uncertainty characterizing the combustion and flow processes.

**Approach:** Our team approach is outlined in Figure 1. *UCI* (Popov, Sideris, and Sirignano) will develop the stochastic framework. They will formulate stochastic partial differential equations in coordination with *Georgia Tech* and *HyPerComp*. *Georgia Tech* (Menon and postdoc) will develop a Large-eddy Simulation (LES) approach and make computations for specified realizations in the stochastic behavior. *HyPerComp* (Munipalli and Ota) will develop reduced



basis models fitting the LES results. These RBMs with their higher computational speeds have the potential to perform parametric studies, and enable inexpensive computations of many realizations for stochastic analysis. *KISS* (Kassoy) will develop and propose thermoacoustic and thermomechanical models to describe relevant combustion phenomena. Some of this modeling will also be done at *UCI* (Sirignano).

**Figure 1. Flow Chart of the Analytical Work.**

Continuing communication and iteration amongst team members will occur through both electronic communication and meetings. The approach and integration of contributions from team members will be tested on model equations as well as with full Navier-Stokes, multicomponent-flow based equations. We will now discuss the various advanced mathematical and computational methods which are integrated in our approach: stochastic processes; asymptotic analysis; large-eddy simulation; reduced-basis modeling.

**Stochastic modeling-Uncertainty quantification:** As a first step in our complex systems approach to the detection of nonlinear triggered instabilities in LPREs, we will develop a

stochastic model of the combustion process to be used for *uncertainty quantification*. Stochastic processes may relate to uncertainty in propellant flow rates, small-scale mixing and chemical reaction, fluid properties, and flow turbulence. Furthermore, uncertainty is also introduced through the Large Eddy Simulation (LES) approach employed by *Georgia Tech*, in particular, during the subgrid modeling phase. Stochastic terms will be incorporated in the deterministic modeling of the process as initial conditions, boundary conditions, or directly into differential equations as forcing functions or coefficients. In the remainder of this section, we summarize our statistical modeling approach.

We consider the general system of stochastic differential equation  $\mathcal{L}(\mathbf{x}, t, \omega; \mathbf{u}) = \mathbf{f}(\mathbf{x}, t, \omega)$  where  $\mathbf{u}(\mathbf{x}, t, \omega)$  is the solution and  $\mathbf{f}(\mathbf{x}, t, \omega)$  is a forcing function.  $\mathcal{L}$  is a (possibly) nonlinear differential operator,  $t \in [0, T]$  is the time variable,  $\mathbf{x} \in D$  collects all spatial variables taking values in a domain  $D$ , and  $\omega \in \Omega$  signifies dependence on random quantities. Stochastic boundary and initial conditions can also be considered. Then, we approximate the solution in terms of a truncated (generalized) Polynomial Chaos Expansion [3] (PCE):  $\mathbf{u}(\mathbf{x}, t, \omega) \cong \sum_{i=0}^N \mathbf{u}_i(\mathbf{x}, t) \Phi_i(Z(\omega))$ , where  $Z = (Z_1, \dots, Z_d)$  are orthonormal Random Variables (RV) and the  $\Phi_i$  are multi-dimensional orthogonal polynomials satisfying  $\langle \Phi_i(Z) \rangle \equiv \int_{I_Z} \Phi_i(z) p_Z(z) dz = \delta_{i0}$  and  $\langle \Phi_i(Z), \Phi_j(Z) \rangle \equiv \int_{I_Z} \Phi_i(z) \Phi_j(z) p_Z(z) dz = \delta_{ij} \gamma_i$ . where  $\delta_{ij} = 1$  if  $i = j$  and 0 otherwise. Here,  $I_Z \subset \mathcal{R}^d$  is the range of values of  $Z$  and  $p_Z(z)$  denotes the Probability Density Function (PDF) of  $Z$ . Several families of  $\Phi_i$  corresponding to some well-known PDF's are documented and can be readily computed symbolically or recursively. For example, for the Gaussian and uniform PDFs the  $\Phi_i$  are the familiar Hermite and Legendre orthogonal polynomials (for  $d = 1$ ), respectively. The number  $N$  of basis polynomials satisfies  $N + 1 = \binom{d+k}{k}$  where  $k$  is the highest degree polynomial  $\Phi_i$  used to approximate  $\mathbf{u}$ . The expansion here is distinct from the RBM expansion (discussed later); the two expansions occur in sequence.

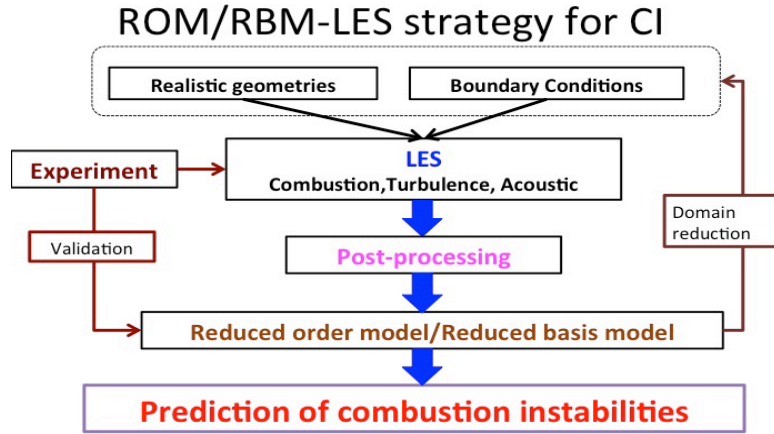
In the Stochastic Galerkin (SG) approach which we are exploring, the coefficients  $\mathbf{u}_i(\mathbf{x}, t)$ , which are deterministic functions, are obtained by requiring that the truncation error is orthogonal to the space spanned by the RV's  $\Phi_i$ . Specifically, we require  $\langle \mathcal{L}(\mathbf{x}, t, \omega; \sum_{i=0}^N \mathbf{u}_i \Phi_i), \Phi_k \rangle = \langle \mathbf{f}(\mathbf{x}, t, \omega), \Phi_k \rangle$ ,  $k = 0, 1, \dots, N$ . These conditions amount to a system of coupled deterministic PDE's for the  $\mathbf{u}_i(\mathbf{x}, t)$  that can be solved with boundary and initial conditions resulting by a similar projection of boundary and initial conditions of the original PDE. This reduction is straightforward when  $\mathcal{L}$  is a linear operator (or contains polynomial nonlinearities) and not much harder otherwise after using numerical quadrature rules to approximate the Expectation “ $\langle \rangle$ ” operator.

In stochastic collocation approaches, the truncation error is forced to be zero at the mesh (nodal) points. The orthogonality of the  $\Phi_i$  implies  $\mathbf{u}_i(\mathbf{x}, t) = \frac{1}{\gamma_i} \langle \mathbf{u}(\mathbf{x}, t, \omega), \Phi_i(Z(\omega)) \rangle \cong \frac{1}{\gamma_i} \sum_{j=1}^M \mathbf{u}(\mathbf{x}, t, \omega^{(j)}) \Phi_i(z^{(j)}) w^{(j)}$  where the  $z^{(j)}$ ,  $j = 1, \dots, M$  are specified samples in  $I_Z$  (quadrature nodes), and  $w^{(j)} \equiv p_Z(z^{(j)})$ . Then, the  $u_i$  can be obtained from the solutions of the  $M$  deterministic PDE's:  $\mathcal{L}(\mathbf{x}, t, \omega^{(j)}; \mathbf{u}^{(j)}) = \mathbf{f}(\mathbf{x}, t, \omega^{(j)})$ ,  $j = 1, \dots, M$  with corresponding boundary and initial conditions.

In both the SG and SC methods, it is necessary to solve *deterministic* PDE's. While in SG only one large system is solved, the SC approach requires the solution of a large number of systems obtained as realizations of the original system by sampling the random quantities. Thus, the simulation approach of *Georgia Tech* and *HyPerComp* can be used without essential modification. The SC approach relates to Monte Carlo (MC) approaches where the random quantities are sampled and statistics are obtained from the PDE solutions to each of these samples. Once the PCE expansion for the solution  $\mathbf{u}(\mathbf{x}, t, \omega)$  has been determined via the SG or the SC methods, statistics on  $\mathbf{u}$  can be readily obtained. For example, it can be shown that  $\langle \mathbf{u}(\mathbf{x}, t, \omega) \rangle \cong \mathbf{u}_0(\mathbf{x}, t)$  and  $\text{CoVar}[\mathbf{u}(\mathbf{x}, t, \omega)] \cong \sum_{i=1}^N \mathbf{u}_i(\mathbf{x}, t) \mathbf{u}_i(\mathbf{x}, t)^T \langle \Phi_i^2 \rangle$ . We will use these statistics to verify power laws and establish that the PDE modeling and LES/RBM simulation approaches preserve the complex system character of the problem. We will then develop machine learning tools for predicting the probability of the emergent behavior (combustion instability) as a function of the location, duration, magnitude, and type of disturbance or variation to the system.

**Large Eddy Simulation (LES):** This is a multi-scale, multi-dimensional problem with a probabilistic character that requires many realizations to develop an understanding as well as an insight for modeling. A brute-force computational effort is impractical and so advanced mathematical analysis leading to cost effective reduced models that can be combined with some limited high fidelity simulation of the spatio-temporal evolution in the combustor is considered as the current strategy. However, in order to develop these reduced order models, a large database of representative test geometries and under realistic test conditions is needed. In particular, since combustion instability is the result of non-linear coupling between unsteady heat release, vortical motion and nonlinear chamber acoustics subject to the system boundary conditions, the simulation strategy is designed to simulate and characterize these nonlinear wave motions in the 3D flow field.

To develop the comprehensive understanding and also to use this database for reduced order modeling, we will carry out LES for configurations that are nominal representation of realistic liquid rocket engine combustors. A well-established code called LESLIE3D (Large-Eddy Simulation with Linear Eddy in 3D) will be employed for all the planned studies. LESLIE3D is a massively parallel multi-block solver for simulating turbulent combustion in realistic geometries. The baseline solver is second-order accurate in time and up to fourth-order accurate in space. The full compressible LES equations are solved using localized dynamic closure for the sub-grid terms, and novel sub-grid closures for turbulence-chemistry interactions are present in this code. Full characteristic based boundary conditions, real gas equation of state and wall heat transfer are included. Recent extensions have focused on developing near-wall models, an implicit capability, and a generalize space-time integration strategy that will eventually allow using even higher order schemes. These capabilities are being developed using other sponsored research but their capabilities will be available for the current planned research.



**Figure 2. Combined LES-ROM/RBM strategy for this research.**

Figure 2 shows schematically our strategy to use LES within the ROM/RBM model development and application to combustion instability in LRE. As shown, the LES model for the realistic configuration will be conducted under experimentally specified conditions and validated wherever possible. The results will be post processed to provide inputs for reduced order modeling (ROM) or reduced basis modeling (RBM). The results of the RBM model will be then used to revisit the simulated conditions and new conditions will be simulated to determine if the ROM/RBM is capable of maintaining predictive accuracy. Iteration between LES, post-processing, ROM/RBM development will form the basis of the initial strategy.

In this project, we plan to study LRE systems under both stable and unstable operating conditions. Also, both single injector and multi-injector shear coaxial systems will be considered for the simulations. Earlier, LES of GOX-GH2, LOX-GH2 and LOX-CH4 supercritical mixing and combustion in single injector conditions have been conducted using the existing LES model, and the results have demonstrated the ability of the code to simulate and predict combustion instability and limit cycle behavior in a high pressure combustion system. This LES capability is currently being extended to study multi-injector flow fields, albeit for a limited set of conditions and number of injectors. These earlier LES results and new ones to be obtained under this effort will form the database for the planned model development for LRP1-GOX in the proposed effort. The single injector case will be a test model to initially study LRP1-GOX supercritical combustion but, eventually, multi-injector test case will be defined based on available configuration.

The LES results currently available are being processed to identify the finite set of basis functions that can be used to describe the spatial behavior of the nonlinear problem. Preliminary analysis of the data from a GOX-GH2 high-pressure combustion instability LES has been used to carry out proper orthogonal decomposition (POD) analysis and to evaluate the approach for RBM analysis. Additionally some classical  $n$ - $\tau$  model analysis for the LES data has been carried out to determine the strengths and limitations of the classical modeling approach for these new complex dynamical systems. Basis functions for RBM analysis are also being extracted. With the use of the basis functions, the governing equations can be reduced to a system of second-order ordinary differential equations (ODEs). Albeit that these ODEs are nonlinear and strongly coupled, their solutions require a relatively insignificant amount of computational resources. These ODEs will then be used to obtain many realizations required for a useful stochastic analysis.

Another goal of this effort is also to identify what is the minimal set of injectors that needs to be simulated to understand injector-to-injector interactions. Injectors may interact only with their nearest neighbors but the exact number needed to predict instability is not clear. Transverse instability can interact with the walls of the chamber and hence may require a full-scale treatment. Shear-coaxial and impinging type injectors have different near-field dynamics and so there are no general rules for single-to-multi-injector scaling. Current effort is to carry out LES for a 5-injector assembly of a subscale LRE that was studied in the past with a goal to obtain additional data for model development.

**Reduced Basis Methods for Combustion Dynamics:** Detailed CFD calculations of turbulent and oscillatory combustion dynamics are computationally very demanding due to the small time and length scales that are present, and the highly nonlinear nature of the physical phenomena that are modeled. Using RBM, we hope to develop surrogate models of the system of equations that are solved in full CFD solvers, specifically the LESLIE-3D code developed at *Georgia Tech*. Thus, a system of equations with a large number of unknowns (proportional to  $K \cdot N_T$  where  $K$  is the number of grid points and the  $N_T$  is the number of time steps needed for a calculation) can be represented by a system with far fewer unknowns, say  $N$ , such that  $N \ll K \cdot N_T$ . It is to be noted that this representation will not linearize the system, or perform other such simplifying approximations of problem physics. Rather, a formal mathematical procedure is performed which minimizes the error incurred in approximating the solution in terms of carefully selected basis functions.

The applications of the RBM that are of interest in this project are:

- Parametric calculations, control, optimization: To use RBM to explore a large parameter space efficiently in large scale computations (e.g.,  $Re$ , mass flow rate, perturbation frequency). This can help in designing control laws, and automatic optimization. Significantly, RBM will provide rapid system performance estimates for stochastic estimates – an important aspect of this project.
- Geometric similarity: To use RBM with parameterized geometries to model topologically similar domains efficiently.
- Surrogate models in complex systems: RBMs can be used to represent subsystems such as injectors when interfacing with more complex combustor models - a network of interoperating RBMs may be used.

The commonly used method to analyze combustion dynamics was developed over many decades, beginning with the efforts of Zinn and Powell [6], Sirignano [7] and recently summarized by Culick [8] in an AGARD monograph. The method involves first expanding the complete system of Navier-Stokes equations using a two-parameter perturbation series. The resulting equations are recast in terms of perturbation pressure, and a nonlinear second order wave equation is derived:

$$\begin{aligned}
& \frac{\partial \rho'}{\partial t} + \bar{\rho} \nabla \cdot \bar{\mathbf{M}}' = -\rho_s + \mathbf{W}' \\
& \boxed{\begin{aligned} \bar{\rho} \frac{\partial \bar{\mathbf{M}}'}{\partial t} + \nabla p' &= -\bar{\mathbf{M}}_s + \mathbf{F}' \\ \bar{\rho} C_v \frac{\partial T'}{\partial t} + \bar{p} \nabla \cdot \bar{\mathbf{M}}' &= -T_s + \mathbf{Q}' \end{aligned}} \quad (M' = V/a_r) \\
& \boxed{\frac{\partial p'}{\partial t} + \gamma \bar{p} \nabla \cdot \bar{\mathbf{M}}' = -p_s + \mathbf{P}'} \\
& \bar{T} \frac{\partial s'}{\partial t} = -s_s + \mathbf{S}'
\end{aligned}
\longrightarrow
\begin{aligned}
& \nabla^2 p' - \frac{1}{\bar{a}^2} \frac{\partial^2 p'}{\partial t^2} = h \\
& \hat{n} \cdot \nabla p' = -f
\end{aligned}$$

At this point, the Galerkin approximation is used to represent the pressure perturbation in terms of a set of position-dependent basis functions as follows:

$$\begin{aligned}
p'(x, t) &= p_r \sum_{n=1}^N \eta_n(t) \psi_n(x), \text{ where the time-dependent coefficients are obtained by projection:} \\
\ddot{\eta}_n + \omega_n^2 \eta_n &= F_n
\end{aligned}$$

**Summary of the RBM formulation:** In this work, we will retain the full set of Favre-filtered Navier-Stokes equations describing turbulent combustion, as presented by Menon et al. [9] and used in LESLIE-3D. These can be written concisely as:

$$\frac{\partial Q}{\partial t} + F(Q) = W,$$

where  $Q$  is the vector of conserved quantities such as mass, momentum and energy,  $F$  is the flux of these quantities and  $W$  is a source term which accounts for species production and depletion rates and so forth.  $Q$  is then approximated as  $Q_{RBM}$  using a modal expansion (Galerkin technique)

in terms of modes  $\psi_n$  as:  $Q_{RBM}(x, t) = \sum_{n=1}^N Q_R(t) \psi_n(x)$ , where the modes are obtained such that

such an expansion minimizes an appropriately defined numerical error:  $\|Q(x, t) - Q_{RBM}(x, t)\| \leq \varepsilon$ .

The coefficients  $Q_R$  are obtained from a system of 1<sup>st</sup> order ordinary differential equations:

$$\frac{d Q_R(t)}{d t} = A F(P^T \psi_n(x) Q_R(t)) + W(\psi_n(x) Q_R(t))$$

where the matrices  $A$  and  $P$  are determined as part of the model reduction process. The calculation proceeds in two parts: The first is performed “offline”, when the solution to the full set of equations is obtained at a set of carefully selected values of time (and parameter if any). The basis functions  $\psi_n$  are computed and a computationally efficient representation of the flux  $F$  is obtained – this is an important aspect in nonlinear calculations.

The second part in RBM is an “online” calculation which involves the solution to the ODEs mentioned above. These equations are solved to obtain temporal coefficients of the basis

functions, which by requirement are far fewer than the full number of unknowns, as mentioned earlier.

**RBM challenges and strategy:** We are seeking to utilize a method that has gained much momentum in the applied mathematics community in solving various canonical problems in physics [4,5,10]. An extension to flows as complex as we are addressing here, has never been attempted thus far. Issues pertaining to stability and computational efficiency in strongly nonlinear dynamics must be addressed and refined as these methods mature. *HyPerComp* personnel have ongoing collaborations with leading researchers in this area and have begun to address these concerns using a broad generalized approach to RBM development. A paper on “blackbox” RBM for nonlinear time marching problems is presently under preparation and relevant results to this project will be summarized in a forthcoming report.

**RBM Status:** A proof-of-concept software has been written in MATLAB (for demonstration purposes) to design appropriate RBM techniques for unsteady nonlinear PDEs. At its core are a number of variants of the Discrete Empirical Interpolation Method (DEIM) [10]. Simultaneously, a software suite named HDrbm is being built using the SCALAPACK libraries to extend these developments to high performance computing applications. Demonstration cases in 1-D and 2-D have shown promising results. A full implementation with an interface to LESLIE-3D will be made after confidence has been established in the accuracy, efficiency and stability aspects of the method. We expect to begin demonstrations for realistic LPRE cases in early 2013.

**Use of Model Equations:** Clearly, we have both a challenging, multifaceted mathematical problem and a multi-step, advanced mathematical approach. For that reason, it is sensible to test first our approach using a system of equations that retains essential aspects of the original problem before we move fully to the original problem.

The model equations should retain essential physics for the combustion dynamics but eliminate much of the secondary physics which could be added in later studies. This model should allow the testing of our statistical approaches before we engage in a full analysis. We chose first to develop a model which focuses on transverse oscillations in a cylindrical chamber allowing averaging over the axial direction and reduction to a two-dimensional problem in the transverse coordinates. Kinematic waves are neglected leaving only the longer acoustic waves. These kinematic waves travel primarily in the axial direction and because of larger gradients (i.e., shorter wavelengths) are more likely to be vitiated by turbulent mixing. Viscosity, heat-conductive, and turbulent-mixing effects on the longer acoustic waves are neglected but they can remain effective for the shorter-scale combustion process modelling. A simplified “short” multi-orifice nozzle boundary condition is used.

The resulting coupled nonlinear wave equations for pressure  $p$ , tangential velocity component  $u_\theta$ , and radial velocity component  $u_r$  follow.



$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} + A p^{\frac{\gamma-1}{2\gamma}} \frac{\partial p}{\partial t} - B p^{\frac{\gamma-1}{\gamma}} \left[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} \right] = & \frac{(\gamma-1)}{\gamma} \frac{1}{p} \left( \frac{\partial p}{\partial t} \right)^2 + (\gamma-1) \frac{\partial E}{\partial t} \\ & + \gamma p^{\frac{\gamma-1}{\gamma}} \left[ \frac{\partial^2 (p^{\frac{1}{\gamma}} u_r^2)}{\partial r^2} + \frac{2}{r} \frac{\partial (p^{\frac{1}{\gamma}} u_r^2)}{\partial r} \right. \\ & + \frac{2}{r} \frac{\partial^2 (p^{\frac{1}{\gamma}} u_r u_\theta)}{\partial r \partial \theta} + \frac{2}{r^2} \frac{\partial (p^{\frac{1}{\gamma}} u_r u_\theta)}{\partial \theta} \\ & \left. + \frac{1}{r^2} \frac{\partial^2 (p^{\frac{1}{\gamma}} u_\theta^2)}{\partial \theta^2} - \frac{1}{r} \frac{\partial (p^{\frac{1}{\gamma}} u_\theta^2)}{\partial r} \right] \end{aligned}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_\theta \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + \frac{C}{r p^{\frac{1}{\gamma}}} \frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + \frac{C}{p^{\frac{1}{\gamma}}} \frac{\partial p}{\partial r} = 0$$

$E$  is the energy per unit volume per unit time released by the combustion process. Modelling of  $E$  is required. Some discussion of that aspect will be given in the next section. Stochastic terms may enter the analysis as initial conditions, boundary conditions, or directly into the above partial differential equations as forcing functions or coefficients. The solution will focus on the emergent structures of special interest which are large-amplitude acoustic oscillations resulting from combustion instability.

We plan to solve the model system of equations in several ways: (i) exact numerical solution, (ii) use of the RBM method, and (iii) perturbation expansion. The numerical solution is the analog of the LES for the full original problem; so, methods (i) and (ii) will be intertwined. The perturbation method will not be as accurate quantitatively but it will order the interactions amongst various portions of the essential physics. Potentially, useful guidance will result.

**Modelling of the Combustion Process:** For the LES subgrid analysis and for the model equations, it will be necessary to consider models of the combustion process. Several approaches will be followed here. Previously used LES subgrid models such as the linear eddy mixing model (LEM) will be used. Other asymptotic models for mixing in the vortices might be considered if time allows. A certain asymptotic approach related to “hot spots” is discussed in the next section. We are also developing a mixing and reaction model for co-axial injector jets operating in an oscillating field. Through the conservation equations, these combustion models will be coupled to the flow during the initiation of resonant oscillations and during the limit-cycle behavior.

**Asymptotic Analysis of Combustion Process:** *KISS* (Kassoy) is developing a consistent, systematic nondimensional formulation of the conservation equations for a reactive supercritical fluid in order to explain the origin of mechanical disturbances arising from spatially distributed transient chemical energy addition (combustion) in a rocket chamber. The analyses identify nondimensional parameters used to characterize the thermophysics and demonstrate that the

generation of acoustic and other disturbances is sensitive to the ratio of the local energy addition time scale to the local acoustic time scale and to the energy added. A model based on spatially distributed, transient energy addition occurring in an isolated hot spot (the near field), valid for a small time-scale ratio, is characterized by nearly constant density heat addition and pressure increasing with temperature. Local gas expansion is the source of gasdynamic disturbances in the unheated environment (the far field). In contrast, large-ratio thermomechanical phenomena in a hot spot are characterized by a nearly constant pressure process, where density varies inversely with the rising temperature. Vanishingly small pressure gradients are the source of extremely weak mechanical disturbances in the far field. A distributed, transient energy addition model, based on an  $O(1)$  time-scale ratio is used to describe first and second order linear, nonhomogeneous acoustic wave equations in a rectangle and a cylinder. Solutions include a component forced by the heat addition term and all the eigenmodes appropriate to the geometry and boundary conditions. Nonlinear forcing terms in the second order equations can be used to determine the modes most likely to be affected by nonlinear processes on a longer time scale. Asymptotic analysis of the rocket combustion solution sensitivity may inform stochastic inputs to the larger complex system analysis.

**Summary:** An innovative approach has been outlined to explore the triggering mechanism of the instability and the driving mechanism for the ultimate nonlinear oscillation by addressing the multi-injector rocket combustion chamber as a complex system with many semi-autonomous components that affect the nonlinear oscillatory macro-behavior. The research is designed to identify the sources that serve as “triggers” of nonlinear instabilities in liquid-propellant rocket engines as well as the sources that serve as drivers of the limit-cycle oscillation and the transient to that cycle. The key relations amongst the initiation process, growth of the nonlinear resonant oscillation, and transient to the limit-cycle will be established. The combination of new methodologies and emerging methodologies may not only aid in addressing the liquid-propellant rocket instability but can have other broader applications.

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